MATHEMATICS

Textbook for Class XII

PART I

12079

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Textbook for Class XII

PART I

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
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Foreword

The National Curriculum Framework, 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

NCERT appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisor for this book, Professor P.K. Jain for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

Director
National Council of Educational
Research and Training

New Delhi
20 December 2005

Preface

The National Council of Educational Research and Training (NCERT) had constituted 21 Focus Groups on Teaching of various subjects related to School Education, to review the National Curriculum Framework for School Education - 2000 (NCFSE - 2000) in face of new emerging challenges and transformations occurring in the fields of content and pedagogy under the contexts of National and International spectrum of school education. These Focus Groups made general and specific comments in their respective areas. Consequently, based on these reports of Focus Groups, National Curriculum Framework (NCF)-2005 was developed.

NCERT designed the new syllabi and constituted Textbook Development Teams for Classes XI and XII to prepare textbooks in mathematics under the new guidelines and new syllabi. The textbook for Class XI is already in use, which was brought in 2005.

The first draft of the present book (Class XII) was prepared by the team consisting of NCERT faculty, experts and practicing teachers. The draft was refined by the development team in different meetings. This draft of the book was exposed to a group of practicing teachers teaching mathematics at higher secondary stage in different parts of the country, in a review workshop organised by the NCERT at Delhi. The teachers made useful comments and suggestions which were incorporated in the draft textbook. The draft textbook was finalised by an editorial board constituted out of the development team. Finally, the Advisory Group in Science and Mathematics and the Monitoring Committee constituted by the HRD Ministry, Government of India have approved the draft of the textbook.

In the fitness of things, let us cite some of the essential features dominating the textbook. These characteristics have reflections in almost all the chapters. The existing textbook contain 13 main chapters and two appendices. Each Chapter contain the followings:

- Introduction: Highlighting the importance of the topic; connection with earlier studied topics; brief mention about the new concepts to be discussed in the chapter.
- Organisation of chapter into sections comprising one or more concepts/sub concepts.
- Motivating and introducing the concepts/sub concepts. Illustrations have been provided wherever possible.

- Proofs/problem solving involving deductive or inductive reasoning, multiplicity of approaches wherever possible have been inducted.
- Geometric viewing / visualisation of concepts have been emphasised whenever needed.
- Applications of mathematical concepts have also been integrated with allied subjects like science and social sciences.
- Adequate and variety of examples/exercises have been given in each section.
- For refocusing and strengthening the understanding and skill of problem solving and applicabilities, miscellaneous types of examples/exercises have been provided involving two or more sub concepts at a time at the end of the chapter. The scope of challenging problems to talented minority have been reflected conducive to the recommendation as reflected in NCF-2005.
- For more motivational purpose, brief historical background of topics have been provided at the end of the chapter and at the beginning of each chapter relevant quotation and photograph of eminent mathematician who have contributed significantly in the development of the topic undertaken, are also provided.
- Lastly, for direct recapitulation of main concepts, formulas and results, brief summary of the chapter has also been provided.

I am thankful to Professor Krishan Kumar, Director, NCERT who constituted the team and invited me to join this national endeavor for the improvement of mathematics education. He has provided us with an enlightened perspective and a very conducive environment. This made the task of preparing the book much more enjoyable and rewarding. I express my gratitude to Professor J.V. Narlikar, Chairperson of the Advisory Group in Science and Mathematics, for his specific suggestions and advice towards the improvement of the book from time to time. I, also, thank Prof. G. Ravindra, Joint Director, NCERT for his help from time to time.

I express my sincere thanks to Professor Hukum Singh, Chief Coordinator and Head DESM, Dr. V. P. Singh, Coordinator and Professor S. K. Singh Gautam who have been helping for the success of this project academically as well as administratively. Also, I would like to place on records my appreciation and thanks to all the members of the team and the teachers who have been associated with this noble cause in one or the other form.

PAWAN K. JAIN
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THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

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The Council gratefully acknowledges the valuable contributions of the following participants of the Textbook Review Workshop: Jagdish Saran, *Professor*, Deptt. of Statistics, University of Delhi; Quddus Khan, *Lecturer*, Shibli National P.G. College Azamgarh (U.P.); P.K. Tewari, *Assistant Commissioner* (Retd.), Kendriya Vidyalaya Sangathan; S.B. Tripathi, *Lecturer*, R.P.V.V. Surajmal Vihar, Delhi; O.N. Singh, *Reader*, RIE, Bhubaneswar, Orissa; Miss Saroj, *Lecturer*, Govt. Girls Senior Secondary School No.1, Roop Nagar, Delhi; P. Bhaskar Kumar, *PGT*, Jawahar Navodaya Vidyalaya, Lepakshi, Anantapur, (A.P.); Mrs. S. Kalpagam, *PGT*, K.V. NAL Campus, Bangalore; Rahul Sofat, *Lecturer*, Air Force Golden Jubilee Institute, Subroto Park, New Delhi; Vandita Kalra, *Lecturer,* Sarvodaya Kanya Vidyalaya, Vikaspuri, District Centre, New Delhi; Janardan Tripathi, *Lecturer,* Govt. R.H.S.S. Aizawl, Mizoram and Ms. Sushma Jaireth, *Reader,* DWS, NCERT, New Delhi.

The Council acknowledges the efforts of Deepak Kapoor, *Incharge*, Computer Station, Sajjad Haider Ansari, Rakesh Kumar and Nargis Islam, *D.T.P. Operators,* Monika Saxena,*Copy Editor* and Abhimanu Mohanty, *Proof Reader.*

The Contribution of APC-Office, administration of DESM and Publication Department is also duly acknowledged.

Constitution of India

Part IV A (Article 51 A)

Fundamental Duties

It shall be the duty of every citizen of India —

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wildlife and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

*(k) who is a parent or guardian, to provide opportunities for education to his child or, as the case may be, ward between the age of six and fourteen years.

Note: The Article 51A containing Fundamental Duties was inserted by the Constitution (42nd Amendment) Act, 1976 (with effect from 3 January 1977).

*(k) was inserted by the Constitution (86th Amendment) Act, 2002 (with effect from 1 April 2010).

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Part I

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Chapter 1

RELATIONS AND FUNCTIONS

❖ *There is no permanent place in the world for ugly mathematics ... . It may be very hard to define mathematical beauty but that is just as true of beauty of any kind, we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it. — G. H. HARDY* ❖

1.1 Introduction

Recall that the notion of relations and functions, domain, co-domain and range have been introduced in Class XI along with different types of specific real valued functions and their graphs. The concept of the term 'relation' in mathematics has been drawn from the meaning of relation in English language, according to which two objects or quantities are related if there is a recognisable connection or link between the two objects or quantities. Let A be the set of students of Class XII of a school and B be the set of students of Class XI of the same school. Then some of the examples of relations from A to B are

Lejeune Dirichlet (1805-1859)

(i) $\{(a, b) \in A \times B$: a is brother of $b\}$,

(ii) $\{(a, b) \in A \times B$: a is sister of $b\}$,

(iii) $\{(a, b) \in A \times B$: age of a is greater than age of $b\}$,

(iv) $\{(a, b) \in A \times B$: total marks obtained by a in the final examination is less than the total marks obtained by b in the final examination$\}$,

(v) $\{(a, b) \in A \times B$: a lives in the same locality as $b\}$. However, abstracting from this, we define mathematically a relation R from A to B as an arbitrary subset of $A \times B$.

If $(a, b) \in R$, we say that a is related to b under the relation R and we write as a R b. In general, $(a, b) \in R$, we do not bother whether there is a recognisable connection or link between a and b. As seen in Class XI, functions are special kind of relations.

In this chapter, we will study different types of relations and functions, composition of functions, invertible functions and binary operations.

1.2 Types of Relations

In this section, we would like to study different types of relations. We know that a relation in a set A is a subset of A × A. Thus, the empty set ϕ and A × A are two extreme relations. For illustration, consider a relation R in the set A = {1, 2, 3, 4} given by R = $\{(a, b): a - b = 10\}$. This is the empty set, as no pair (a, b) satisfies the condition $a - b = 10$. Similarly, $R' = \{(a, b) : |a - b| \geq 0\}$ is the whole set A × A, as all pairs (a, b) in A × A satisfy $|a - b| \geq 0$. These two extreme examples lead us to the following definitions.

Definition 1 A relation R in a set A is called *empty relation*, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

Definition 2 A relation R in a set A is called *universal relation*, if each element of A is related to every element of A, i.e., $R = A \times A$.

Both the empty relation and the universal relation are some times called *trivial relations*.

Example 1 Let A be the set of all students of a boys school. Show that the relation R in A given by $R = \{(a, b) : a$ is sister of $b\}$ is the empty relation and $R' = \{(a, b)$: the difference between heights of a and b is less than 3 meters$\}$ is the universal relation.

Solution Since the school is boys school, no student of the school can be sister of any student of the school. Hence, $R = \phi$, showing that R is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 meters. This shows that $R' = A \times A$ is the universal relation.

Remark In Class XI, we have seen two ways of representing a relation, namely raster method and set builder method. However, a relation R in the set {1, 2, 3, 4} defined by R = $\{(a, b) : b = a + 1\}$ is also expressed as a R b if and only if $b = a + 1$ by many authors. We may also use this notation, as and when convenient.

If $(a, b) \in R$, we say that a is related to b and we denote it as a R b.

One of the most important relation, which plays a significant role in Mathematics, is an *equivalence relation*. To study equivalence relation, we first consider three types of relations, namely reflexive, symmetric and transitive.

Definition 3 A relation R in a set A is called

(i) *reflexive*, if $(a, a) \in R$, for every $a \in A$,

(ii) *symmetric*, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

(iii) *transitive*, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

Definition 4 A relation R in a set A is said to be an *equivalence relation* if R is reflexive, symmetric and transitive.

Example 2 Let T be the set of all triangles in a plane with R a relation in T given by R = $\{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Solution R is reflexive, since every triangle is congruent to itself. Further, $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to $T_2 \Rightarrow T_2$ is congruent to $T_1 \Rightarrow (T_2, T_1) \in R$. Hence, R is symmetric. Moreover, $(T_1, T_2), (T_2, T_3) \in R \Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to $T_3 \Rightarrow T_1$ is congruent to $T_3 \Rightarrow (T_1, T_3) \in R$. Therefore, R is an equivalence relation.

Example 3 Let L be the set of all lines in a plane and R be the relation in L defined as R = $\{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$. Show that R is symmetric but neither reflexive nor transitive.

Solution R is not reflexive, as a line L_1 can not be perpendicular to itself, i.e., $(L_1, L_1) \notin R$. R is symmetric as $(L_1, L_2) \in R$

$\Rightarrow$ L_1 is perpendicular to L_2

$\Rightarrow$ L_2 is perpendicular to L_1

$\Rightarrow$ $(L_2, L_1) \in R$.

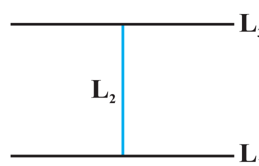


Fig 1.1

R is not transitive. Indeed, if L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 can never be perpendicular to L_3. In fact, L_1 is parallel to L_3, i.e., $(L_1, L_2) \in R$, $(L_2, L_3) \in R$ but $(L_1, L_3) \notin R$.

Example 4 Show that the relation R in the set {1, 2, 3} given by R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} is reflexive but neither symmetric nor transitive.

Solution R is reflexive, since (1, 1), (2, 2) and (3, 3) lie in R. Also, R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$. Similarly, R is not transitive, as $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

Example 5 Show that the relation R in the set **Z** of integers given by

$$R = \{(a, b) : 2 \text{ divides } a - b\}$$

is an equivalence relation.

Solution R is reflexive, as 2 divides $(a - a)$ for all $a \in \mathbf{Z}$. Further, if $(a, b) \in R$, then 2 divides $a - b$. Therefore, 2 divides $b - a$. Hence, $(b, a) \in R$, which shows that R is symmetric. Similarly, if $(a, b) \in R$ and $(b, c) \in R$, then $a - b$ and $b - c$ are divisible by 2. Now, $a - c = (a - b) + (b - c)$ is even (Why?). So, $(a - c)$ is divisible by 2. This shows that R is transitive. Thus, R is an equivalence relation in **Z**.

In Example 5, note that all even integers are related to zero, as (0, ± 2), (0, ± 4) etc., lie in R and no odd integer is related to 0, as (0, ± 1), (0, ± 3) etc., do not lie in R. Similarly, all odd integers are related to one and no even integer is related to one. Therefore, the set E of all even integers and the set O of all odd integers are subsets of **Z** satisfying following conditions:

(i) All elements of E are related to each other and all elements of O are related to each other.

(ii) No element of E is related to any element of O and vice-versa.

(iii) E and O are disjoint and $\mathbf{Z} = \mathrm{E} \cup \mathrm{O}$.

The subset E is called the *equivalence class containing zero* and is denoted by [0]. Similarly, O is the equivalence class containing 1 and is denoted by [1]. Note that $[0] \neq [1]$, $[0] = [2r]$ and $[1] = [2r + 1]$, $r \in \mathbf{Z}$. Infact, what we have seen above is true for an arbitrary equivalence relation R in a set X. Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets A_i called partitions or subdivisions of X satisfying:

(i) all elements of A_i are related to each other, for all i.

(ii) no element of A_i is related to any element of A_j, $i \neq j$.

(iii) $\cup \mathrm{A}_j = \mathrm{X}$ and $\mathrm{A}_i \cap \mathrm{A}_j = \phi$, $i \neq j$.

