In the previous chapter, we have studied about matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like

\[ \begin{align*}
a_1x + b_1y &= c_1 \\
a_2x + b_2y &= c_2
\end{align*} \]

can be represented as

\[ \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \]

Now, this system of equations has a unique solution or not, is determined by the number \( a_1b_2 - a_2b_1 \). (Recall that if \( a_1b_2 - a_2b_1 \neq 0 \), then the system of linear equations has a unique solution). The number \( a_1b_2 - a_2b_1 \) which determines uniqueness of solution is associated with the matrix

\[ \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \]

and is called the determinant of \( A \) or \( \det A \). Determinants have wide applications in Engineering, Science, Economics, Social Science, etc.

In this chapter, we shall study determinants up to order three only with real entries. Also, we will study various properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and solution of linear equations in two or three variables using inverse of a matrix.

**4.2 Determinant**

To every square matrix \( A = [a_{ij}] \) of order \( n \), we can associate a number (real or complex) called determinant of the square matrix \( A \), where \( a_{ij} = (i, j)^{th} \) element of \( A \).
This may be thought of as a function which associates each square matrix with a unique number (real or complex). If $M$ is the set of square matrices, $K$ is the set of numbers (real or complex) and $f : M \to K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of $A$. It is also denoted by $|A|$ or $\det A$ or $\Delta$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of $A$ is written as $|A| = \det (A)$.

**Remarks**

(i) For matrix $A$, $|A|$ is read as determinant of $A$ and not modulus of $A$.

(ii) Only square matrices have determinants.

### 4.2.1 Determinant of a matrix of order one

Let $A = \begin{bmatrix} a \end{bmatrix}$ be the matrix of order 1, then determinant of $A$ is defined to be equal to $a$.

### 4.2.2 Determinant of a matrix of order two

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ be a matrix of order $2 \times 2$, then the determinant of $A$ is defined as:

$$
\det (A) = |A| = a_{11}a_{22} - a_{21}a_{12}
$$

**Example 1**

Evaluate $\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}$.

**Solution**

We have $\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix} = 2(2) - 4(-1) = 4 + 4 = 8$.

**Example 2**

Evaluate $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

**Solution**

We have $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x \begin{bmatrix} x \end{bmatrix} - \begin{bmatrix} x + 1 \\ x - 1 \end{bmatrix} = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$.

### 4.2.3 Determinant of a matrix of order $3 \times 3$

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order $3 \times 3$. 

$$
\begin{vmatrix}
  x & x + 1 \\
  x - 1 & x
\end{vmatrix} = x(x) - (x + 1)(x - 1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1
$$
3 corresponding to each of three rows (R₁, R₂ and R₃) and three columns (C₁, C₂ and C₃) giving the same value as shown below.

Consider the determinant of square matrix \( A = [a_{ij}]_{3 \times 3} \)

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

i.e.,

**Expansion a**

**Step 1** Multiply first element \( a_{11} \) of \( R_1 \) by \((-1)^{1+1}(–1)\) sum of suffixes in \( a_{11} \) and with the second order determinant obtained by deleting the elements of first row (R₁) and first column (C₁) of \( |A| \), i.e., 

\[
(–1)^{1+1} a_{11}
\]

**Step 2** Multiply 2nd element \( a_{12} \) of \( R_1 \) by \((-1)^{1+2}\) sum of suffixes in \( a_{12} \) and the second order determinant obtained by deleting elements of first row (R₁) and 2nd column (C₂) of \( |A| \), i.e., 

\[
(–1)^{1+2} a_{12}
\]

**Step 3** Multiply third element \( a_{13} \) of \( R_1 \) by \((-1)^{1+3}\) sum of suffixes in \( a_{13} \) and the second order determinant obtained by deleting elements of first row (R₁) and 3rd column (C₃) of \( |A| \), i.e., 

\[
(–1)^{1+3} a_{13}
\]

**Step 4** Now the expansion of determinant of \( A \), i.e., \( |A| \) written as sum of all three terms obtained

\[
\det A = |A| = a_{11} |a_{32} & a_{33} \ldots \ldots \ldots \ldots |a_{31} & a_{33} | \\
+ (-1)^1 a_{12} |a_{21} & a_{22} |a_{31} & a_{32} |
\]

or

\[
|A| = a_{11} (a_{22} a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{31} a_{23}) \\
+ a_{13} (a_{21} a_{32} - a_{31} a_{22})
\]
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} - a_{11} a_{32} a_{23} - a_{12} a_{21} a_{33} + a_{12} a_{31} a_{23} + a_{13} a_{21} a_{32}$$

$$- a_{13} a_{31} a_{22} \quad \ldots (1)$$

**Note** We shall apply all four steps together.

Expandin

$$|A| = a_{23} a_{11} a_{32} - a_{21} a_{11} a_{33}$$

$$a_{13} a_{31} a_{22} \quad \ldots (2)$$

By expanding along the first column $C_1$,

$$|A| = a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21} (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ a_{31} (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11} (a_{22} a_{33} - a_{23} a_{32}) - a_{21} (a_{12} a_{33} - a_{13} a_{32}) + a_{31} (a_{12} a_{23} - a_{13} a_{22})$$
\[ \det A = a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} \]

\[ \ldots (3) \]

Clearly, values of \( \det A \) in (1), (2), and (3) are equal. It is left as an exercise to the reader to verify that the values of \( \det A \) by expanding along \( R_3 \), \( C_2 \), and \( C_3 \) are equal to the value of \( \det A \) obtained in (1), (2) or (3).

**Remarks**

(i) For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.

(ii) While expanding, instead of multiplying by \((-1)^{i+j}\), we can multiply by +1 or –1 according as \((i+j)\) is even or odd.

(iii) Let \( A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \). Then, it is easy to verify that \( A = 2B \). Also

\[ \det A = 0 - 8 = -8 \text{ and } \det B = 0 - 2 = -2. \]

Observe that, \( \det A = 4(–2) = 2^2 \det B \) or \( \det A = 2^n \det B \), where \( n = 2 \) is the order of square matrices \( A \) and \( B \).

In general, if \( A = kB \) where \( A \) and \( B \) are square matrices of order \( n \), then \( \det A = k^n \).