The subsets A_i are called *equivalence classes*. The interesting part of the situation is that we can go reverse also. For example, consider a subdivision of the set **Z** given by three mutually disjoint subsets A_1, A_2 and A_3 whose union is **Z** with

$$\mathrm{A}_1 = \{x \in \mathbf{Z} : x \text{ is a multiple of } 3\} = \{..., -6, -3, 0, 3, 6, ...\}$$
$$\mathrm{A}_2 = \{x \in \mathbf{Z} : x - 1 \text{ is a multiple of } 3\} = \{..., -5, -2, 1, 4, 7, ...\}$$
$$\mathrm{A}_3 = \{x \in \mathbf{Z} : x - 2 \text{ is a multiple of } 3\} = \{..., -4, -1, 2, 5, 8, ...\}$$

Define a relation R in **Z** given by $\mathrm{R} = \{(a, b) : 3 \text{ divides } a - b\}$. Following the arguments similar to those used in Example 5, we can show that R is an equivalence relation. Also, A_1 coincides with the set of all integers in **Z** which are related to zero, A_2 coincides with the set of all integers which are related to 1 and A_3 coincides with the set of all integers in **Z** which are related to 2. Thus, $\mathrm{A}_1 = [0]$, $\mathrm{A}_2 = [1]$ and $\mathrm{A}_3 = [2]$. In fact, $\mathrm{A}_1 = [3r]$, $\mathrm{A}_2 = [3r + 1]$ and $\mathrm{A}_3 = [3r + 2]$, for all $r \in \mathbf{Z}$.

Example 6 Let R be the relation defined in the set A = {1, 2, 3, 4, 5, 6, 7} by $\mathrm{R} = \{(a, b) : \text{both } a \text{ and } b \text{ are either odd or even}\}$. Show that R is an equivalence relation. Further, show that all the elements of the subset {1, 3, 5, 7} are related to each other and all the elements of the subset {2, 4, 6} are related to each other, but no element of the subset {1, 3, 5, 7} is related to any element of the subset {2, 4, 6}.

Solution Given any element a in A, both a and a must be either odd or even, so that $(a, a) \in$ R. Further, $(a, b) \in$ R $\Rightarrow$ both a and b must be either odd or even $\Rightarrow (b, a) \in$ R. Similarly, $(a, b) \in$ R and $(b, c) \in$ R $\Rightarrow$ all elements a, b, c, must be either even or odd simultaneously $\Rightarrow (a, c) \in$ R. Hence, R is an equivalence relation. Further, all the elements of {1, 3, 5, 7} are related to each other, as all the elements of this subset are odd. Similarly, all the elements of the subset {2, 4, 6} are related to each other, as all of them are even. Also, no element of the subset {1, 3, 5, 7} can be related to any element of {2, 4, 6}, as elements of {1, 3, 5, 7} are odd, while elements of {2, 4, 6} are even.

EXERCISE 1.1

1. Determine whether each of the following relations are reflexive, symmetric and transitive:
   (i) Relation R in the set A = {1, 2, 3, ..., 13, 14} defined as
   $$R = \{(x, y) : 3x - y = 0\}$$
   (ii) Relation R in the set **N** of natural numbers defined as
   $$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$
   (iii) Relation R in the set A = {1, 2, 3, 4, 5, 6} as
   $$R = \{(x, y) : y \text{ is divisible by } x\}$$
   (iv) Relation R in the set **Z** of all integers defined as
   $$R = \{(x, y) : x - y \text{ is an integer}\}$$
   (v) Relation R in the set A of human beings in a town at a particular time given by
   (a) R = {(x, y) : x and y work at the same place}
   (b) R = {(x, y) : x and y live in the same locality}
   (c) R = {(x, y) : x is exactly 7 cm taller than y}
   (d) R = {(x, y) : x is wife of y}
   (e) R = {(x, y) : x is father of y}
2. Show that the relation R in the set **R** of real numbers, defined as
   $R = \{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.
3. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as
   $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.
4. Show that the relation R in **R** defined as $R = \{(a, b) : a \leq b\}$, is reflexive and transitive but not symmetric.
5. Check whether the relation R in **R** defined by $R = \{(a, b) : a \leq b^3\}$ is reflexive, symmetric or transitive.

6. Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

7. Show that the relation R in the set A of all the books in a library of a college, given by R = {(x, y) : x and y have same number of pages} is an equivalence relation.

8. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by

R = {(a, b) : $|a - b|$ is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

9. Show that each of the relation R in the set A = $\{x \in \mathbf{Z} : 0 \leq x \leq 12\}$, given by

(i) R = {(a, b) : $|a - b|$ is a multiple of 4}

(ii) R = {(a, b) : $a = b$}

is an equivalence relation. Find the set of all elements related to 1 in each case.

10. Give an example of a relation. Which is

(i) Symmetric but neither reflexive nor transitive.

(ii) Transitive but neither reflexive nor symmetric.

(iii) Reflexive and symmetric but not transitive.

(iv) Reflexive and transitive but not symmetric.

(v) Symmetric and transitive but not reflexive.

11. Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P $\neq$ (0, 0) is the circle passing through P with origin as centre.

12. Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2) : T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

13. Show that the relation R defined in the set A of all polygons as R = {(P_1, P_2) : P_1 and P_2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

14. Let L be the set of all lines in XY plane and R be the relation in L defined as R = {(L_1, L_2) : L_1 is parallel to L_2}. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

15. Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation.

16. Let R be the relation in the set **N** given by $R = \{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer.

(A) $(2, 4) \in R$ (B) $(3, 8) \in R$ (C) $(6, 8) \in R$ (D) $(8, 7) \in R$

1.3 Types of Functions

The notion of a function along with some special functions like identity function, constant function, polynomial function, rational function, modulus function, signum function etc. along with their graphs have been given in Class XI.

Addition, subtraction, multiplication and division of two functions have also been studied. As the concept of function is of paramount importance in mathematics and among other disciplines as well, we would like to extend our study about function from where we finished earlier. In this section, we would like to study different types of functions.

Consider the functions f_1, f_2, f_3 and f_4 given by the following diagrams.

In Fig 1.2, we observe that the images of distinct elements of X_1 under the function f_1 are distinct, but the image of two distinct elements 1 and 2 of X_1 under f_2 is same, namely b. Further, there are some elements like e and f in X_2 which are not images of any element of X_1 under f_1, while all elements of X_3 are images of some elements of X_1 under f_3. The above observations lead to the following definitions:

Definition 5 A function $f : X \to Y$ is defined to be *one-one* (or *injective*), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called *many-one*.

The function f_1 and f_4 in Fig 1.2 (i) and (iv) are one-one and the function f_2 and f_3 in Fig 1.2 (ii) and (iii) are many-one.

Definition 6 A function $f : X \to Y$ is said to be *onto* (or *surjective*), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

The function f_3 and f_4 in Fig 1.2 (iii), (iv) are onto and the function f_1 in Fig 1.2 (i) is not onto as elements e, f in X_2 are not the image of any element in X_1 under f_1.

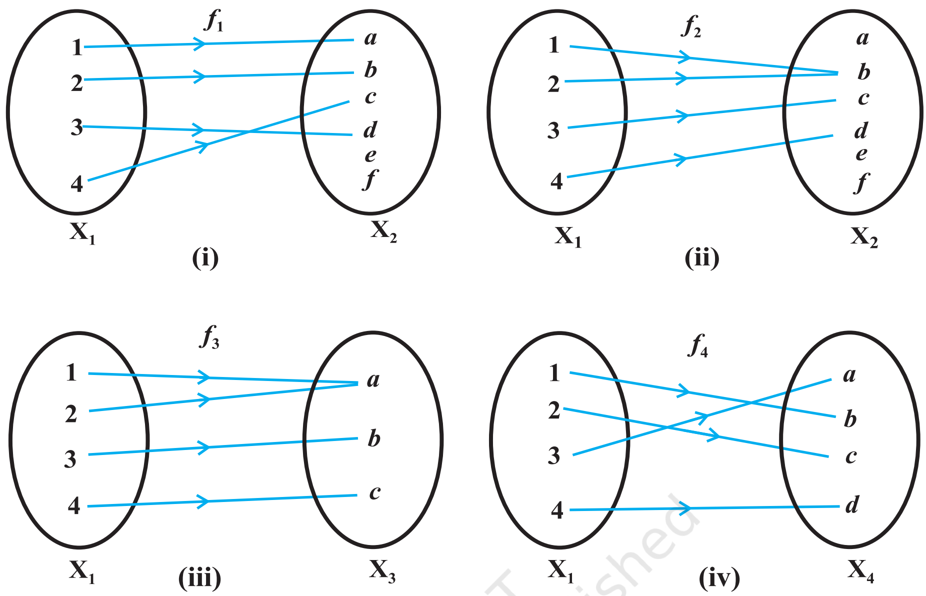


Fig 1.2 (i) to (iv)

Remark $f : \mathrm{X} \to \mathrm{Y}$ is onto if and only if Range of $f = \mathrm{Y}$.

Definition 7 A function $f : \mathrm{X} \to \mathrm{Y}$ is said to be *one-one* and *onto* (or *bijective*), if f is both one-one and onto.

The function f_4 in Fig 1.2 (iv) is one-one and onto.

Example 7 Let A be the set of all 50 students of Class X in a school. Let $f : \mathrm{A} \to \mathbf{N}$ be function defined by $f(x)$ = roll number of the student x. Show that f is one-one but not onto.

Solution No two different students of the class can have same roll number. Therefore, f must be one-one. We can assume without any loss of generality that roll numbers of students are from 1 to 50. This implies that 51 in **N** is not roll number of any student of the class, so that 51 can not be image of any element of X under f. Hence, f is not onto.

Example 8 Show that the function $f : \mathbf{N} \to \mathbf{N}$, given by $f(x) = 2x$, is one-one but not onto.

Solution The function f is one-one, for $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Further, f is not onto, as for $1 \in \mathbf{N}$, there does not exist any x in **N** such that $f(x) = 2x = 1$.

Example 9 Prove that the function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = 2x$, is one-one and onto.

Solution f is one-one, as $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Also, given any real number y in R, there exists $\frac{y}{2}$ in R such that $f(\frac{y}{2}) = 2 \,.\, (\frac{y}{2}) = y$. Hence, f is onto.

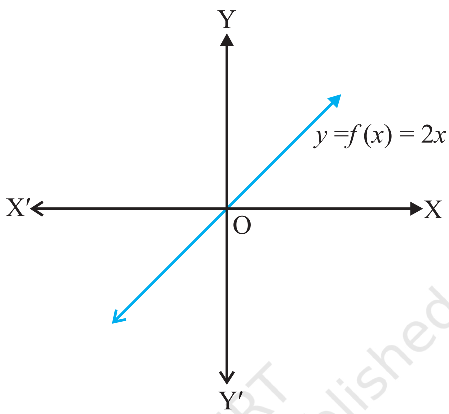


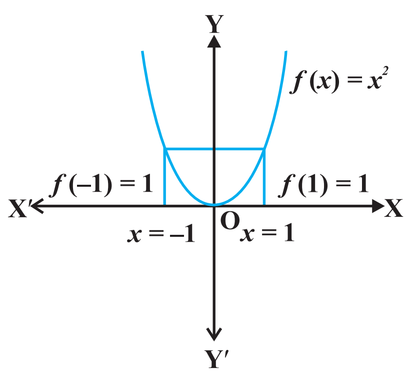
Fig 1.3

Example 10 Show that the function $f : \mathbf{N} \to \mathbf{N}$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$, for every $x > 2$, is onto but not one-one.

Solution f is not one-one, as $f(1) = f(2) = 1$. But f is onto, as given any $y \in \mathbf{N}$, $y \neq 1$, we can choose x as $y + 1$ such that $f(y + 1) = y + 1 - 1 = y$. Also for $1 \in \mathbf{N}$, we have $f(1) = 1$.

Example 11 Show that the function $f : \mathbf{R} \to \mathbf{R}$, defined as $f(x) = x^2$, is neither one-one nor onto.

Solution Since $f(-1) = 1 = f(1)$, f is not one-one. Also, the element -2 in the co-domain $\mathbf{R}$ is not image of any element x in the domain $\mathbf{R}$ (Why?). Therefore f is not onto.



The image of 1 and –1 under f is 1.

Fig 1.4

Example 12 Show that $f : \mathbf{N} \to \mathbf{N}$, given by

$$f(x) = \begin{cases} x+1, \text{if } x \text{ is odd,} \\ x-1, \text{if } x \text{ is even} \end{cases}$$

is both one-one and onto.

Solution Suppose $f(x_1) = f(x_2)$. Note that if x_1 is odd and x_2 is even, then we will have $x_1 + 1 = x_2 - 1$, i.e., $x_2 - x_1 = 2$ which is impossible. Similarly, the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument. Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd. Then $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$. Similarly, if both x_1 and x_2 are even, then also $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$. Thus, f is one-one. Also, any odd number $2r + 1$ in the co-domain **N** is the image of $2r + 2$ in the domain **N** and any even number $2r$ in the co-domain **N** is the image of $2r - 1$ in the domain **N**. Thus, f is onto.

Example 13 Show that an onto function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.

Solution Suppose f is not one-one. Then there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same. Also, the image of 3 under f can be only one element. Therefore, the range set can have at the most two elements of the co-domain $\{1, 2, 3\}$, showing that f is not onto, a contradiction. Hence, f must be one-one.

Example 14 Show that a one-one function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.

Solution Since f is one-one, three elements of $\{1, 2, 3\}$ must be taken to 3 different elements of the co-domain $\{1, 2, 3\}$ under f. Hence, f has to be onto.

Remark The results mentioned in Examples 13 and 14 are also true for an arbitrary finite set X, i.e., a one-one function $f : X \to X$ is necessarily onto and an onto map $f : X \to X$ is necessarily one-one, for every finite set X. In contrast to this, Examples 8 and 10 show that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

EXERCISE 1.2

1. Show that the function $f : \mathbf{R}_* \to \mathbf{R}_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where $\mathbf{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R}_*$ is replaced by **N** with co-domain being same as $\mathbf{R}_*$?
2. Check the injectivity and surjectivity of the following functions:
   (i) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$
   (ii) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$
   (iii) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$
   (iv) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$
   (v) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^3$
3. Prove that the Greatest Integer Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

4. Show that the Modulus Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

5. Show that the Signum Function $f : \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ \;\;1, \text{ if } x < 0 \end{cases}$$

is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$

(ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$

8. Let A and B be sets. Show that $f : \text{A} \times \text{B} \to \text{B} \times \text{A}$ such that $f(a, b) = (b, a)$ is bijective function.

9. Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, \text{ if } n \text{ is odd} \\ \dfrac{n}{2}, \text{ if } n \text{ is even} \end{cases}$ for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

10. Let A = $\mathbf{R}$ – {3} and B = $\mathbf{R}$ – {1}. Consider the function $f : \text{A} \to \text{B}$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

11. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto.

12. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 3x$. Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto.

1.4 Composition of Functions and Invertible Function

In this section, we will study composition of functions and the inverse of a bijective function. Consider the set A of all students, who appeared in Class X of a Board Examination in 2006. Each student appearing in the Board Examination is assigned a roll number by the Board which is written by the students in the answer script at the time of examination. In order to have confidentiality, the Board arranges to deface the roll numbers of students in the answer scripts and assigns a fake code number to each roll number. Let B $\subset$ **N** be the set of all roll numbers and C $\subset$ **N** be the set of all code numbers. This gives rise to two functions $f : A \to B$ and $g : B \to C$ given by $f(a)$ = the roll number assigned to the student a and $g(b)$ = the code number assigned to the roll number b. In this process each student is assigned a roll number through the function f and each roll number is assigned a code number through the function g. Thus, by the combination of these two functions, each student is eventually attached a code number.

This leads to the following definition:

Definition 8 Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by

$$gof(x) = g(f(x)), \ \forall \ x \in A.$$

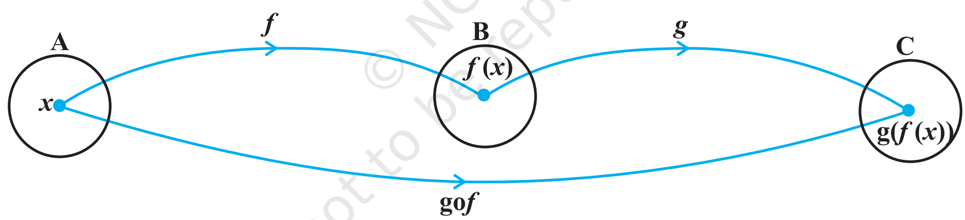


Fig 1.5

Example 15 Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Solution We have $gof(2) = g(f(2)) = g(3) = 7$, $gof(3) = g(f(3)) = g(4) = 7$, $gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(5) = 11$.

Example 16 Find gof and fog, if $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution We have $gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$. Note that $3\cos^2 x \neq \cos 3x^2$, for $x = 0$. Hence, $gof \neq fog$.

Example 17 Show that if $f:\mathbf{R}-\left\{\frac{7}{5}\right\}\to\mathbf{R}-\left\{\frac{3}{5}\right\}$ is defined by $f(x)=\frac{3x+4}{5x-7}$ and $g:\mathbf{R}-\left\{\frac{3}{5}\right\}\to\mathbf{R}-\left\{\frac{7}{5}\right\}$ is defined by $g(x)=\frac{7x+4}{5x-3}$, then $fog=\mathrm{I}_\mathrm{A}$ and $gof=\mathrm{I}_\mathrm{B}$, where, $\mathrm{A}=\mathbf{R}-\left\{\frac{3}{5}\right\}$, $\mathrm{B}=\mathbf{R}-\left\{\frac{7}{5}\right\}$; $\mathrm{I}_\mathrm{A}(x)=x,\ \forall\, x\in \mathrm{A}$, $\mathrm{I}_\mathrm{B}(x)=x,\ \forall\, x\in \mathrm{B}$ are called identity functions on sets A and B, respectively.

Solution We have

$$gof(x)=g\left(\frac{3x+4}{5x-7}\right)=\frac{7\left(\frac{(3x+4)}{(5x-7)}\right)+4}{5\left(\frac{(3x+4)}{(5x-7)}\right)-3}=\frac{21x+28+20x-28}{15x+20-15x+21}=\frac{41x}{41}=x$$

Similarly, $$fog(x)=f\left(\frac{7x+4}{5x-3}\right)=\frac{3\left(\frac{(7x+4)}{(5x-3)}\right)+4}{5\left(\frac{(7x+4)}{(5x-3)}\right)-7}=\frac{21x+12+20x-12}{35x+20-35x+21}=\frac{41x}{41}=x$$

Thus, $gof(x)=x,\ \forall\, x\in \mathrm{B}$ and $fog(x)=x,\ \forall\, x\in \mathrm{A}$, which implies that $gof=\mathrm{I}_\mathrm{B}$ and $fog=\mathrm{I}_\mathrm{A}$.

Example 18 Show that if $f:\mathrm{A}\to\mathrm{B}$ and $g:\mathrm{B}\to\mathrm{C}$ are one-one, then $gof:\mathrm{A}\to\mathrm{C}$ is also one-one.

Solution Suppose $gof(x_1) = gof(x_2)$

$\Rightarrow$ $g(f(x_1)) = g(f(x_2))$

$\Rightarrow$ $f(x_1) = f(x_2)$, as g is one-one

$\Rightarrow$ $x_1 = x_2$, as f is one-one

Hence, gof is one-one.

Example 19 Show that if $f:\mathrm{A}\to\mathrm{B}$ and $g:\mathrm{B}\to\mathrm{C}$ are onto, then $gof:\mathrm{A}\to\mathrm{C}$ is also onto.

Solution Given an arbitrary element $z\in \mathrm{C}$, there exists a pre-image y of z under g such that $g(y)=z$, since g is onto. Further, for $y\in \mathrm{B}$, there exists an element x in A with $f(x) = y$, since f is onto. Therefore, $gof(x) = g(f(x)) = g(y) = z$, showing that gof is onto.

Example 20 Consider functions f and g such that composite gof is defined and is one-one. Are f and g both necessarily one-one.

Solution Consider $f : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4, 5, 6\}$ defined as $f(x) = x, \ \forall x$ and $g : \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2, 3, 4, 5, 6\}$ as $g(x) = x$, for $x = 1, 2, 3, 4$ and $g(5) = g(6) = 5$. Then, $gof(x) = x \ \forall x$, which shows that gof is one-one. But g is clearly not one-one.

Example 21 Are f and g both necessarily onto, if gof is onto?

Solution Consider $f : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$ and $g : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3\}$ defined as $f(1) = 1, f(2) = 2, f(3) = f(4) = 3, g(1) = 1, g(2) = 2$ and $g(3) = g(4) = 3$. It can be seen that gof is onto but f is not onto.

Remark It can be verified in general that gof is one-one implies that f is one-one. Similarly, gof is onto implies that g is onto.

Now, we would like to have close look at the functions f and g described in the beginning of this section in reference to a Board Examination. Each student appearing in Class X Examination of the Board is assigned a roll number under the function f and each roll number is assigned a code number under g. After the answer scripts are examined, examiner enters the mark against each code number in a mark book and submits to the office of the Board. The Board officials decode by assigning roll number back to each code number through a process reverse to g and thus mark gets attached to roll number rather than code number. Further, the process reverse to f assigns a roll number to the student having that roll number. This helps in assigning mark to the student scoring that mark. We observe that while composing f and g, to get gof, first f and then g was applied, while in the reverse process of the composite gof, first the reverse process of g is applied and then the reverse process of f.

Example 22 Let $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$ be one-one and onto function given by $f(1) = a, f(2) = b$ and $f(3) = c$. Show that there exists a function $g : \{a, b, c\} \rightarrow \{1, 2, 3\}$ such that $gof = I_X$ and $fog = I_Y$, where, $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Solution Consider $g : \{a, b, c\} \rightarrow \{1, 2, 3\}$ as $g(a) = 1, g(b) = 2$ and $g(c) = 3$. It is easy to verify that the composite $gof = I_X$ is the identity function on X and the composite $fog = I_Y$ is the identity function on Y.

Remark The interesting fact is that the result mentioned in the above example is true for an arbitrary one-one and onto function $f : X \rightarrow Y$. Not only this, even the converse is also true , i.e., if $f : X \rightarrow Y$ is a function such that there exists a function $g : Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$, then f must be one-one and onto.

The above discussion, Example 22 and Remark lead to the following definition:

Definition 9 A function $f : X \rightarrow Y$ is defined to be *invertible*, if there exists a function $g : Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the *inverse of f* and is denoted by f^{-1}.

Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible. This fact significantly helps for proving a function f to be invertible by showing that f is one-one and onto, specially when the actual inverse of f is not to be determined.

Example 23 Let $f : \mathbf{N} \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where, $Y = \{y \in \mathbf{N} : y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$. Show that f is invertible. Find the inverse.

Solution Consider an arbitrary element y of Y. By the definition of Y, $y = 4x + 3$, for some x in the domain $\mathbf{N}$. This shows that $x = \frac{(y-3)}{4}$. Define $g : Y \rightarrow \mathbf{N}$ by $g(y) = \frac{(y-3)}{4}$. Now, $gof(x) = g(f(x)) = g(4x + 3) = \frac{(4x+3-3)}{4} = x$ and

$$fog(y) = f(g(y)) = f\left(\frac{(y-3)}{4}\right) = \frac{4(y-3)}{4} + 3 = y - 3 + 3 = y.$$ This shows that $gof = I_N$ and $fog = I_Y$, which implies that f is invertible and g is the inverse of f.

Example 24 Let $Y = \{n^2 : n \in \mathbf{N}\} \subset \mathbf{N}$. Consider $f : \mathbf{N} \rightarrow Y$ as $f(n) = n^2$. Show that f is invertible. Find the inverse of f.

Solution An arbitrary element y in Y is of the form n^2, for some $n \in \mathbf{N}$. This implies that $n = \sqrt{y}$. This gives a function $g : Y \rightarrow \mathbf{N}$, defined by $g(y) = \sqrt{y}$. Now, $gof(n) = g(n^2) = \sqrt{n^2} = n$ and $fog(y) = f\left(\sqrt{y}\right) = \left(\sqrt{y}\right)^2 = y$, which shows that $gof = I_N$ and $fog = I_Y$. Hence, f is invertible with $f^{-1} = g$.

Example 25 Let $f' : \mathbf{N} \rightarrow \mathbf{R}$ be a function defined as $f'(x) = 4x^2 + 12x + 15$. Show that $f : \mathbf{N} \rightarrow S$, where, S is the range of f, is invertible. Find the inverse of f.

Solution Let y be an arbitrary element of range f. Then $y = 4x^2 + 12x + 15$, for some x in $\mathbf{N}$, which implies that $y = (2x + 3)^2 + 6$. This gives $x = \frac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}$, as $y \geq 6$.

Let us define $g : \mathrm{S} \to \mathbf{N}$ by $g(y) = \dfrac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}$.

Now $\quad gof(x) = g(f(x)) = g(4x^2 + 12x + 15) = g((2x + 3)^2 + 6)$

$$= \frac{\left(\left(\sqrt{(2x+3)^2+6-6}\right)-3\right)}{2} = \frac{(2x+3-3)}{2} = x$$

and $\quad fog\ (y) = f\left(\dfrac{\left(\left(\sqrt{y-6}\right)-3\right)}{2}\right) = \left(\dfrac{2\left(\left(\sqrt{y-6}\right)-3\right)}{2}+3\right)^2 + 6$

$$= \left(\left(\left(\sqrt{y-6}\right)-3+3\right)\right)^2 + 6 = \left(\sqrt{y-6}\right)^2 + 6 = y - 6 + 6 = y.$$

Hence, $\quad gof = \mathrm{I_N}$ and $fog = \mathrm{I_S}$. This implies that f is invertible with $f^{-1} = g$.

Example 26 Consider $f : \mathbf{N} \to \mathbf{N}$, $g : \mathbf{N} \to \mathbf{N}$ and $h : \mathbf{N} \to \mathbf{R}$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$, $\forall\, x, y$ and z in N. Show that $ho(gof) = (hog)\, of$.

Solution We have

$$ho(gof)\ (x) = h(gof\ (x)) = h(g(f(x))) = h(g(2x))$$
$$= h(3(2x) + 4) = h(6x + 4) = \sin(6x + 4)\ \ \forall\, x \in \mathbf{N}.$$

Also, $\quad ((hog)of)\ (x) = (hog)\ (f(x)) = (hog)\ (2x) = h(g(2x))$

$$= h(3(2x) + 4) = h(6x + 4) = \sin(6x + 4),\ \ \forall\, x \in \mathbf{N}.$$

This shows that $ho(gof) = (hog)\, of$.

This result is true in general situation as well.

Theorem 1 If $f : \mathrm{X} \to \mathrm{Y}$, $g : \mathrm{Y} \to \mathrm{Z}$ and $h : \mathrm{Z} \to \mathrm{S}$ are functions, then

$$ho(gof) = (hog)\, of.$$

Proof We have

$$ho(gof)\ (x) = h(gof(x)) = h(g(f(x))),\ \ \forall\, x \text{ in X}$$

and $\quad (hog)\ of\ (x) = hog(f(x)) = h(g(f(x))),\ \ \forall\, x$ in X.

Hence, $\quad ho(gof) = (hog)\, of.$

Example 27 Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ and $g : \{a, b, c\} \to \{\text{apple, ball, cat}\}$ defined as $f(1) = a$, $f(2) = b$, $f(3) = c$, $g(a) =$ apple, $g(b) =$ ball and $g(c) =$ cat. Show that f, g and gof are invertible. Find out f^{-1}, g^{-1} and $(gof)^{-1}$ and show that $(gof)^{-1} = f^{-1}o\, g^{-1}$.