**Example 3**

Evaluate the determinant

\[ \Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}. \]

**Solution**

Note that in the third column, two entries are zero. So expanding along third column (\( C_3 \)), we get

\[ \Delta = -1 3 1 2 1 2 \]

\[ 4 - 0 0 \]

\[ 4 1 -1 3 \]

\[ = 4(-1 - 12) - 0 + 0 = -52 \]

**Example 4**

Evaluate

\[ \Delta = \begin{vmatrix} 0 & \sin \alpha & -\cos \alpha \\ -\sin \alpha & 0 & \sin \beta \\ \cos \alpha & -\sin \beta & 0 \end{vmatrix}. \]
Solution Expanding along R₁, we get

\[
\Delta = 0 \begin{vmatrix}
0 & \sin \beta & -\sin \alpha & \sin \beta & -\cos \alpha & -\sin \alpha & 0 \\
-\sin \beta & 0 & -\sin \alpha & \sin \beta & \cos \alpha & -\sin \alpha & 0 \\
\end{vmatrix}
\]

\[
= 0 - \sin \alpha (0 - \sin \beta \cos \alpha) - \cos \alpha (\sin \alpha \sin \beta - 0)
\]

Example 5 F

Solution We

i.e.

i.e.

Hence

Evaluate the determinants

1. \[
\begin{vmatrix}
2 & \\
-5 & \\
\end{vmatrix}
\]

2. (i) \[
\begin{vmatrix}
\cos \theta & \sin \theta \\
\sin \theta & \cos \theta \\
\end{vmatrix}
\]

(ii) \[
\begin{vmatrix}
2 & -1 & -1 \\
1 & 1 & -2 \\
\end{vmatrix}
\]

3. If \( A = \[
\begin{bmatrix}
1 & 2 \\
4 & 2 \\
\end{bmatrix}
\]

Then show that \( |2A| = 4 |A| \)

4. If \( A = \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
0 & 0 & 4 \\
\end{bmatrix}
\]

Then show that \( |3A| = 27 |A| \)

5. Evaluate the determinants

(i) \[
\begin{vmatrix}
3 & -1 & -2 \\
0 & 0 & -1 \\
3 & -5 & 0 \\
\end{vmatrix}
\]

(ii) \[
\begin{vmatrix}
3 & -4 & 5 \\
1 & 1 & -2 \\
2 & 3 & 1 \\
\end{vmatrix}
\]
6. If \( A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix} \) find \( |A| \).

7. Find values of \( x \), if

(i) \( \begin{pmatrix} 2 & 4 & 2 \\ 5 & 1 & 6 \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \)

(ii) \( \begin{pmatrix} 2 & 3 & 3 \\ 4 & 5 & 2 \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \)

8. If \( \begin{pmatrix} 2 & 6 & 2 \\ 18 & 18 & 6 \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \), then \( x \) is equal to

(A) 6 (B) ± 6 (C) – 6 (D) 0

4.3 Properties of Determinants

In this section, we will study some properties of determinants which simplifies its evaluation by obtaining maximum number of zeros in a row or a column. These properties are true for determinants of any order. However, we shall restrict ourselves to determinants of order 3 only.

**Property 1**

The value of the determinant remains unchanged if its rows and columns are interchanged.

**Verification**

Expandin
Expanding $\Delta_1$ along first column, we get

$$\Delta_1 = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Hence $\Delta = \Delta_1$

**Remark** It follows from above property that if $A$ is a square matrix, then $\det(A) = \det(A')$, where $A'$ = transpose of $A$.

**Note** If $R_i = i^{th}$ row and $C_i = i^{th}$ column, then for interchange of row and columns, we will symbolically write $C_i \leftrightarrow R_i$

Let us verify the above property by example.

**Example 6**

**Solution** Exp

By interchanging

Clearly

Hence, Property 1 is verified.

**Property 2** If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

**Verification** Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
Expanding along first row, we get
\[ \Delta = a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1) \]

Interchanging first and third rows, the new determinant obtained is given by
\[ \Delta = \begin{vmatrix} c_1 & c_2 & c_3 \end{vmatrix} \]

Expanding along third row, we get
\[ \Delta = a_1 (c_2 b_3 - b_2 c_3) - a_2 (c_1 b_3 - b_2 c_1) + a_3 (c_1 b_2 - b_2 c_2) \]

Clearly \[ \Delta = - \Delta \]

Similarly, we can verify the result by interchanging any two columns.

**Example 7**

**Solution** \( \Delta = \)

Interchanging

Expanding the determinant \( \Delta_1 \) along first row, we have
\[ \Delta_1 = 2 \begin{vmatrix} 5 & -7 & 1 \\ 0 & 4 & 1 \\ 6 & 4 & 6 \end{vmatrix} \]

\[ = 2 (20 - 0) + 3 (4 + 42) + 5 (0 - 30) \]

\[ = 40 + 138 - 150 = 28 \]
Clearly \( \Delta_1 = -\Delta \)

Hence, Property 2 is verified.

**Property 3** If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

**Proof** If we interchange the identical rows (or columns) of the determinant \( \Delta \), then \( \Delta \) does not change. However, by Property 2, it follows that \( \Delta \) has changed its sign.

Therefore, or

Let us verify the above property by an example.

**Example 8**

**Solution** Exp

Here \( R_1 \) and \( R_3 \)

**Property 4** If each element of a row (or a column) of a determinant is multiplied by a constant \( k \), then its value gets multiplied by \( k \).

**Verification**

and \( \Delta_1 \), be the determinant obtained by multiplying the elements of the first row by \( k \).

Then

\[
\begin{vmatrix}
\end{vmatrix}
\]

Expanding along first row, we get

\[
\Delta_1 = k \begin{vmatrix} a_2 & c_2 & b_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & a_3 \end{vmatrix} + k \begin{vmatrix} a_2 & b_3 - b_2 a_3 \end{vmatrix}
\]

\[
= k \left[ a_1 \begin{vmatrix} b_3 & c_2 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_3 - b_2 a_3 \end{vmatrix} \right]
\]

\[
= k \Delta
\]
Hence
\[
\begin{vmatrix}
  k & a_1 & k b_1 & k c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix} = k
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}
\]

Remarks
(i) By this property, we can take out any common factor from any one row or any column of a given determinant.

(ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero. For example
\[
\Delta = \begin{vmatrix}
  1 & 2 & 3 \\
  1 & 2 & 3 \\
  1 & 2 & 3 \\
\end{vmatrix} = 0
\begin{vmatrix}
  a & a & a \\
  b & b & b \\
  \lambda & \lambda & \lambda \\
\end{vmatrix}
\]

Example 9 E

Solution
\[
\begin{vmatrix}
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} = 0
\]

Property 5
If as sum of two (or more) determinants are expressed as sum of two (or more) determinants.

For example,
\[
\begin{vmatrix}
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} = \begin{vmatrix}
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} + \begin{vmatrix}
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

Verification
\[
L.H.S. = \begin{vmatrix}
  a_1 + \lambda_1 & a_2 + \lambda_2 & a_3 + \lambda_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

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Expanding the determinants along the first row, we get
\[ \Delta = (a_1 + \lambda_1) (b_2 c_3 - c_2 b_3) - (a_2 + \lambda_2) (b_1 c_3 - b_3 c_1) + (a_3 + \lambda_3) (b_1 c_2 - b_2 c_1) \]

(by rearranging terms)

Similarly, we

**Example 10**

**Solution** We

\[
\begin{vmatrix} c \\ 2z \\ z \end{vmatrix} \]

(by Property 5)

and Property 4)

**Property 6** If, of corresponding elements of determinant remains the same, of determinant the operation

\[ R_i \rightarrow R_i + k \]

**Verification**

Let \[ \Delta = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \] and \[ \Delta_1 = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \],

where \( \Delta_1 \) is obtained by the operation \( R_i \rightarrow R_i + kR_j \).

Here, we have multiplied the elements of the third row \( R_3 \) by a constant \( k \) and added them to the corresponding elements of the first row \( R_1 \).

Symbolically, we write this operation as \( R_i \rightarrow R_i + kR_j \).
Now, again

\[ \Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k c_1 & k c_2 & k c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \]  

(Using Property 5)

Hence

**Remarks**

(i) If \( \Delta_1 \) is the determinant obtained by applying \( R_i \to kR_i \) or \( C_i \to kC_i \) to the determinant \( \Delta \), then

\[ \Delta_1 = k \Delta. \]

(ii) If more than one operation like \( R_i \to R_i + kR_j \) is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another.

**Example 11**

**Solution** Applying operations \( R_2 \to R_2 - 2R_1 \) and \( R_3 \to R_3 - 3R_1 \) to the given determinant \( \Delta \), we have

\[ \Delta = \begin{vmatrix} 0 & 2 & 3 \\ 0 & 3 & 6 \\ a & a & b \end{vmatrix} + \begin{vmatrix} 0 & 2 & 3 \\ 0 & 7 & 6 \\ a & a & b \end{vmatrix} \]

Now applying \( R_3 \to R_3 - 3R_2 \), we get

\[ \Delta = \begin{vmatrix} 0 & 2 & 3 \\ 0 & 0 & 3 \\ a & a & b \end{vmatrix} \]

Expanding along \( C_1 \), we obtain

\[ \Delta = a \begin{vmatrix} a & 2a + b \\ 0 & a \end{vmatrix} + 0 + 0 \]

\[ = a (a^2 - 0) = a (a^2) = a^3 \]
Example 12 Without expanding, prove that
\[
\Delta = \begin{vmatrix}
x + y & y + z & z + x \\
z & x & y \\
1 & 1 & 1
\end{vmatrix} = 0
\]

Solution Apply

Since the entering

Example 13

Solution Apply

Taking factors \((b - a)\) and \((c - a)\) common from \(R_2\) and \(R_3\), respectively, we get

\[
\Delta = (b - a)(c - a)[(-b + c)]
\]

(Expanding along first column)

Example 14 Prove that
\[
\begin{vmatrix}
b & c + a & b \\
c & c & a + b \\
b + c & a & a
\end{vmatrix} = 4abc
\]

Solution Let \(\Delta = \begin{vmatrix}
b + c & a & a \\
b & c + a & b \\
c & c & a + b
\end{vmatrix}\)
Applying \( R_1 \rightarrow R_1 - R_2 - R_3 \) to \( \Delta \), we get

\[
\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}
\]

Expanding along \( R_1 \), we obtain

\[
\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} + \begin{vmatrix} b & c+a & b \\ c & c & a+b \\ 0 & -2c & -2b \end{vmatrix} \]

\[
= 2c(a - b) - 2b(c - a) + 2bc(a + b) - 2ab(c + b) + 4abc
\]

Example 15

show that \( 1 + \)

Solution We

\[
\Delta = \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} x \\ y \\ z \end{vmatrix}
\]

\[
= (-1)^3 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \end{vmatrix}
\]

\[
= (-1)^3 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \end{vmatrix}
\]

\[
= (1 + xyz)
\]

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\[ \Delta = (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y - x & y^2 - x^2 \\ 0 & z - x & z^2 - x^2 \end{vmatrix} \]

(Using \( R_2 \rightarrow R_2 - R_1 \) and \( R_3 \rightarrow R_3 - R_1 \))

Taking out common factor \((y - x)\) from \( R_2 \) and \((z - x)\) from \( R_3 \), we get

\[ \Delta = 2x^2 \begin{vmatrix} 1 + x & & \\ y - x & 1 & \\ z - x & 1 \end{vmatrix} \]

Since \( \Delta = 0 \) and \( x, y, z \) are all different, i.e., \( x - y \neq 0, \ y - z \neq 0, \ z - x \neq 0 \), we get

\[ 1 + xyz = 0 \]

Example 16

\[ \begin{vmatrix} 1 + a & & \\ 1 & 1 & \\ 1 & 1 & \\ 1 & 1 & \end{vmatrix} \]

Solution Taki

L.H.S.