Solution Note that by definition, f and g are bijective functions. Let f^{-1}: $\{a, b, c\} \to (1, 2, 3\}$ and g^{-1} : {apple, ball, cat} $\to \{a, b, c\}$ be defined as $f^{-1}\{a\} = 1, f^{-1}\{b\} = 2,\ f^{-1}\{c\} = 3,\ g^{-1}\{\text{apple}\} = a,\ g^{-1}\{\text{ball}\} = b$ and $g^{-1}\{\text{cat}\} = c$. It is easy to verify that $f^{-1}of = I_{\{1, 2, 3\}}, f\, o\, f^{-1} = I_{\{a, b, c\}}, g^{-1}og = I_{\{a, b, c\}}$ and $g\, o\, g^{-1} = I_D$, where, D = {apple, ball, cat}. Now, gof : {1, 2, 3} $\to$ {apple, ball, cat} is given by $gof(1)$ = apple, $gof(2)$ = ball, $gof(3)$ = cat. We can define

$(gof)^{-1}$: {apple, ball, cat} $\to$ {1, 2, 3} by $(gof)^{-1}$ (apple) = 1, $(gof)^{-1}$ (ball) = 2 and $(g\,of)^{-1}$ (cat) = 3. It is easy to see that $(g\,o\,f)^{-1}\ o\ (g\,o\,f) = I_{\{1, 2, 3\}}$ and $(gof)\ o\ (gof)^{-1} = I_D$. Thus, we have seen that f, g and gof are invertible.

Now, $f^{-1}og^{-1}$ (apple)= $f^{-1}(g^{-1}$(apple)) = $f^{-1}(a) = 1 = (gof)^{-1}$ (apple)

$f^{-1}og^{-1}$ (ball) = $f^{-1}(g^{-1}$(ball)) = $f^{-1}(b) = 2 = (gof)^{-1}$ (ball) and

$f^{-1}og^{-1}$ (cat) = $f^{-1}(g^{-1}$(cat)) = $f^{-1}(c) = 3 = (gof)^{-1}$ (cat).

Hence $(gof)^{-1} = f^{-1}og^{-1}$.

The above result is true in general situation also.

Theorem 2 Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$.

Proof To show that gof is invertible with $(gof)^{-1} = f^{-1}og^{-1}$, it is enough to show that $(f^{-1}og^{-1})o(gof) = I_X$ and $(gof)o(f^{-1}og^{-1}) = I_Z$.

Now,
$$\begin{aligned}(f^{-1}og^{-1})o(gof) &= ((f^{-1}og^{-1})\ og)\ of, \text{ by Theorem 1}\\ &= (f^{-1}o(g^{-1}og))\ of, \text{ by Theorem 1}\\ &= (f^{-1}\,o\,I_Y)\ of, \text{ by definition of } g^{-1}\\ &= I_X.\end{aligned}$$

Similarly, it can be shown that $(gof\,)\,o\,(f^{-1}\,o\,g^{-1}) = I_Z$.

Example 28 Let S = {1, 2, 3}. Determine whether the functions f : S $\to$ S defined as below have inverses. Find f^{-1}, if it exists.

(a) $f = \{(1, 1), (2, 2), (3, 3)\}$
(b) $f = \{(1, 2), (2, 1), (3, 1)\}$
(c) $f = \{(1, 3), (3, 2), (2, 1)\}$

Solution

(a) It is easy to see that f is one-one and onto, so that f is invertible with the inverse f^{-1} of f given by $f^{-1} = \{(1, 1), (2, 2), (3, 3)\} = f$.

(b) Since $f(2) = f(3) = 1$, f is not one-one, so that f is not invertible.

(c) It is easy to see that f is one-one and onto, so that f is invertible with $f^{-1} = \{(3, 1), (2, 3), (1, 2)\}$.

EXERCISE 1.3

1. Let $f : \{1, 3, 4\} \to \{1, 2, 5\}$ and $g : \{1, 2, 5\} \to \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof.

2. Let f, g and h be functions from **R** to **R**. Show that

$$(f + g) \circ h = foh + goh$$

$$(f \, . \, g) \circ h = (foh) \, . \, (goh)$$

3. Find gof and fog, if

(i) $f(x) = | \, x \, |$ and $g(x) = | \, 5x - 2 \, |$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$.

4. If $f(x) = \dfrac{(4x+3)}{(6x-4)}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$, for all $x \neq \dfrac{2}{3}$. What is the inverse of f?

5. State with reason whether following functions have inverse

(i) $f : \{1, 2, 3, 4\} \to \{10\}$ with
$f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(ii) $g : \{5, 6, 7, 8\} \to \{1, 2, 3, 4\}$ with
$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(iii) $h : \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$ with
$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

6. Show that $f : [-1, 1] \to \mathbf{R}$, given by $f(x) = \dfrac{x}{(x+2)}$ is one-one. Find the inverse of the function $f : [-1, 1] \to$ Range f.

(Hint: For $y \in$ Range f, $y = f(x) = \dfrac{x}{x+2}$, for some x in $[-1, 1]$, i.e., $x = \dfrac{2y}{(1-y)}$)

7. Consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

8. Consider $f : \mathbf{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

9. Consider $f : \mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$.

10. Let $f : X \to Y$ be an invertible function. Show that f has unique inverse. (Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$, $fog_1(y) = 1_Y(y) = fog_2(y)$. Use one-one ness of f).

11. Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

12. Let $f: X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

13. If $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = (3-x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $x^{\frac{1}{3}}$ (B) x^3 (C) x (D) $(3 - x^3)$.

14. Let $f : \mathbf{R} - \left\{-\frac{4}{3}\right\} \to \mathbf{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is the map g : Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$ (B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$ (D) $g(y) = \frac{3y}{4-3y}$

1.5 Binary Operations

Right from the school days, you must have come across four fundamental operations namely addition, subtraction, multiplication and division. The main feature of these operations is that given any two numbers a and b, we associate another number $a + b$ or $a - b$ or ab or $\frac{a}{b}$, $b \neq 0$. It is to be noted that only two numbers can be added or multiplied at a time. When we need to add three numbers, we first add two numbers and the result is then added to the third number. Thus, addition, multiplication, subtraction and division are examples of binary operation, as 'binary' means two. If we want to have a general definition which can cover all these four operations, then the set of numbers is to be replaced by an arbitrary set X and then general binary operation is nothing but association of any pair of elements a, b from X to another element of X. This gives rise to a general definition as follows:

Definition 10 A binary operation $*$ on a set A is a function $* : A \times A \to A$. We denote $* (a, b)$ by $a * b$.

Example 29 Show that addition, subtraction and multiplication are binary operations on $\mathbf{R}$, but division is not a binary operation on $\mathbf{R}$. Further, show that division is a binary operation on the set $\mathbf{R}_*$ of nonzero real numbers.

Solution $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by

$(a, b) \to a + b$

$- : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by

$(a, b) \to a - b$

$\times : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by

$(a, b) \to ab$

Since '+', '–' and '×' are functions, they are binary operations on $\mathbf{R}$.

But $\div: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, given by $(a, b) \to \frac{a}{b}$, is not a function and hence not a binary operation, as for $b = 0$, $\frac{a}{b}$ is not defined.

However, $\div : \mathbf{R}_* \times \mathbf{R}_* \to \mathbf{R}_*$, given by $(a, b) \to \frac{a}{b}$ is a function and hence a binary operation on $\mathbf{R}_*$.

Example 30 Show that subtraction and division are not binary operations on $\mathbf{N}$.

Solution $- : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$, given by $(a, b) \to a - b$, is not binary operation, as the image of (3, 5) under '–' is $3 - 5 = -2 \notin \mathbf{N}$. Similarly, $\div : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$, given by $(a, b) \to a \div b$ is not a binary operation, as the image of (3, 5) under $\div$ is $3 \div 5 = \frac{3}{5} \notin \mathbf{N}$.

Example 31 Show that $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(a, b) \to a + 4b^2$ is a binary operation.

Solution Since $*$ carries each pair (a, b) to a unique element $a + 4b^2$ in $\mathbf{R}$, $*$ is a binary operation on $\mathbf{R}$.

Example 32 Let P be the set of all subsets of a given set X. Show that $\cup : P \times P \to P$ given by $(A, B) \to A \cup B$ and $\cap : P \times P \to P$ given by $(A, B) \to A \cap B$ are binary operations on the set P.

Solution Since union operation $\cup$ carries each pair (A, B) in $P \times P$ to a unique element $A \cup B$ in P, $\cup$ is binary operation on P. Similarly, the intersection operation $\cap$ carries each pair (A, B) in $P \times P$ to a unique element $A \cap B$ in P, $\cap$ is a binary operation on P.

Example 33 Show that the $\vee : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(a, b) \to \max\{a, b\}$ and the $\wedge : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(a, b) \to \min\{a, b\}$ are binary operations.

Solution Since $\vee$ carries each pair (a, b) in $\mathbf{R} \times \mathbf{R}$ to a unique element namely maximum of a and b lying in **R**, $\vee$ is a binary operation. Using the similar argument, one can say that $\wedge$ is also a binary operation.

Remark $\vee(4, 7) = 7$, $\vee(4, -7) = 4$, $\wedge(4, 7) = 4$ and $\wedge(4, -7) = -7$.

When number of elements in a set A is small, we can express a binary operation $*$ on the set A through a table called the *operation table* for the operation $*$. For example consider $A = \{1, 2, 3\}$. Then, the operation $\vee$ on A defined in Example 33 can be expressed by the following operation table (Table 1.1) . Here, $\vee(1, 3) = 3$, $\vee(2, 3) = 3$, $\vee(1, 2) = 2$.

Table 1.1

V	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

Here, we are having 3 rows and 3 columns in the operation table with (i, j) the entry of the table being maximum of i^{th} and j^{th} elements of the set A. This can be generalised for general operation $* : A \times A \to A$. If $A = \{a_1, a_2, ..., a_n\}$. Then the operation table will be having n rows and n columns with $(i, j)^{th}$ entry being $a_i * a_j$. Conversely, given any operation table having n rows and n columns with each entry being an element of $A = \{a_1, a_2, ..., a_n\}$, we can define a binary operation $* : A \times A \to A$ given by $a_i * a_j$ = the entry in the i^{th} row and j^{th} column of the operation table.

One may note that 3 and 4 can be added in any order and the result is same, i.e., $3 + 4 = 4 + 3$, but subtraction of 3 and 4 in different order give different results, i.e., $3 - 4 \neq 4 - 3$. Similarly, in case of multiplication of 3 and 4, order is immaterial, but division of 3 and 4 in different order give different results. Thus, addition and multiplication of 3 and 4 are meaningful, but subtraction and division of 3 and 4 are meaningless. For subtraction and division we have to write 'subtract 3 from 4', 'subtract 4 from 3', 'divide 3 by 4' or 'divide 4 by 3'.

This leads to the following definition:

Definition 11 A binary operation $*$ on the set X is called *commutative*, if $a * b = b * a$, for every $a, b \in$ X.

Example 34 Show that $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\times : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are commutative binary operations, but $- : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\div : \mathbf{R}_* \times \mathbf{R}_* \to \mathbf{R}_*$ are not commutative.

Solution Since $a + b = b + a$ and $a \times b = b \times a$, $\forall\, a, b \in \mathbf{R}$, '+' and '×' are commutative binary operation. However, '–' is not commutative, since $3 - 4 \neq 4 - 3$. Similarly, $3 \div 4 \neq 4 \div 3$ shows that '÷' is not commutative.

Example 35 Show that $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by $a * b = a + 2b$ is not commutative.

Solution Since $3 * 4 = 3 + 8 = 11$ and $4 * 3 = 4 + 6 = 10$, showing that the operation $*$ is not commutative.

If we want to associate three elements of a set X through a binary operation on X, we encounter a natural problem. The expression $a * b * c$ may be interpreted as $(a * b) * c$ or $a * (b * c)$ and these two expressions need not be same. For example, $(8 - 5) - 2 \neq 8 - (5 - 2)$. Therefore, association of three numbers 8, 5 and 3 through the binary operation 'subtraction' is meaningless, unless bracket is used. But in case of addition, $8 + 5 + 2$ has the same value whether we look at it as $(8 + 5) + 2$ or as $8 + (5 + 2)$. Thus, association of 3 or even more than 3 numbers through addition is meaningful without using bracket. This leads to the following:

Definition 12 A binary operation $* : A \times A \to A$ is said to be *associative* if

$$(a * b) * c = a * (b * c), \ \forall\, a, b, c, \in A.$$

Example 36 Show that addition and multiplication are associative binary operation on **R**. But subtraction is not associative on **R**. Division is not associative on $\mathbf{R}_*$.

Solution Addition and multiplication are associative, since $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$ $\forall\ a, b, c \in$ R. However, subtraction and division are not associative, as $(8 - 5) - 3 \neq 8 - (5 - 3)$ and $(8 \div 5) \div 3 \neq 8 \div (5 \div 3)$.

Example 37 Show that $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $a * b \to a + 2b$ is not associative.

Solution The operation $*$ is not associative, since

$$(8 * 5) * 3 = (8 + 10) * 3 = (8 + 10) + 6 = 24,$$

while $$8 * (5 * 3) = 8 * (5 + 6) = 8 * 11 = 8 + 22 = 30.$$

Remark Associative property of a binary operation is very important in the sense that with this property of a binary operation, we can write $a_1 * a_2 * ... * a_n$ which is not ambiguous. But in absence of this property, the expression $a_1 * a_2 * ... * a_n$ is ambiguous unless brackets are used. Recall that in the earlier classes brackets were used whenever subtraction or division operations or more than one operation occurred.

For the binary operation '+' on **R**, the interesting feature of the number zero is that $a + 0 = a = 0 + a$, i.e., any number remains unaltered by adding zero. But in case of multiplication, the number 1 plays this role, as $a \times 1 = a = 1 \times a$, $\forall$ a in **R**. This leads to the following definition:

Definition 13 Given a binary operation $* : A \times A \to A$, an element $e \in A$, if it exists, is called *identity* for the operation $*$, if $a * e = a = e * a$, $\forall$ $a \in A$.

Example 38 Show that zero is the identity for addition on **R** and 1 is the identity for multiplication on R. But there is no identity element for the operations

$$- : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \text{ and } \div : \mathbf{R}_* \times \mathbf{R}_* \to \mathbf{R}_*.$$

Solution $a + 0 = 0 + a = a$ and $a \times 1 = a = 1 \times a$, $\forall a \in \mathbf{R}$ implies that 0 and 1 are identity elements for the operations '+' and '×' respectively. Further, there is no element e in **R** with $a - e = e - a$, $\forall a$. Similarly, we can not find any element e in $\mathbf{R}_*$ such that $a \div e = e \div a$, $\forall a$ in $\mathbf{R}_*$. Hence, '–' and '÷' do not have identity element.

Remark Zero is identity for the addition operation on **R** but it is not identity for the addition operation on **N**, as $0 \notin \mathbf{N}$. In fact the addition operation on **N** does not have any identity.

One further notices that for the addition operation $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, given any $a \in \mathbf{R}$, there exists $-a$ in **R** such that $a + (-a) = 0$ (identity for '+') $= (-a) + a$. Similarly, for the multiplication operation on **R**, given any $a \neq 0$ in **R**, we can choose $\frac{1}{a}$ in **R** such that $a \times \frac{1}{a} = 1$(identity for '×') $= \frac{1}{a} \times a$. This leads to the following definition:

Definition 14 Given a binary operation $* : A \times A \to A$ with the identity element e in A, an element $a \in A$ is said to be *invertible* with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the *inverse of* a and is denoted by a^{-1}.

Example 39 Show that $-a$ is the inverse of a for the addition operation '+' on **R** and $\frac{1}{a}$ is the inverse of $a \neq 0$ for the multiplication operation '×' on **R**.

Solution As $a + (-a) = a - a = 0$ and $(-a) + a = 0$, $-a$ is the inverse of a for addition. Similarly, for $a \neq 0$, $a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$ implies that $\frac{1}{a}$ is the inverse of a for multiplication.

Example 40 Show that $-a$ is not the inverse of $a \in \mathbf{N}$ for the addition operation + on $\mathbf{N}$ and $\frac{1}{a}$ is not the inverse of $a \in \mathbf{N}$ for multiplication operation × on $\mathbf{N}$, for $a \neq 1$.

Solution Since $-a \notin \mathbf{N}$, $-a$ can not be inverse of a for addition operation on $\mathbf{N}$, although $-a$ satisfies $a + (-a) = 0 = (-a) + a$.

Similarly, for $a \neq 1$ in $\mathbf{N}$, $\frac{1}{a} \notin \mathbf{N}$, which implies that other than 1 no element of $\mathbf{N}$ has inverse for multiplication operation on $\mathbf{N}$.

Examples 34, 36, 38 and 39 show that addition on $\mathbf{R}$ is a commutative and associative binary operation with 0 as the identity element and $-a$ as the inverse of a in $\mathbf{R}$ $\forall a$.

EXERCISE 1.4

1. Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.
   (i) On $\mathbf{Z}^+$, define $*$ by $a * b = a - b$
   (ii) On $\mathbf{Z}^+$, define $*$ by $a * b = ab$
   (iii) On $\mathbf{R}$, define $*$ by $a * b = ab^2$
   (iv) On $\mathbf{Z}^+$, define $*$ by $a * b = |a - b|$
   (v) On $\mathbf{Z}^+$, define $*$ by $a * b = a$
2. For each operation $*$ defined below, determine whether $*$ is binary, commutative or associative.
   (i) On $\mathbf{Z}$, define $a * b = a - b$
   (ii) On $\mathbf{Q}$, define $a * b = ab + 1$
   (iii) On $\mathbf{Q}$, define $a * b = \frac{ab}{2}$
   (iv) On $\mathbf{Z}^+$, define $a * b = 2^{ab}$
   (v) On $\mathbf{Z}^+$, define $a * b = a^b$
   (vi) On $\mathbf{R} - \{-1\}$, define $a * b = \frac{a}{b+1}$
3. Consider the binary operation $\wedge$ on the set $\{1, 2, 3, 4, 5\}$ defined by $a \wedge b = \min \{a, b\}$. Write the operation table of the operation $\wedge$.

4. Consider a binary operation $*$ on the set $\{1, 2, 3, 4, 5\}$ given by the following multiplication table (Table 1.2).
   (i) Compute $(2 * 3) * 4$ and $2 * (3 * 4)$
   (ii) Is $*$ commutative?
   (iii) Compute $(2 * 3) * (4 * 5)$.
   (Hint: use the following table)

Table 1.2

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

5. Let $*'$ be the binary operation on the set $\{1, 2, 3, 4, 5\}$ defined by $a *' b$ = H.C.F. of a and b. Is the operation $*'$ same as the operation $*$ defined in Exercise 4 above? Justify your answer.
6. Let $*$ be the binary operation on **N** given by $a * b$ = L.C.M. of a and b. Find
   (i) $5 * 7$, $20 * 16$
   (ii) Is $*$ commutative?
   (iii) Is $*$ associative?
   (iv) Find the identity of $*$ in **N**
   (v) Which elements of **N** are invertible for the operation $*$?
7. Is $*$ defined on the set $\{1, 2, 3, 4, 5\}$ by $a * b$ = L.C.M. of a and b a binary operation? Justify your answer.
8. Let $*$ be the binary operation on **N** defined by $a * b$ = H.C.F. of a and b. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on **N**?
9. Let $*$ be a binary operation on the set **Q** of rational numbers as follows:
   (i) $a * b = a - b$
   (ii) $a * b = a^2 + b^2$
   (iii) $a * b = a + ab$
   (iv) $a * b = (a - b)^2$
   (v) $a * b = \dfrac{ab}{4}$
   (vi) $a * b = ab^2$

Find which of the binary operations are commutative and which are associative.

10. Find which of the operations given above has identity.
11. Let A = **N** × **N** and $*$ be the binary operation on A defined by

$$(a, b) * (c, d) = (a + c, b + d)$$

Show that $*$ is commutative and associative. Find the identity element for $*$ on A, if any.

12. State whether the following statements are true or false. Justify.

(i) For an arbitrary binary operation $*$ on a set **N**, $a * a = a \ \forall a \in \mathbf{N}$.

(ii) If $*$ is a commutative binary operation on **N**, then $a * (b * c) = (c * b) * a$

13. Consider a binary operation $*$ on **N** defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is $*$ both associative and commutative?

(B) Is $*$ commutative but not associative?

(C) Is $*$ associative but not commutative?

(D) Is $*$ neither commutative nor associative?

Miscellaneous Examples

Example 41 If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is also an equivalence relation.

Solution Since R_1 and R_2 are equivalence relations, $(a, a) \in R_1$, and $(a, a) \in R_2 \ \forall a \in A$. This implies that $(a, a) \in R_1 \cap R_2, \ \forall a$, showing $R_1 \cap R_2$ is reflexive. Further, $(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2 \Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2 \Rightarrow (b, a) \in R_1 \cap R_2$, hence, $R_1 \cap R_2$ is symmetric. Similarly, $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2 \Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2 \Rightarrow (a, c) \in R_1 \cap R_2$. This shows that $R_1 \cap R_2$ is transitive. Thus, $R_1 \cap R_2$ is an equivalence relation.

Example 42 Let R be a relation on the set A of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if $xv = yu$. Show that R is an equivalence relation.

Solution Clearly, (x, y) R (x, y), $\forall (x, y) \in A$, since $xy = yx$. This shows that R is reflexive. Further, (x, y) R $(u, v) \Rightarrow xv = yu \Rightarrow uy = vx$ and hence (u, v) R (x, y). This shows that R is symmetric. Similarly, (x, y) R (u, v) and (u, v) R $(a, b) \Rightarrow xv = yu$ and $ub = va \Rightarrow xv\dfrac{a}{u} = yu\dfrac{a}{u} \Rightarrow xv\dfrac{b}{v} = yu\dfrac{a}{u} \Rightarrow xb = ya$ and hence (x, y) R (a, b). Thus, R is transitive. Thus, R is an equivalence relation.

Example 43 Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Let R_1 be a relation in X given by $R_1 = \{(x, y) : x - y$ is divisible by $3\}$ and R_2 be another relation on X given by $R_2 = \{(x, y): \{x, y\} \subset \{1, 4, 7\}\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$. Show that $R_1 = R_2$.

Solution Note that the characteristic of sets {1, 4, 7}, {2, 5, 8} and {3, 6, 9} is that difference between any two elements of these sets is a multiple of 3. Therefore, $(x, y) \in R_1 \Rightarrow x - y$ is a multiple of $3 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$. Hence, $R_1 \subset R_2$. Similarly, $\{x, y\} \in R_2 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow x - y$ is divisible by $3 \Rightarrow \{x, y\} \in R_1$. This shows that $R_2 \subset R_1$. Hence, $R_1 = R_2$.

Example 44 Let $f : X \rightarrow Y$ be a function. Define a relation R in X given by $R = \{(a, b): f(a) = f(b)\}$. Examine whether R is an equivalence relation or not.

Solution For every $a \in X$, $(a, a) \in R$, since $f(a) = f(a)$, showing that R is reflexive. Similarly, $(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$. Therefore, R is symmetric. Further, $(a, b) \in R$ and $(b, c) \in R \Rightarrow f(a) = f(b)$ and $f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow (a, c) \in R$, which implies that R is transitive. Hence, R is an equivalence relation.

Example 45 Determine which of the following binary operations on the set R are associative and which are commutative.

(a) $a * b = 1 \ \forall \ a, b \in R$ (b) $a * b = \dfrac{(a+b)}{2} \ \forall \ a, b \in R$

Solution

(a) Clearly, by definition $a * b = b * a = 1$, $\forall a, b \in R$. Also $(a * b) * c = (1 * c) = 1$ and $a * (b * c) = a * (1) = 1$, $\forall \ a, b, c \in R$. Hence R is both associative and commutative.

(b) $a * b = \dfrac{a+b}{2} = \dfrac{b+a}{2} = b * a$, shows that $*$ is commutative. Further,

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c.$$

$$= \frac{\left(\frac{a+b}{2}\right) + c}{2} = \frac{a+b+2c}{4}.$$

But $$a * (b * c) = a * \left(\frac{b+c}{2}\right)$$

$$= \frac{a + \frac{b+c}{2}}{2} = \frac{2a+b+c}{4} \neq \frac{a+b+2c}{4} \text{ in general.}$$

Hence, $*$ is not associative.

Example 46 Find the number of all one-one functions from set A = {1, 2, 3} to itself.

Solution One-one function from {1, 2, 3} to itself is simply a permutation on three symbols 1, 2, 3. Therefore, total number of one-one maps from {1, 2, 3} to itself is same as total number of permutations on three symbols 1, 2, 3 which is 3! = 6.

Example 47 Let A = {1, 2, 3}. Then show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric is three.

Solution The smallest relation R_1 containing (1, 2) and (2, 3) which is reflexive and transitive but not symmetric is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}. Now, if we add the pair (2, 1) to R_1 to get R_2, then the relation R_2 will be reflexive, transitive but not symmetric. Similarly, we can obtain R_3 by adding (3, 2) to R_1 to get the desired relation. However, we can not add two pairs (2, 1), (3, 2) or single pair (3, 1) to R_1 at a time, as by doing so, we will be forced to add the remaining pair in order to maintain transitivity and in the process, the relation will become symmetric also which is not required. Thus, the total number of desired relations is three.

Example 48 Show that the number of equivalence relation in the set {1, 2, 3} containing (1, 2) and (2, 1) is two.

Solution The smallest equivalence relation R_1 containing (1, 2) and (2, 1) is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}. Now we are left with only 4 pairs namely (2, 3), (3, 2), (1, 3) and (3, 1). If we add any one, say (2, 3) to R_1, then for symmetry we must add (3, 2) also and now for transitivity we are forced to add (1, 3) and (3, 1). Thus, the only equivalence relation bigger than R_1 is the universal relation. This shows that the total number of equivalence relations containing (1, 2) and (2, 1) is two.

Example 49 Show that the number of binary operations on {1, 2} having 1 as identity and having 2 as the inverse of 2 is exactly one.

Solution A binary operation $*$ on {1, 2} is a function from {1, 2} × {1, 2} to {1, 2}, i.e., a function from {(1, 1), (1, 2), (2, 1), (2, 2)} → {1, 2}. Since 1 is the identity for the desired binary operation $*$, $*$ (1, 1) = 1, $*$ (1, 2) = 2, $*$ (2, 1) = 2 and the only choice left is for the pair (2, 2). Since 2 is the inverse of 2, i.e., $*$ (2, 2) must be equal to 1. Thus, the number of desired binary operation is only one.

Example 50 Consider the identity function $I_N : \mathbf{N} \to \mathbf{N}$ defined as $I_N(x) = x \ \forall x \in \mathbf{N}$. Show that although I_N is onto but $I_N + I_N : \mathbf{N} \to \mathbf{N}$ defined as

$$(I_N + I_N)(x) = I_N(x) + I_N(x) = x + x = 2x \text{ is not onto.}$$

Solution Clearly I_N is onto. But $I_N + I_N$ is not onto, as we can find an element 3 in the co-domain **N** such that there does not exist any x in the domain **N** with $(I_N + I_N)(x) = 2x = 3$.

Example 51 Consider a function $f : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $f(x) = \sin x$ and $g : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $g(x) = \cos x$. Show that f and g are one-one, but $f + g$ is not one-one.

Solution Since for any two distinct elements x_1 and x_2 in $\left[0, \frac{\pi}{2}\right]$, $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$, both f and g must be one-one. But $(f + g)(0) = \sin 0 + \cos 0 = 1$ and $(f + g)\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$. Therefore, $f + g$ is not one-one.

Miscellaneous Exercise on Chapter 1

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 10x + 7$. Find the function $g : \mathbf{R} \to \mathbf{R}$ such that $g \circ f = f \circ g = 1_{\mathbf{R}}$.
2. Let $f : \mathrm{W} \to \mathrm{W}$ be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.
3. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.
4. Show that the function $f : \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1 + |x|}$, $x \in \mathbf{R}$ is one one and onto function.
5. Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ is injective.
6. Give examples of two functions $f : \mathbf{N} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ such that $g \circ f$ is injective but g is not injective.