Applying \( R_1 \rightarrow \)

\[ \Delta = abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix} \]
\[ \det \begin{vmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{b} & \frac{1}{b} + 1 \\ 1 & \frac{1}{b} & \frac{1}{b} \end{vmatrix} \]

Now applying \( C_2 \rightarrow C_2 - C_1 \), \( C_3 \rightarrow C_3 - C_1 \), we get

\[ \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \]

Note

\[ A \]

\[ C_1 \rightarrow C_1 - a \]

EXERCISE 4.2

Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

1. \[ \begin{vmatrix} x & a \\ y & b \\ z & c \end{vmatrix} + \begin{vmatrix} x & a \\ y & b \\ z & c \end{vmatrix} = 0 \]

2. \[ \begin{vmatrix} a & b & b \\ b & c & c \\ c & a & a \end{vmatrix} - \begin{vmatrix} a & b & b \\ b & c & c \\ c & a & a \end{vmatrix} = 0 \]

3. \[ \begin{vmatrix} 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0 \]

4. \[ \begin{vmatrix} 1 & ca & b(c + a) \\ 1 & ab & c(a + b) \end{vmatrix} = 0 \]

5. \[ \begin{vmatrix} b + c & q + r & y + z \\ c + a & r + p & z + x \\ a + b & p + q & x + y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \]
6. \[ \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0 \]

7. \[ \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2 \]

By using properties of determinants, in Exercises 8 to 14, show that:

8. (i) \[ \begin{vmatrix} x & x^2 \\ y & y^2 \\ z & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)(xy + yz + zx) \]

9. \[ \begin{vmatrix} a & a^3 \\ 1 & 1 \\ x & x^2 \end{vmatrix} \]

10. (i) \[ \begin{vmatrix} 2; & 2; \\ 2; & 2; \end{vmatrix} \]

11. (i) \[ \begin{vmatrix} 2b & b - c - a & 2b \\ 2c & 2c & c - a - b \end{vmatrix} = (a + b + c) \]

12. (i) \[ \begin{vmatrix} x + y + 2z & x & y \\ z & y + z + 2x & y \\ z & x & z + x + 2y \end{vmatrix} = 2(x + y + z)^3 \]
12. \[
\begin{vmatrix}
1 & x & x^2 \\
x^2 & 1 & x \\
x & x^2 & 1
\end{vmatrix} = (1 - x^3)^2
\]

13. \[
\begin{vmatrix}
1 + a^2 & -2ab & 2b \\
2ab & -1 & -a \\
a^2 + 1 & ab & ca
\end{vmatrix}
\]

Choose the correct answer in Exercises 15 and 16.

15. Let \( A \) be a square matrix of order 3 \( \times \) 3, then \( |kA| \) is equal to
   (A) \( k|A| \) (B) \( k^2|A| \) (C) \( k^3|A| \) (D) \( 3|A| \)

16. Which of the following is correct
   (A) Determinant is a square matrix.
   (B) Determinant is a number associated to a matrix.
   (C) Determinant is a number associated to a square matrix.
   (D) None of these

4.4 Area of a Triangle

In earlier classes, we have studied that the area of a triangle whose vertices are
\((x_1, y_1), (x_2, y_2), (x_3, y_3)\), is given by the expression
\[
\Delta = \frac{1}{2} \begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1
\end{vmatrix}
\]

Remarks
(i) Since area is a positive quantity, we always take the absolute value of the determinant in (1).
(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

**Example 17** Find the area of the triangle whose vertices are (3, 8), (–4, 2) and (5, 1).

**Solution** The area of triangle is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix} = \frac{1}{2} [1 \times (3 \times 1 - 2 \times 1) - 2 \times (-4 \times 1 - 5 \times 1) + 1 \times (-4 \times 2 - 3 \times 1)]$$

$$= \frac{1}{2} [1 \times 1 - 2 \times (-4 - 5) + 1 \times (-8 - 3)] = \frac{1}{2} [1 + 14 - 11] = \frac{1}{2} \times 4 = 2$$

**Example 18** Find the equation of the line joining A(1, 3) and B(0, 0) using determinants and find k if D(k, 0) is a point such that area of triangle ABD is 3 sq units.

**Solution** Let P(x, y) be any point on AB. Then, area of triangle ABP is zero (Why?). So

$$0 = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ x & y & 1 \end{vmatrix}$$

This gives

$$x - 3y = 0$$

which is the equation of the required line AB.

Also, since the area of the triangle ABD is 3 sq units, we have

$$1 \begin{vmatrix} 2 & 7 & 1 \\ 0 & 1 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

This gives,

$$\frac{3}{2} = \pm 3, \text{ i.e., } k = \mp 2.$$

**EXERCISE 4.3**

1. Find area of the triangle with vertices at the point given in each of the following:
   (i) (1, 0), (6, 0), (4, 3)  
   (ii) (2, 7), (1, 1), (10, 8)  
   (iii) (–2, –3), (3, 2), (–1, –8)
2. Show that points 
   \( A (a, b + c), B (b, c + a), C (c, a + b) \) are collinear.

3. Find values of \( k \) if area of triangle is 4 sq. units and vertices are 
   (i) \((k, 0), (4, 0), (0, 2)\)  
   (ii) \((-2, 0), (0, 4), (0, k)\)

4. (i) Find equation of line joining \( A (1, 2), B (3, 6) \).
   (ii) Find equation of line joining \( A (3, 1), B (9, 3) \).

5. If area of triangle with vertices \( (2, -6), (5, 4), (k, 4) \) is 35 sq units, then \( k \) is 
   (A) 12 (B) -2 (C) -12, -2 (D) 12, -2

4.5 Minors

In this section, we will learn to write the expansion of a determinant in compact form using minors.

**Definition 1**

Minor of an element \( a_{ij} \) of a determinant is the determinant obtained by deleting its \( i \)th row and \( j \)th column in which element \( a_{ij} \) lies. Minor of an element \( a_{ij} \) is denoted by \( M_{ij} \).

**Remark** Minor of an element of a determinant of order \( n \) \((n \geq 2)\) is a determinant of order \( n - 1 \).

**Example 19**

Find the minor of element 6 in the determinant

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix}
\]

**Solution** Since 6 lies in the second row and third column, its minor \( M_{23} \) is given by

\[
M_{23} = \begin{vmatrix}
1 & 2 \\
7 & 8 \\
\end{vmatrix} = 8 - 14 = -6
\]

**Definition 2**

Cofactor of an element \( a_{ij} \), denoted by \( A_{ij} \) is defined by

\[
A_{ij} = (-1)^{i+j} M_{ij}
\]

**Example 20** Find minors and cofactors of all the elements of the determinant

\[
\begin{vmatrix}
1 & -2 \\
4 & 3 \\
\end{vmatrix}
\]

**Solution** Minor of the element \( a_{ij} \) is \( M_{ij} \)

Here \( a_{11} = 1 \). So \( M_{11} = \text{Minor of } a_{11} = 3 \)

\( M_{12} = \text{Minor of the element } a_{12} = 4 \)

\( M_{21} = \text{Minor of the element } a_{21} = -2 \)
Minor of the element $a_{22} = 1$

Now, cofactor of $a_{ij}$ is $A_{ij}$. So

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (4) = -4$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-2) = 2$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (1) = 1$

Example 21

Solution

Minor of

Cofactor

Minor of

Cofactor

Remark

Expansion of determinant

Expa $a_{22}$

ve $a_{32}$

$\Delta = (-1)^1$

$= a_{11} A$

$= \text{sum of product of elements of any row (or column) with their corresponding cofactors.}$

Note

If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example,
$\Delta = a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23}$

$= a_{11} (-1)^{1+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} (-1)^{1+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} (-1)^{1+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly,

Example 22

\[
\begin{pmatrix}
2 & -3 \\
6 & 0 \\
1 & 5
\end{pmatrix}
\]

Solution We

\[M_{11} = 0\]
\[M_{12} = \begin{vmatrix} 1 & 5 \\ 1 & 5 \end{vmatrix} = -4\]
\[M_{13} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13;\]
\[A_{21} = (-1)^{2+1} (13) = -13\]
\[M_{21} = 4\]
\[M_{22} = -19\]
\[M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13;\]
\[A_{23} = (-1)^{2+3} (13) = -13\]
\[M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12;\]
\[A_{31} = (-1)^{3+1} (-12) = -12\]
Now \( a_{11} \):
So \( a_{11} = 2 \)

Write Minors

1. (i) \[
\begin{vmatrix}
2 \\
0
\end{vmatrix}
\]

2. (i) \[
\begin{vmatrix}
1 \\
0 \\
0
\end{vmatrix}
\]

3. Using \( \xi \) \[
\begin{vmatrix}
8 \\
1 \\
3 \\
yz \\
zx \\
xy
\end{vmatrix}
\]

4. Using \( \zeta \) \[
\begin{vmatrix}
A_{11} \\
A_{21} \\
A_{31}
\end{vmatrix}
\]

5. If \( \Delta = \)
is given by

(A) \( a_{11} A_{11} + a_{12} A_{31} + a_{13} A_{33} \)
(B) \( a_{11} A_{11} + a_{12} A_{21} + a_{13} A_{31} \)
(C) \( a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13} \)
(D) \( a_{11} A_{11} + a_{12} A_{21} + a_{31} A_{31} \)

4.6 Adjoint and Inverse of a Matrix

In the previous chapter, we have studied inverse of a matrix. In this section, we shall discuss the condition for existence of inverse of a matrix.

To find inverse of a matrix \( A \), i.e., \( A^{-1} \) we shall first define adjoint of a matrix.
4.6.1 *Adjoint of a matrix*

**Definition 3** The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A^*]_{n \times n}$, where $A^*_{ij}$ is the cofactor of the element $a_{ij}$. Adjoint of the matrix $A$ is denoted by $adj \ A$.

Let

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Then $adj \ A$

**Example 23**

**Solution** We

Hence

**Remark** For $n 	imes n$ square matrices,

The $adj \ A$

of $a_{12}$ and $a_{21}$

hanging signs

Change sign

Interchange

We state the following theorem without proof.

**Theorem 1** If $A$ be any given square matrix of order $n$, then

$$A(adj \ A) = (adj \ A)A = |A|I,$$

where $I$ is the identity matrix of order $n$. 

2021-22
Verification

Let \( A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \), then \( \text{adj} \ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \).

Since the sum of the product of elements of a row (or a column) with corresponding cofactors is equal to \( |A| \) and otherwise zero, we have

\[
A \text{ (adj } A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Similarly, we have

\[
(\text{adj } A) A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Hence \( A \text{ (adj } A) = (\text{adj } A) A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

Definition 4

A square matrix \( A \) is said to be singular if \( A = 0 \).

For example, the determinant of matrix \( A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \) is zero.

Hence \( A \) is a singular matrix.

Definition 5

A square matrix \( A \) is said to be non-singular if \( A \neq 0 \).

Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \). Then \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2 \neq 0 \).

Hence \( A \) is a nonsingular matrix.

We state the following theorems without proof.

Theorem 2

If \( A \) and \( B \) are nonsingular matrices of the same order, then \( AB \) and \( BA \) are also nonsingular matrices of the same order.

Theorem 3

The determinant of the product of matrices is equal to the product of their respective determinants, that is, \( |AB| = |A| \cdot |B| \), where \( A \) and \( B \) are square matrices of the same order.

Remark

We know that \( (\text{adj } A) A = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} \cdot |A| \neq 0 \).
Writing determinants of matrices on both sides, we have

\[
| (\text{adj } A) A | = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}
\]

i.e.

(Why?)

In general, if

**Theorem 4**

A square matrix $A$ is invertible if and only if $A$ is a nonsingular matrix.

**Proof**

Let $A$ be an invertible matrix of order $n$ and $I$ be the identity matrix of order $n$.

Then, there exists a square matrix $B$ of order $n$ such that $AB = BA = I$.

Now $AB = I$.

So $AB = I$ or $A B = I$ (since $I 1, AB = A B = I$)

This gives

Thus

Conversely, let $A$ be nonsingular. Then $A \neq 0$.

Now $A (\text{adj } A) = (\text{adj } A) A = A I$ (Theorem 1)

or $A^{-1} = 1_{|A|}$

Thus

$|A|$

**Example 24**

If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that $A \text{adj } A = |A| I$. Also find $A^{-1}$.

**Solution**

We have $|A| = 1 (16 - 9) - 3 (4 - 3) + 3 (3 - 4) = 1 \neq 0$
Now $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$, $A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, $A_{33} = 1$

Therefore $\text{adj} A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ . & . & . \end{bmatrix}$

Now

$$\begin{bmatrix} -3 \\ -3 \\ -4 \end{bmatrix} = |A|. I$$

Also

$$\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Example 25

Solution We

Since, $|AB|$

$$(AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB) = -\frac{1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Further, $|A| = -11 \neq 0$ and $|B| = 1 \neq 0$. Therefore, $A^{-1}$ and $B^{-1}$ both exist and are given by

$$A^{-1} = \frac{1}{11} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$
Therefore \( B^{-1}A^{-1} = \frac{-1}{11} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{-1}{11} \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} \)

Hence \((AB)^{-1} = B^{-1} A^{-1}\)

**Example 26**
where \(I\) is a \(2 \times 2\) identity matrix and \(O\) is a \(2 \times 2\) zero matrix.

**Solution**

We have

\[
\begin{align*}
\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^2 - 4A + I &= O,
\end{align*}
\]

Hence

\[
\begin{align*}
A^{-1} &= 4I - A
\end{align*}
\]

Find adjoint of each of the matrices in Exercises 1 and 2.