   (Hint : Consider $f(x) = x$ and $g(x) = |x|$).
7. Give examples of two functions $f : \mathbf{N} \to \mathbf{N}$ and $g : \mathbf{N} \to \mathbf{N}$ such that $g \circ f$ is onto but f is not onto.

   (Hint : Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x - 1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$
8. Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify your answer.

9. Given a non-empty set X, consider the binary operation $* : P(X) \times P(X) \to P(X)$ given by $A * B = A \cap B \;\; \forall$ A, B in P(X), where P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation $*$.

10. Find the number of all onto functions from the set $\{1, 2, 3, ... , n\}$ to itself.

11. Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

(i) $F = \{(a, 3), (b, 2), (c, 1)\}$ (ii) $F = \{(a, 2), (b, 1), (c, 1)\}$

12. Consider the binary operations $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $o : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined as $a * b = |a - b|$ and $a \text{ o } b = a, \; \forall \, a, b \in \mathbf{R}$. Show that $*$ is commutative but not associative, o is associative but not commutative. Further, show that $\forall \, a, b, c \in \mathbf{R}$, $a * (b \text{ o } c) = (a * b) \text{ o } (a * c)$. [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

13. Given a non-empty set X, let $* : P(X) \times P(X) \to P(X)$ be defined as $A * B = (A - B) \cup (B - A), \; \forall \, A, B \in P(X)$. Show that the empty set ϕ is the identity for the operation $*$ and all the elements A of P(X) are invertible with $A^{-1} = A$. (Hint : $(A - \phi) \cup (\phi - A) = A$ and $(A - A) \cup (A - A) = A * A = \phi$).

14. Define a binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ as

$$a * b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a.

15. Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g : A \to B$ be functions defined by $f(x) = x^2 - x, \; x \in A$ and $g(x) = 2\left|x - \dfrac{1}{2}\right| - 1, \; x \in A$. Are f and g equal? Justify your answer. (Hint: One may note that two functions $f : A \to B$ and $g : A \to B$ such that $f(a) = g(a) \;\; \forall \, a \in A$, are called equal functions).

16. Let $A = \{1, 2, 3\}$. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

17. Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

18. Let $f : \mathbf{R} \to \mathbf{R}$ be the Signum Function defined as

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and $g : \mathrm{R} \to \mathrm{R}$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then, does fog and gof coincide in $(0, 1]$?

19. Number of binary operations on the set $\{a, b\}$ are

(A) 10 (B) 16 (C) 20 (D) 8

Summary

In this chapter, we studied different types of relations and equivalence relation, composition of functions, invertible functions and binary operations. The main features of this chapter are as follows:

- *Empty relation* is the relation R in X given by $\mathrm{R} = \phi \subset \mathrm{X} \times \mathrm{X}$.
- *Universal relation* is the relation R in X given by $\mathrm{R} = \mathrm{X} \times \mathrm{X}$.
- *Reflexive relation* R in X is a relation with $(a, a) \in \mathrm{R} \ \forall \ a \in \mathrm{X}$.
- *Symmetric relation* R in X is a relation satisfying $(a, b) \in \mathrm{R}$ implies $(b, a) \in \mathrm{R}$.
- *Transitive relation* R in X is a relation satisfying $(a, b) \in \mathrm{R}$ and $(b, c) \in \mathrm{R}$ implies that $(a, c) \in \mathrm{R}$.
- *Equivalence relation* R in X is a relation which is reflexive, symmetric and transitive.
- *Equivalence class* $[a]$ containing $a \in \mathrm{X}$ for an equivalence relation R in X is the subset of X containing all elements b related to a.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *one-one* (or *injective*) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in \mathrm{X}$.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *onto* (or *surjective*) if given any $y \in \mathrm{Y}, \exists \ x \in \mathrm{X}$ such that $f(x) = y$.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *one-one and onto* (or *bijective*), if f is both one-one and onto.
- The *composition* of functions $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ is the function $gof : \mathrm{A} \to \mathrm{C}$ given by $gof(x) = g(f(x)) \ \forall \ x \in \mathrm{A}$.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *invertible* if $\exists \ g : \mathrm{Y} \to \mathrm{X}$ such that $gof = \mathrm{I}_\mathrm{X}$ and $fog = \mathrm{I}_\mathrm{Y}$.
- A function $f : \mathrm{X} \to \mathrm{Y}$ is *invertible* if and only if f is one-one and onto.

- Given a finite set X, a function $f : X \to X$ is one-one (respectively onto) if and only if f is onto (respectively one-one). This is the characteristic property of a finite set. This is not true for infinite set
- A ***binary operation*** $*$ on a set A is a function $*$ from $A \times A$ to A.
- An element $e \in X$ is the ***identity*** element for binary operation $* : X \times X \to X$, if $a * e = a = e * a \ \forall \ a \in X$.
- An element $a \in X$ is ***invertible*** for binary operation $* : X \times X \to X$, if there exists $b \in X$ such that $a * b = e = b * a$ where, e is the identity for the binary operation $*$. The element b is called ***inverse*** of a and is denoted by a^{-1}.
- An operation $*$ on X is ***commutative*** if $a * b = b * a \ \forall \ a, b$ in X.
- An operation $*$ on X is ***associative*** if $(a * b) * c = a * (b * c) \ \forall \ a, b, c$ in X.

Historical Note

The concept of function has evolved over a long period of time starting from R. Descartes (1596-1650), who used the word 'function' in his manuscript "*Geometrie*" in 1637 to mean some positive integral power x^n of a variable x while studying geometrical curves like hyperbola, parabola and ellipse. James Gregory (1636-1675) in his work " *Vera Circuli et Hyperbolae Quadratura*" (1667) considered function as a quantity obtained from other quantities by successive use of algebraic operations or by any other operations. Later G. W. Leibnitz (1646-1716) in his manuscript "*Methodus tangentium inversa, seu de functionibus*" written in 1673 used the word 'function' to mean a quantity varying from point to point on a curve such as the coordinates of a point on the curve, the slope of the curve, the tangent and the normal to the curve at a point. However, in his manuscript "*Historia*" (1714), Leibnitz used the word 'function' to mean quantities that depend on a variable. He was the first to use the phrase 'function of x'. John Bernoulli (1667-1748) used the notation ϕx for the first time in 1718 to indicate a function of x. But the general adoption of symbols like f, F, ϕ, ψ ... to represent functions was made by Leonhard Euler (1707-1783) in 1734 in the first part of his manuscript "*Analysis Infinitorium*". Later on, Joseph Louis Lagrange (1736-1813) published his manuscripts "*Theorie des functions analytiques*" in 1793, where he discussed about analytic function and used the notion $f(x)$, $F(x)$, $\phi(x)$ etc. for different function of x. Subsequently, Lejeunne Dirichlet (1805-1859) gave the definition of function which was being used till the set theoretic definition of function presently used, was given after set theory was developed by Georg Cantor (1845-1918). The set theoretic definition of function known to us presently is simply an abstraction of the definition given by Dirichlet in a rigorous manner.

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Chapter 2

INVERSE TRIGONOMETRIC FUNCTIONS

❖ *Mathematics, in general, is fundamentally the science of self-evident things.* — *FELIX KLEIN* ❖

2.1 Introduction

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In Chapter 1, we have studied that the inverse of a function f, denoted by f^{-1}, exists if f is one-one and onto. There are many functions which are not one-one, onto or both and hence we can not talk of their inverses. In Class XI, we studied that trigonometric functions are not one-one and onto over their natural domains and ranges and hence their inverses do not exist. In this chapter, we shall study about the restrictions on domains and ranges of trigonometric functions which ensure the existence of their inverses and observe their behaviour through graphical representations. Besides, some elementary properties will also be discussed.

The inverse trigonometric functions play an important role in calculus for they serve to define many integrals. The concepts of inverse trigonometric functions is also used in science and engineering.

2.2 Basic Concepts

In Class XI, we have studied trigonometric functions, which are defined as follows:

sine function, i.e., sine : $\mathbf{R} \rightarrow [-1, 1]$

cosine function, i.e., cos : $\mathbf{R} \rightarrow [-1, 1]$

tangent function, i.e., tan : $\mathbf{R} - \{ x : x = (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\} \rightarrow \mathbf{R}$

cotangent function, i.e., cot : $\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z}\} \rightarrow \mathbf{R}$

secant function, i.e., sec : $\mathbf{R} - \{ x : x = (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\} \rightarrow \mathbf{R} - (-1, 1)$

cosecant function, i.e., cosec : $\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z}\} \rightarrow \mathbf{R} - (-1, 1)$

We have also learnt in Chapter 1 that if f : X→Y such that $f(x) = y$ is one-one and onto, then we can define a unique function g : Y→X such that $g(y) = x$, where $x \in$ X and $y = f(x)$, $y \in$ Y. Here, the domain of g = range of f and the range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}. Further, g is also one-one and onto and inverse of g is f. Thus, $g^{-1} = (f^{-1})^{-1} = f$. We also have

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

and $$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$$

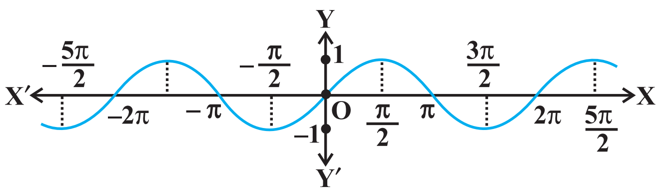
Since the domain of sine function is the set of all real numbers and range is the closed interval [–1, 1]. If we restrict its domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then it becomes one-one and onto with range [– 1, 1]. Actually, sine function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc., is one-one and its range is [–1, 1]. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is [– 1, 1] and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and so on. Corresponding to each such interval, we get a *branch* of the function $\sin^{-1}$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal value branch, whereas other intervals as range give different branches of* $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is [–1, 1] and range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. We write $\sin^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

From the definition of the inverse functions, it follows that $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. In other words, if $y = \sin^{-1} x$, then $\sin y = x$.

Remarks

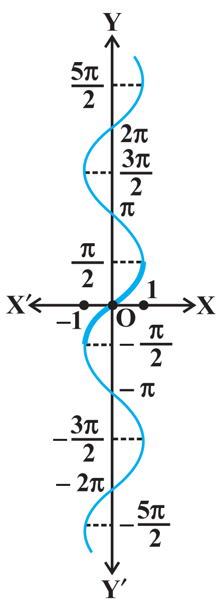
(i) We know from Chapter 1, that if $y = f(x)$ is an invertible function, then $x = f^{-1}(y)$. Thus, the graph of $\sin^{-1}$ function can be obtained from the graph of original function by interchanging x and y axes, i.e., if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse of sine function. Thus, the graph of the function $y = \sin^{-1} x$ can be obtained from the graph of $y = \sin x$ by interchanging x and y axes. The graphs of $y = \sin x$ and $y = \sin^{-1} x$ are as given in Fig 2.1 (i), (ii), (iii). The dark portion of the graph of $y = \sin^{-1} x$ represent the principal value branch.

(ii) It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$. This can be visualised by looking the graphs of $y = \sin x$ and $y = \sin^{-1} x$ as given in the same axes (Fig 2.1 (iii)).



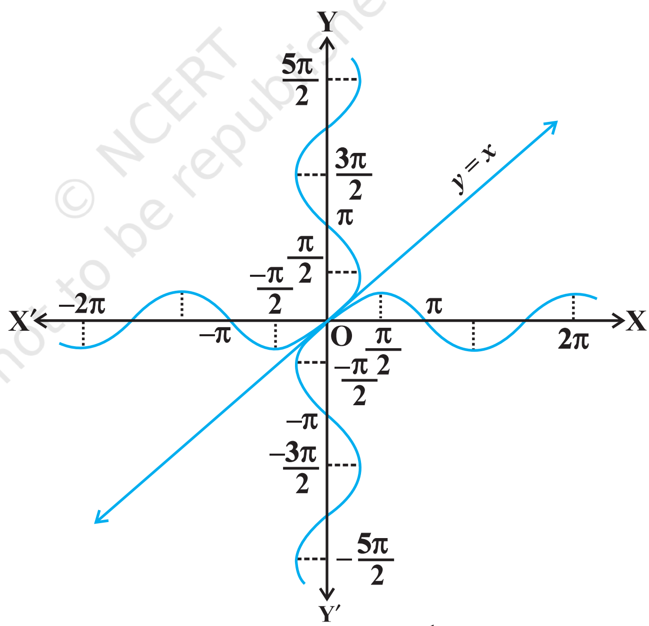
$y = \sin x$

Fig 2.1 (i)



$y = \sin^{-1} x$

Fig 2.1 (ii)



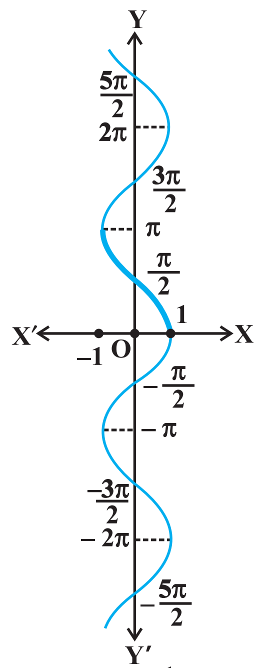
$y = \sin x$ and $y = \sin^{-1} x$

Fig 2.1 (iii)

Like sine function, the cosine function is a function whose domain is the set of all real numbers and range is the set [–1, 1]. If we restrict the domain of cosine function to [0, π], then it becomes one-one and onto with range [–1, 1]. Actually, cosine function restricted to any of the intervals $[-\pi, 0]$, $[0,\pi]$, $[\pi, 2\pi]$ etc., is bijective with range as $[-1, 1]$. We can, therefore, define the inverse of cosine function in each of these intervals. We denote the inverse of the cosine function by $\cos^{-1}$ (arc cosine function). Thus, $\cos^{-1}$ is a function whose domain is $[-1, 1]$ and range could be any of the intervals $[-\pi, 0]$, $[0, \pi]$, $[\pi, 2\pi]$ etc. Corresponding to each such interval, we get a branch of the function $\cos^{-1}$. The branch with range $[0, \pi]$ is called the *principal value branch* of the function $\cos^{-1}$. We write

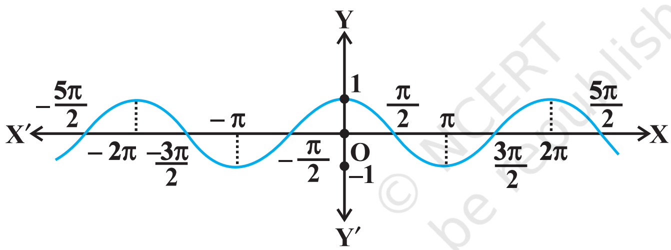
$$\cos^{-1} : [-1, 1] \to [0, \pi].$$

The graph of the function given by $y = \cos^{-1} x$ can be drawn in the same way as discussed about the graph of $y = \sin^{-1} x$. The graphs of $y = \cos x$ and $y = \cos^{-1} x$ are given in Fig 2.2 (i) and (ii).