1. \[
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}
\]

Verify \(A(adjA) = (adjA)A = |A|I\) in Exercises 3 and 4

3. \[
\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}
\]

4. \[
\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}
\]
Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

5. \[
\begin{bmatrix}
2 & -2 \\
4 & 3
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
-1 & 5 \\
-3 & 2
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 2 & 4 \\
0 & 0 & 5
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & 0 \\
3 & 3 \\
5 & 2
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha \\
0 & \sin \alpha & -\cos \alpha
\end{bmatrix}
\]

12. Let \( A = \)

13. If \( A = \)

14. For the \( \pm aA + bI = O \).

15. For the

16. If \( A = \)

Verify that \( A^3 - 6A^2 + 9A - 4I = O \) and hence find \( A^{-1} \).

17. Let \( A \) be a nonsingular square matrix of order \( 3 \times 3 \). Then \( |\text{adj} \, A| \) is equal to
   (A) \( |A| \)    (B) \( |A|^2 \)    (C) \( |A|^3 \)    (D) \( 3|A| \)

18. If \( A \) is an invertible matrix of order 2, then \( \det (A^{-1}) \) is equal to
   (A) \( \det (A) \)    (B) \( \frac{1}{\det (A)} \)    (C) 1    (D) 0
4.7 Applications of Determinants and Matrices

In this section, we shall discuss application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

Consistent system A system of equations is said to be consistent if its solution (one or more) exists.

Inconsistent system A system of equations is said to be inconsistent if its solution does not exist.

Note: In this chapter, we restrict ourselves to the system of linear equations having unique solutions only.

4.7.1 Solution of system of linear equations using inverse of a matrix

Let us express the system of linear equations as matrix equations and solve them using inverse of the coefficient matrix.

Consider the system of equations:

\[ a_1 x + b_1 y + c_1 z = d_1 \]
\[ a_2 x + b_2 y + c_2 z = d_2 \]
\[ a_3 x + b_3 y + c_3 z = d_3 \]

Let \( A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \), \( X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) and \( B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \).

Then, the system of equations can be written as \( AX = B \), i.e.,

\[ \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \]

Case I If \( A \) is nonsingular, then its inverse exists. Now

\[ AX = B \]
or \( A^{-1}(AX) = A^{-1}B \) (premultiplying by \( A^{-1} \))

or \( (A^{-1}A)X = A^{-1}B \) (by associative property)

or \( IX = A^{-1}B \)

or \( X = A^{-1}B \)

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.
Case II If A is a singular matrix, then \(|A| = 0\).

In this case, we calculate \((adj\ A)\ B\).

If \((adj\ A)\ B \neq O\), (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If \((adj\ A)\ B = O\), then system may be either consistent or inconsistent according as the system

**Example 27**: 

**Solution** The

Now, \(|A| = \) \(\neq 0\), Hence, A is nonsingular matrix and so has a unique solution.

Note that

Therefore

i.e.

Hence

**Example 28**: 

**Solution** The

where

We see that

\[|A| = 3 (2 - 3) + 2(4 + 4) + 3 (-6 - 4) = -17 \neq 0\]
Hence, A is nonsingular and so its inverse exists. Now

\[
\begin{align*}
A_{11} &= -1, & A_{12} &= -8, & A_{13} &= -10 \\
A_{21} &= -5, & A_{22} &= -6, & A_{23} &= 1 \\
A_{31} &= -1, & A_{32} &= 9, & A_{33} &= 7
\end{align*}
\]

Therefore

\[
\begin{bmatrix}
8 \\
1 \\
4
\end{bmatrix}
\]

i.e.

Hence

Example 29

\[
\text{by 3 and add double of the matrix method.}
\]

Solution Let

Then, according to given conditions, we have

\[
\begin{align*}
x + y + z &= 6 \\
y + 3z &= 11 \\
x + z &= 2 \\
x - 2y + z &= 0
\end{align*}
\]

This system can be written as

\[
AX = B
\]

where

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 3 \\
1 & 2 & 1
\end{bmatrix}, \quad X = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}, \quad B = \begin{bmatrix}
6 \\
11 \\
0
\end{bmatrix}
\]

Here \(|A| = 1(1 + 6) - (0 - 3) + (0 - 1) = 9 \neq 0\). Now we find \(adj A\)

\[
\begin{align*}
A_{11} &= 1(1 + 6) = 7, & A_{12} &= -(0 - 3) = 3, & A_{13} &= -1 \\
A_{21} &= -(1 + 2) = -3, & A_{22} &= 0, & A_{23} &= -(-2 - 1) = 3 \\
A_{31} &= (3 - 1) = 2, & A_{32} &= -(3 - 0) = -3, & A_{33} &= (1 - 0) = 1
\end{align*}
\]
Hence

\[
adj \ A = \begin{bmatrix}
7 & -3 & 2 \\
3 & 0 & -3 \\
-1 & 3 & 1
\end{bmatrix}
\]

Thus

\[
A^{-1} = \frac{1}{\det(A)} \begin{bmatrix}
7 & 3 & 2 \\
3 & 0 & 3 \\
-1 & 3 & 1
\end{bmatrix}
\]

Since

\[
x = 1, \quad y = 2, \quad z = 3
\]

EXERCISE 4.6

Examine the consistency of the system of equations in Exercises 1 to 6.

1. \(x + 2y = 2\)
   
2. \(2x - y = 5\)
   
3. \(x + 3y = 5\)
   
4. \(x + 2y = 3\)
   
5. \(2x + y + 3z = 7\)
   
6. \(x - 2y + z = 4\)

Solve system of linear equations, using matrix method, in Exercises 7 to 14.