$y = \cos^{-1} x$

Fig 2.2 (ii)



$y = \cos x$

Fig 2.2 (i)

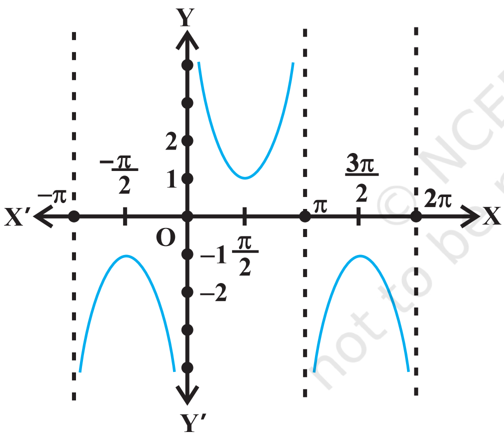
Let us now discuss $\text{cosec}^{-1}x$ and $\sec^{-1}x$ as follows:

Since, $\text{cosec } x = \dfrac{1}{\sin x}$, the domain of the cosec function is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and the range is the set $\{y : y \in \mathbf{R}, y \geq 1$ or $y \leq -1\}$ i.e., the set $\mathbf{R} - (-1, 1)$. It means that $y = \text{cosec } x$ assumes all real values except $-1 < y < 1$ and is not defined for integral multiple of π. If we restrict the domain of cosec function to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$, then it is one to one and onto with its range as the set $\mathbf{R} - (-1, 1)$. Actually, cosec function restricted to any of the intervals $\left[\dfrac{-3\pi}{2}, \dfrac{-\pi}{2}\right] - \{-\pi\}$, $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$, $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] - \{\pi\}$ etc., is bijective and its range is the set of all real numbers $\mathbf{R} - (-1, 1)$.

Thus cosec^{-1} can be defined as a function whose domain is $\mathbf{R} - (-1, 1)$ and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{-\pi\}$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$ etc. The function corresponding to the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the *principal value branch* of cosec^{-1}. We thus have principal branch as

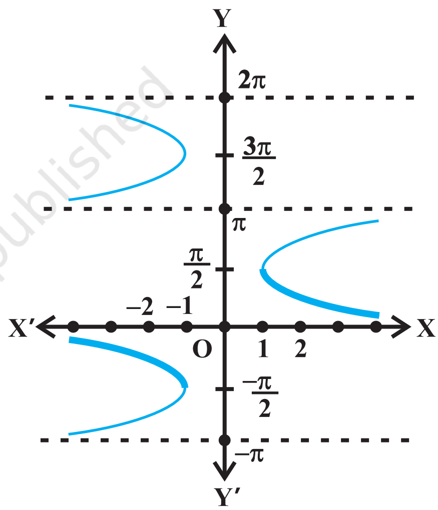
$$\text{cosec}^{-1} : \mathbf{R} - (-1, 1) \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

The graphs of $y = \text{cosec } x$ and $y = \text{cosec}^{-1} x$ are given in Fig 2.3 (i), (ii).



$y = \text{cosec } x$

Fig 2.3 (i)



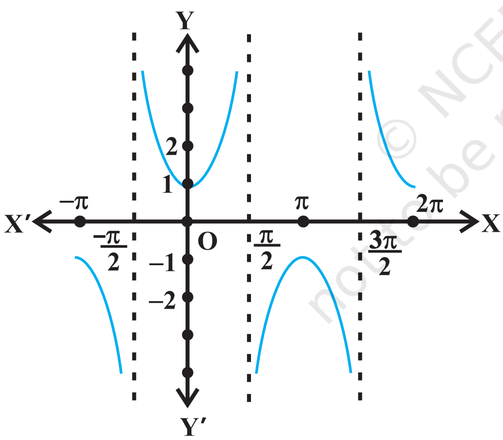
$y = \text{cosec}^{-1} x$

Fig 2.3 (ii)

Also, since $\sec x = \frac{1}{\cos x}$, the domain of $y = \sec x$ is the set $\mathbf{R} - \{x : x = (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set $\mathbf{R} - (-1, 1)$. It means that sec (secant function) assumes all real values except $-1 < y < 1$ and is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of secant function to $[0, \pi] - \{\frac{\pi}{2}\}$, then it is one-one and onto with its range as the set **R** – (–1, 1). Actually, secant function restricted to any of the intervals $[-\pi, 0] - \{\frac{-\pi}{2}\}$, $[0, \pi] - \left\{\frac{\pi}{2}\right\}$, $[\pi, 2\pi] - \{\frac{3\pi}{2}\}$ etc., is bijective and its range is $\mathbf{R} - \{-1, 1\}$. Thus $\sec^{-1}$ can be defined as a function whose domain is $\mathbf{R} - (-1, 1)$ and range could be any of the intervals $[-\pi, 0] - \{\frac{-\pi}{2}\}$, $[0, \pi] - \{\frac{\pi}{2}\}$, $[\pi, 2\pi] - \{\frac{3\pi}{2}\}$ etc. Corresponding to each of these intervals, we get different branches of the function $\sec^{-1}$. The branch with range $[0, \pi] - \{\frac{\pi}{2}\}$ is called the *principal value branch* of the function $\sec^{-1}$. We thus have

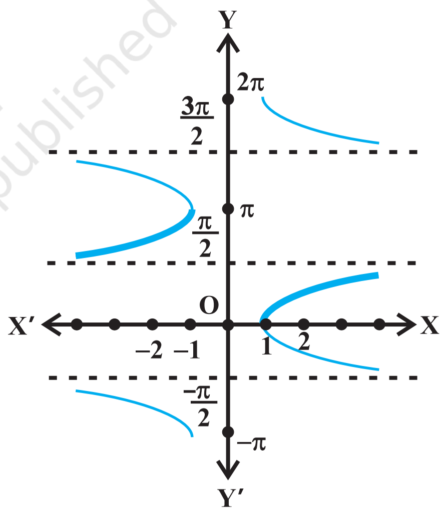
$$\sec^{-1} : \mathbf{R} - (-1,1) \to [0, \pi] - \{\frac{\pi}{2}\}$$

The graphs of the functions $y = \sec x$ and $y = \sec^{-1} x$ are given in Fig 2.4 (i), (ii).



$y = \sec x$

Fig 2.4 (i)



$y = \sec^{-1}x$

Fig 2.4 (ii)

Finally, we now discuss $\tan^{-1}$ and $\cot^{-1}$

We know that the domain of the tan function (tangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq (2n + 1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and the range is **R**. It means that tan function is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of tangent function to

$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then it is one-one and onto with its range as **R**. Actually, tangent function restricted to any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ etc., is bijective and its range is **R**. Thus $\tan^{-1}$ can be defined as a function whose domain is **R** and range could be any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and so on. These intervals give different branches of the function $\tan^{-1}$. The branch with range $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is called the *principal value branch* of the function $\tan^{-1}$.

We thus have

$$\tan^{-1} : \mathbf{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of the function $y = \tan x$ and $y = \tan^{-1}x$ are given in Fig 2.5 (i), (ii).

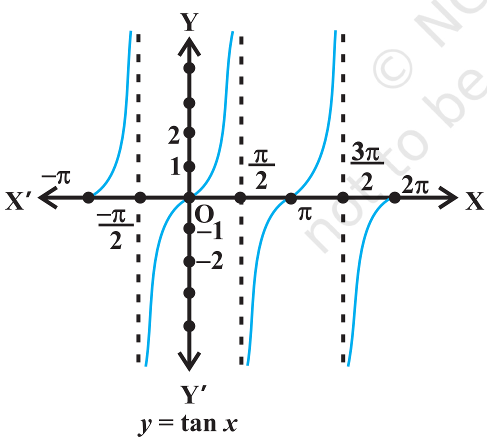


Fig 2.5 (i)

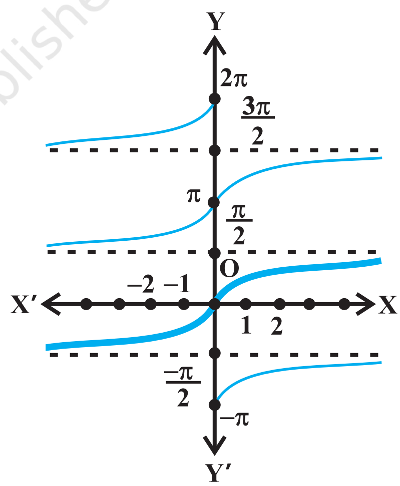


Fig 2.5 (ii)

We know that domain of the cot function (cotangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq n\pi, n \in \mathbf{Z}\}$ and range is **R**. It means that cotangent function is not defined for integral multiples of π. If we restrict the domain of cotangent function to (0, π), then it is bijective with and its range as **R**. In fact, cotangent function restricted to any of the intervals (–π, 0), (0, π), (π, 2π) etc., is bijective and its range is **R**. Thus $\cot^{-1}$ can be defined as a function whose domain is the **R** and range as any of the intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$ etc. These intervals give different branches of the function $\cot^{-1}$. The function with range $(0, \pi)$ is called the *principal value branch* of the function $\cot^{-1}$. We thus have

$$\cot^{-1} : \mathbf{R} \to (0, \pi)$$

The graphs of $y = \cot x$ and $y = \cot^{-1}x$ are given in Fig 2.6 (i), (ii).

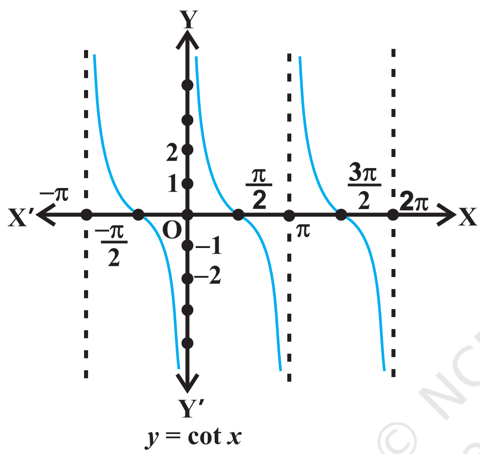


Fig 2.6 (i)

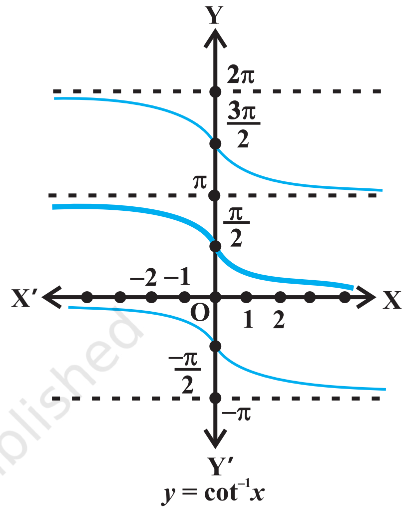


Fig 2.6 (ii)

The following table gives the inverse trigonometric function (principal value branches) along with their domains and ranges.

$\sin^{-1}$	:	$[-1, 1]$	$\to$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	:	$[-1, 1]$	$\to$	$[0, \pi]$
cosec^{-1}	:	$\mathbf{R} - (-1,1)$	$\to$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	:	$\mathbf{R} - (-1, 1)$	$\to$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\tan^{-1}$	:	$\mathbf{R}$	$\to$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	:	$\mathbf{R}$	$\to$	$(0, \pi)$

☛ Note

1. $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.

We now consider some examples:

Example 1 Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Solution Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$. Then, $\sin y = \frac{1}{\sqrt{2}}$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. Therefore, principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$

Example 2 Find the principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution Let $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$. Then,

$$\cot y = \frac{-1}{\sqrt{3}} = -\cot\left(\frac{\pi}{3}\right) = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\left(\frac{2\pi}{3}\right)$$

We know that the range of principal value branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\left(\frac{2\pi}{3}\right) = \frac{-1}{\sqrt{3}}$. Hence, principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$

EXERCISE 2.1

Find the principal values of the following:

1. $\sin^{-1}\left(-\frac{1}{2}\right)$ **2.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **3.** $\operatorname{cosec}^{-1}(2)$

4. $\tan^{-1}(-\sqrt{3})$ **5.** $\cos^{-1}\left(-\frac{1}{2}\right)$ **6.** $\tan^{-1}(-1)$

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **8.** $\cot^{-1}(\sqrt{3})$ **9.** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

10. $\text{cosec}^{-1}(-\sqrt{2})$

Find the values of the following:

11. $\tan^{-1}(1) + \cos^{-1} \ -\frac{1}{2} + \sin^{-1} \ -\frac{1}{2}$ **12.** $\cos^{-1} \ \frac{1}{2} + 2\sin^{-1} \ \frac{1}{2}$

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π (B) $-\frac{\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

2.3 Properties of Inverse Trigonometric Functions

In this section, we shall prove some important properties of inverse trigonometric functions. It may be mentioned here that these results are valid within the principal value branches of the corresponding inverse trigonometric functions and wherever they are defined. Some results may not be valid for all values of the domains of inverse trigonometric functions. In fact, they will be valid only for some values of x for which inverse trigonometric functions are defined. We will not go into the details of these values of x in the domain as this discussion goes beyond the scope of this text book.