7. \(5x + 2y = 3\)
   
8. \(2x - y = -2\)
   
9. \(4x - 3y = 3\)
   
10. \(3x - 5y = 9\)
    
11. \(2x + y + z = 1\)
    
12. \(x + y + z = 4\)
    
13. \(x - 2y - z = \frac{3}{2}\)
    
14. \(2x + y - 3z = 0\)
    
\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

Thus

\[
x = 1, \quad y = 2, \quad z = 3
\]
15. If \( A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \), find \( A^{-1} \). Using \( A^{-1} \) solve the system of equations
\[
\begin{aligned}
2x - 3y + 5z &= 11 \\
3x + 2y - 4z &= -5 \\
x + y - 2z &= -3
\end{aligned}
\]

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹70. Find cost of each item per kg by matrix method.

Example 30

Solution

\[1R_3 \rightarrow R_3 - R_1 \]

\[= -\frac{1}{2} \left( a + b + c \right) \left( 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \right) \]

\[= -\frac{1}{2} \left( a + b + c \right) \left[ \left( a - b \right)^2 + \left( b - c \right)^2 + \left( c - a \right)^2 \right] \]

which is negative (since \( a + b + c > 0 \) and \( \left( a - b \right)^2 + \left( b - c \right)^2 + \left( c - a \right)^2 > 0 \))
Example 31 If \( a, b, c \), are in A.P, find value of
\[
\begin{vmatrix}
2y + 4 & 5y + 7 & 8y + a \\
3y + 5 & 6y + 8 & 9y + b \\
4y + 6 & 7y + 9 & 10y + c \\
\end{vmatrix}
\]

Solution
App
\[
\begin{align*}
\text{Applying } R_1 & \rightarrow R_1 + R_3 - 2R_2 \\
& \text{to the given determinant, we obtain}
\end{align*}
\]
\[
0 0 0 \\
3 5 6 8 9 \\
4 6 7 9 10 \\
+ + +
\]
\[
= 0 \quad \text{(Since } 2b = a + c) \]

Example 32

Solution
App
\[
\begin{align*}
\text{Applying } R_1 & \rightarrow xR_1, R_2 \rightarrow yR_2, R_3 \rightarrow zR_3 \\
& \text{to } \triangle \\
\text{and dividing by } xyz, \text{ we get}
\end{align*}
\]
\[
\begin{vmatrix}
\end{vmatrix}
\]
\[
= 2xyz \left( x + y + z \right)^3
\]

Taking comm

Applying \( C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1 \), we have
\[
\Delta = \begin{vmatrix}
(y + z)^2 & x^2 - (y + z)^2 & x^2 - (y + z)^2 \\
y^2 & (x + z)^2 - y^2 & 0 \\
z^2 & 0 & (x + y)^2 - z^2
\end{vmatrix}
\]
Taking common factor \((x + y + z)\) from \(C_2\) and \(C_3\), we have

\[
\Delta = (x + y + z)^2 \begin{vmatrix}
(y + z) & x - (y + z) & x - (y + z) \\
y^2 & (x + z) - y & 0 \\
z^2 & 0 & (x + y) - z
\end{vmatrix}
\]

Applying \(R_1\).

Applying \(C_2\).

Finally expanding along \(R_1\), we have

\[
\Delta = (x + y + z)^2 (2yz) \left[(x + z)(x + y) - yz\right] = (x + y + z)^3 (2xyz)
\]

Example 33

Use product

\[
\begin{bmatrix}
1 & 1 & 2 \\
0 & 2 & -3 \\
3 & -2 & 4
\end{bmatrix}
\begin{bmatrix}
-2 & 0 & 1 \\
9 & 2 & -3 \\
6 & 1 & -2
\end{bmatrix}
\]

Solution

Consider the product

\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & -3 \\
3 & -2 & 4
\end{bmatrix}
\begin{bmatrix}
-2 & 0 & 1 \\
9 & 2 & -3 \\
6 & 1 & -2
\end{bmatrix}
\]

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\[
\begin{bmatrix}
-2 - 9 + 12 & 0 - 2 + 2 & 1 + 3 - 4 \\
0 + 18 - 18 & 0 + 4 - 3 & 0 - 6 + 6 \\
-6 - 18 + 24 & 0 - 4 + 4 & 3 + 6 - 8 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Hence

Now, given system of equations can be written, in matrix form, as follows

\[
\begin{bmatrix}
1 - 1 & 2 \\
0 & 2 - 3 \\
3 - 2 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
2 \\
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
3 \\
2 \\
\end{bmatrix}
\]

Hence

**Example 34**

**Solution**

\[
\Delta = \begin{vmatrix}
ax + b & cx + d & px + q \\
u & v & w \\
\end{vmatrix}
= (1 - x^2) \begin{vmatrix}
a & c & p \\
ax + b & cx + d & px + q \\
u & v & w \\
\end{vmatrix}
\]
Applying $R_2 \rightarrow R_2 - x \cdot R_1$, we get

\[
\Delta = (1 - x^2) \begin{vmatrix}
    a & c & p \\
    b & d & q \\
    u & v & w
\end{vmatrix}
\]

### 1. Prove that

\[
\sin \cos - \sin - \cos \theta = \cos \theta.
\]

### 2. Without expanding the determinant, prove that

\[
\begin{vmatrix}
    2 & 2 & 3 \\
    2 & 2 & 3 \\
    2 & 2 & 3
\end{vmatrix}
\]

is independent of $\theta$.

### 3. Evaluate

\[
\cos \cos \cos \sin - \sin \sin \cos \sin \sin \cos
\]

### 4. If $a, b, c$ are real numbers, and

\[
\Delta = a^2 + ab + bc + c^2 = 0,
\]

show that either

\[
a + b + c = 0 \quad \text{or} \quad a = b = c.
\]

### 5. Solve the equation

\[
\begin{vmatrix}
    a & c & p \\
    b & d & q \\
    u & v & w
\end{vmatrix} + \begin{vmatrix}
    a & c & p \\
    b & d & q \\
    u & v & w
\end{vmatrix} + \begin{vmatrix}
    a & c & p \\
    b & d & q \\
    u & v & w
\end{vmatrix} = a^2 b^2 c^2.
\]

### 6. Prove that

\[
\begin{vmatrix}
    a^2 + ab & b^2 & ac \\
    ab & b^2 + bc & c^2 \\
    3 & -1 & 1 \\
    -15 & 6 & -5 \\
    5 & -2 & 2
\end{vmatrix} = 4a^2 b^2 c^2.
\]

### 7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.
8. Let \( A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \). Verify that

(i) \( (\text{adj } A)^{-1} = \text{adj } (A^{-1}) \) 
(ii) \( (A^{-1})^{-1} = A \)

9. Evaluate

10. Evaluate

Using properties of determinants in Exercises 11 to 15, prove that:

11. \( \begin{vmatrix} \alpha & \alpha^2 \\ \beta & \beta^2 \\ \gamma & \gamma^2 \end{vmatrix} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma) \)

12. \( \begin{vmatrix} x & x^2 \\ y & y^2 \\ z & z^2 \end{vmatrix} = (1 + xyz)(x - y)(y - z)(z - x) \), where \( p \) is any scalar.

13. \( \begin{vmatrix} 3a \\ -b + a \\ -c + a \end{vmatrix} \)

14. \( \begin{vmatrix} 1 & 1 + 4 + 3p + 2q \\ 2 & 3 + 2p & 4 + 3p + 2q \\ 3 & 6 + 3p & 10 + 6p + 3q \end{vmatrix} = 1 \)

15. \( \begin{vmatrix} \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix} = 0 \)

16. Solve the system of equations

\( \frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4 \)
Choose the correct answer in Exercise 17 to 19.

17. If \( a, b, c \) are in A.P, then the determinant
\[
\begin{vmatrix}
2 & 3 & 2 \\
3 & 4 & 2 \\
4 & 5 & 2 \\
\end{vmatrix}
\]
is
(A) 0 
(B) 1 
(C) \( x \) 
(D) 2

18. If \( x, y, z \) are nonzero real numbers, then the inverse of matrix
\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]
is
(A) \[ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
(B) \[ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
(C) \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

19. Let \( A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix} \), where \( 0 \leq \theta \leq 2\pi \). Then
(A) \( \text{Det}(A) = 0 \) 
(B) \( \text{Det}(A) \in (2, \infty) \) 
(C) \( \text{Det}(A) \in (2, 4) \) 
(D) \( \text{Det}(A) \in [2, 4] \)
Summary

- Determinant of a matrix $A = [a_{ij}]_{1 \times 1}$ is given by $|a_{11}| = a_{11}$

- Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- Determinant of a matrix $A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$ is given by (expanding along $R_1$)

$$\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = a \begin{vmatrix} b & c \\ b & c \end{vmatrix} - b \begin{vmatrix} a & c \\ a & c \end{vmatrix} + c \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

For any square matrix $A$:

- $|A^T| = |A|$, where $A^T$ is the transpose of $A$.

- If we interchange any two rows (or columns), then sign of determinant changes.

- If any two rows (or columns) are identical or proportional, then value of determinant is zero.

- If we multiply each element of a row (or a column) of a determinant by constant $k$, then value of determinant is multiplied by $k$.

- Multiplying a determinant by $k$ means multiply elements of only one row (or one column) by $k$.

- If $A = [a_{ij}]$, then $a_{ij}k = k[a_{ij}]$.

- If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.

- If to each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows or columns are added, then value of determinant remains same.
Area of a triangle with vertices \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\) is given by
\[
\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}
\]

Minor of an element \(a_{ij}\) of the determinant of matrix \(A\) is the determinant obtained by deleting \(i\)th row and \(j\)th column and denoted by \(M_{ij}\).

Cofactor of \(a_{ij}\) of given by \(A_{ij} = (-1)^{i+j} M_{ij}\).

Value of determinant of a matrix \(A\) is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example,
\[
A = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}
\]

If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For example,
\[
a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0
\]

A square matrix \(A\) has inverse if and only if \(A\) is non-singular.

If \(AB = BA = I\), where \(B\) is square matrix, then \(B\) is called inverse of \(A\). Also \(A^{-1} = B\) or \(B^{-1} = A\) and hence \((A^{-1})^{-1} = A\).

\[
A^{-1} = \frac{1}{|A|} \text{adj} A
\]

\[
\text{adj} A = \begin{bmatrix} a_3 & -b_3 & c_3 \\ -a_2 & a_1 & -b_1 \\ b_2 & -b_1 & a_1 \end{bmatrix}
\]

If \(a_1 x + b_1 y + c_1 z = d_1, \ a_2 x + b_2 y + c_2 z = d_2, \ a_3 x + b_3 y + c_3 z = d_3\), then these equations can be written as \(AX = B\), where
\[
A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}
\]
Unique solution of equation $AX = B$ is given by $X = A^{-1}B$, where $|A| \neq 0$.

A system of equation is consistent or inconsistent according as its solution exists or not.

For a square matrix $A$ in matrix equation $AX = B$

(i) $|A| \neq 0$, there exists unique solution

(ii) $|A| = 0$ and $(\text{adj} A)B \neq 0$, then there exists no solution

(iii) $|A| = 0$ and $(\text{adj} A)B = 0$, then system may or may not be consistent.

The Chinese method of representing the coefficients of the unknowns of several linear equations by using rods on a calculating board naturally led to the discovery of a simple method of elimination. The arrangement of rods was precisely that of the rows of a determinant. The Chinese, therefore, early developed the idea of solving a system of linear equations.

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Seki Kowa, the greatest of the Japanese Mathematicians of the seventeenth century in his work ‘Kai Fukudai no Ho’ in 1683 showed that he had the idea of determinant and its expansion. But he used this device only in eliminating a quantity from two equations and not directly in the solution of a set of simultaneous linear equations.

Vendermonde was the first to recognize determinants as independent functions. He may be called the formal founder. Laplace (1772), gave the general method of expanding a determinant in terms of its complementary minors. In 1773 Lagrange treated determinants in his ‘Mécanique Analytique’. For purpose other than the solution of equations, he used determinants in his theory of numbers.

The next great contributor was Jacques-Philippe-Marie Binet, (1812) who stated the theorem relating to the product of two matrices of $m$-columns and $n$-rows, which for the special case of $m = n$ reduces to the multiplication theorem.

Also on the same day, Cauchy (1812) presented one on the same subject. He used the word ‘determinant’ in its present sense. He gave the proof of multiplication theorem more satisfactory than Binet’s.

The greatest contributor to the theory was Carl Gustav Jacob Jacobi, after this the word determinant received its final acceptance.