Let us recall that if $y = \sin^{-1}x$, then $x = \sin y$ and if $x = \sin y$, then $y = \sin^{-1}x$. This is equivalent to

$$\sin(\sin^{-1} x) = x,\ x \in [-1, 1] \text{ and } \sin^{-1}(\sin x) = x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Same is true for other five inverse trigonometric functions as well. We now prove some properties of inverse trigonometric functions.

1. (i) $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1} x,\ x \ge 1 \text{ or } x \le -1$

(ii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x,\ x \ge 1 \text{ or } x \le -1$

(iii) $\tan^{-1}\frac{1}{x} = \cot^{-1} x,\ x > 0$

To prove the first result, we put $\text{cosec}^{-1} x = y$, i.e., $x = \text{cosec}\ y$

Therefore $\frac{1}{x} = \sin y$

Hence $\sin^{-1}\frac{1}{x} = y$

or $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1} x$

Similarly, we can prove the other parts.

2. **(i)** $\sin^{-1}(-x) = -\sin^{-1} x,\ x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x,\ x \in \mathbf{R}$

(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x,\ |x| \geq 1$

Let $\sin^{-1}(-x) = y$, i.e., $-x = \sin y$ so that $x = -\sin y$, i.e., $x = \sin(-y)$.

Hence $\sin^{-1} x = -y = -\sin^{-1}(-x)$

Therefore $\sin^{-1}(-x) = -\sin^{-1} x$

Similarly, we can prove the other parts.

3. **(i)** $\cos^{-1}(-x) = \pi - \cos^{-1} x,\ x \in [-1, 1]$

(ii) $\sec^{-1}(-x) = \pi - \sec^{-1} x,\ |x| \geq 1$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1} x,\ x \in \mathbf{R}$

Let $\cos^{-1}(-x) = y$ i.e., $-x = \cos y$ so that $x = -\cos y = \cos(\pi - y)$

Therefore $\cos^{-1} x = \pi - y = \pi - \cos^{-1}(-x)$

Hence $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Similarly, we can prove the other parts.

4. **(i)** $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},\ x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ x \in \mathbf{R}$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2},\ |x| \geq 1$

Let $\sin^{-1} x = y$. Then $x = \sin y = \cos\left(\frac{\pi}{2} - y\right)$

Therefore $\cos^{-1} x = \frac{\pi}{2} - y = \frac{\pi}{2} - \sin^{-1} x$

Hence $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

Similarly, we can prove the other parts.

5. (i) $\tan^{-1}x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy},\ xy < 1$

(ii) $\tan^{-1}x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy},\ xy > -1$

(iii) $\tan^{-1}x + \tan^{-1} y = \pi + \tan^{-1} \left(\dfrac{x+y}{1-xy}\right),\ xy > 1;\ x, y > 0$

Let $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$. Then $x = \tan \theta$, $y = \tan \phi$

Now $\tan(\theta+\phi) = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = \dfrac{x+y}{1-xy}$

This gives $\theta + \phi = \tan^{-1}\dfrac{x+y}{1-xy}$

Hence $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$

In the above result, if we replace y by $-y$, we get the second result and by replacing y by x, we get the third result as given below.

6. (i) $2\tan^{-1} x = \sin^{-1} \dfrac{2x}{1+x^2},\ |x| \le 1$

(ii) $2\tan^{-1} x = \cos^{-1} \dfrac{1-x^2}{1+x^2},\ x \ge 0$

(iii) $2\tan^{-1} x = \tan^{-1} \dfrac{2x}{1-x^2},\ -1 < x < 1$

Let $\tan^{-1} x = y$, then $x = \tan y$. Now

$$\sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{2\tan y}{1+\tan^2 y}$$
$$= \sin^{-1}(\sin 2y) = 2y = 2\tan^{-1} x$$

Also $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \cos^{-1}\dfrac{1-\tan^2 y}{1+\tan^2 y} = \cos^{-1}(\cos 2y) = 2y = 2\tan^{-1} x$

(iii) Can be worked out similarly.

We now consider some examples.

Example 3 Show that

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x,\ -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x,\ \dfrac{1}{\sqrt{2}} \le x \le 1$

Solution

(i) Let $x = \sin\theta$. Then $\sin^{-1} x = \theta$. We have

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$= 2\sin^{-1} x$$

(ii) Take $x = \cos\theta$, then proceeding as above, we get, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x$

Example 4 Show that $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{2}{11} = \tan^{-1}\dfrac{3}{4}$

Solution By property 5 (i), we have

$$\text{L.H.S.} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}} = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4} = \text{R.H.S.}$$

Example 5 Express $\tan^{-1}\dfrac{\cos x}{1-\sin x},\ \dfrac{-3\pi}{2} < x < \dfrac{\pi}{2}$ in the simplest form.

Solution We write

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right] = \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Alternatively,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right] = \tan^{-1}\left[\frac{\sin\left(\frac{\pi-2x}{2}\right)}{1-\cos\left(\frac{\pi-2x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi-2x}{4}\right)\cos\left(\frac{\pi-2x}{4}\right)}{2\sin^2\left(\frac{\pi-2x}{4}\right)}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{\pi-2x}{4}\right)\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\pi-2x}{4}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Example 6 Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$, $x > 1$ in the simplest form.

Solution Let $x = \sec\theta$, then $\sqrt{x^2-1} = \sqrt{\sec^2\theta-1} = \tan\theta$

Therefore, $\cot^{-1}\frac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1} x$, which is the simplest form.

Example 7 Prove that $\tan^{-1} x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \ |x| < \frac{1}{\sqrt{3}}$

Solution Let $x = \tan\theta$. Then $\theta = \tan^{-1} x$. We have

$$\text{R.H.S.} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1} x = \tan^{-1} x + 2\tan^{-1} x$$

$$= \tan^{-1} x + \tan^{-1}\frac{2x}{1-x^2} = \text{L.H.S. (Why?)}$$

Example 8 Find the value of $\cos(\sec^{-1} x + \text{cosec}^{-1} x)$, $|x| \geq 1$

Solution We have $\cos(\sec^{-1} x + \text{cosec}^{-1} x) = \cos\left(\frac{\pi}{2}\right) = 0$

EXERCISE 2.2

Prove the following:

1. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3), \ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

2. $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x), \ x \in \left[\frac{1}{2}, 1\right]$

3. $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

4. $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Write the following functions in the simplest form:

5. $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, \ x \neq 0$

6. $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, \ |x| > 1$

7. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), \ 0 < x < \pi$

8. $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), \ \frac{-\pi}{4} < x < \frac{3\pi}{4}$

9. $\tan^{-1}\dfrac{x}{\sqrt{a^2-x^2}}$, $|x| < a$

10. $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)$, $a > 0$; $\dfrac{-a}{\sqrt{3}} < x < \dfrac{a}{\sqrt{3}}$

Find the values of each of the following:

11. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right]$

12. $\cot(\tan^{-1}a + \cot^{-1}a)$

13. $\tan\dfrac{1}{2}\left[\sin^{-1}\dfrac{2x}{1+x^2}+\cos^{-1}\dfrac{1-y^2}{1+y^2}\right]$, $|x| < 1$, $y > 0$ and $xy < 1$

14. If $\sin\left(\sin^{-1}\dfrac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x

15. If $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}$, then find the value of x

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$

17. $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

18. $\tan\left(\sin^{-1}\dfrac{3}{5}+\cot^{-1}\dfrac{3}{2}\right)$

19. $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$ is equal to

(A) $\dfrac{7\pi}{6}$ (B) $\dfrac{5\pi}{6}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{6}$

20. $\sin\left(\dfrac{\pi}{3}-\sin^{-1}(-\dfrac{1}{2})\right)$ is equal to

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{4}$ (D) 1

21. $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$ is equal to

(A) π (B) $-\dfrac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Miscellaneous Examples

Example 9 Find the value of $\sin^{-1}(\sin\frac{3\pi}{5})$

Solution We know that $\sin^{-1}(\sin x) = x$. Therefore, $\sin^{-1}(\sin\frac{3\pi}{5}) = \frac{3\pi}{5}$

But $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal branch of $\sin^{-1} x$

However $\sin(\frac{3\pi}{5}) = \sin(\pi - \frac{3\pi}{5}) = \sin\frac{2\pi}{5}$ and $\frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore $\sin^{-1}(\sin\frac{3\pi}{5}) = \sin^{-1}(\sin\frac{2\pi}{5}) = \frac{2\pi}{5}$

Example 10 Show that $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Solution Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$

Therefore $\sin x = \frac{3}{5}$ and $\sin y = \frac{8}{17}$

Now $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$ (Why?)

and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{64}{289}} = \frac{15}{17}$

We have $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$$= \frac{4}{5}\times\frac{15}{17} + \frac{3}{5}\times\frac{8}{17} = \frac{84}{85}$$

Therefore $x - y = \cos^{-1}\frac{84}{85}$

Hence $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Example 11 Show that $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

Solution Let $\sin^{-1}\frac{12}{13} = x, \quad \cos^{-1}\frac{4}{5} = y, \tan^{-1}\frac{63}{16} = z$

Then $\qquad \sin x = \frac{12}{13}, \quad \cos y = \frac{4}{5}, \quad \tan z = \frac{63}{16}$

Therefore $\qquad \cos x = \frac{5}{13}, \ \sin y = \frac{3}{5}, \ \tan x = \frac{12}{5}$ and $\tan y = \frac{3}{4}$

We have $\qquad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{12}{5} \times \frac{3}{4}} = -\frac{63}{16}$

Hence $\qquad \tan(x+y) = -\tan z$

i.e., $\qquad \tan(x + y) = \tan(-z)$ or $\tan(x + y) = \tan(\pi - z)$

Therefore $\qquad x + y = -z$ or $x + y = \pi - z$

Since $\qquad x, y$ and z are positive, $x + y \neq -z$ (Why?)

Hence $\qquad x + y + z = \pi$ or $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

Example 12 Simplify $\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$, if $\frac{a}{b}\tan x > -1$

Solution We have,

$$\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right] = \tan^{-1}\left[\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x + a\sin x}{b\cos x}}\right] = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$$

$$= \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x) = \tan^{-1}\frac{a}{b} - x$$

Example 13 Solve $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

Solution We have $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

or $$\tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right) = \frac{\pi}{4}$$

i.e. $$\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$$

Therefore $$\frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$$

or $$6x^2 + 5x - 1 = 0 \text{ i.e., } (6x - 1)(x + 1) = 0$$

which gives $$x = \frac{1}{6} \text{ or } x = -1.$$

Since $x = -1$ does not satisfy the equation, as the L.H.S. of the equation becomes negative, $x=\frac{1}{6}$ is the only solution of the given equation.

Miscellaneous Exercise on Chapter 2

Find the value of the following:

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Prove that

3. $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

4. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

5. $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

6. $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

7. $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

8. $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Prove that

9. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, $x \in [0, 1]$

10. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2},\ x \in \left(0, \frac{\pi}{4}\right)$

11. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}} \le x \le 1$ [Hint: Put $x = \cos 2\theta$]

12. $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solve the following equations:

13. $2\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

14. $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x > 0)$

15. $\sin(\tan^{-1}x), |x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

16. $\sin^{-1}(1 - x) - 2\sin^{-1}x = \frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

17. $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{3\pi}{4}$

Summary

- The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

Functions	**Domain**	**Range (Principal Value Branches)**
$y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \text{cosec}^{-1} x$	$\mathbf{R} - (-1,1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$\mathbf{R} - (-1, 1)$	$[0, \pi] - \{\frac{\pi}{2}\}$
$y = \tan^{-1} x$	$\mathbf{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	$\mathbf{R}$	$(0, \pi)$

- $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
- The value of an inverse trigonometric functions which lies in its principal value branch is called the *principal value* of that inverse trigonometric functions.

For suitable values of domain, we have

- $y = \sin^{-1} x \Rightarrow x = \sin y$
- $x = \sin y \Rightarrow y = \sin^{-1} x$
- $\sin (\sin^{-1} x) = x$
- $\sin^{-1} (\sin x) = x$
- $\sin^{-1} \frac{1}{x} = \text{cosec}^{-1} x$
- $\cos^{-1} (-x) = \pi - \cos^{-1} x$
- $\cos^{-1} \frac{1}{x} = \sec^{-1}x$
- $\cot^{-1} (-x) = \pi - \cot^{-1} x$
- $\tan^{-1} \frac{1}{x} = \cot^{-1} x$
- $\sec^{-1} (-x) = \pi - \sec^{-1} x$

- $\sin^{-1}(-x) = -\sin^{-1}x$
- $\tan^{-1}(-x) = -\tan^{-1}x$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
- $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$
- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$
- $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$
- $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$
- $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$
- $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy > 1;\ x, y > 0$

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhata (476A.D.), Brahmagupta (598 A.D.), Bhaskara I (600 A.D.) and Bhaskara II (1114 A.D.) got important results of trigonometry. All this knowledge went from India to Arabia and then from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents one of the main contribution of the *siddhantas* (Sanskrit astronomical works) to mathematics.

Bhaskara I (about 600 A.D.) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., were given by Bhaskara II.

The symbols $\sin^{-1}x$, $\cos^{-1}x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The name of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\text{H}}{\text{S}} = \frac{h}{s} = \tan \text{(sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

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Chapter 3

MATRICES

❖ *The essence of Mathematics lies in its freedom. — CANTOR* ❖

3.1 Introduction

The knowledge of matrices is necessary in various branches of mathematics. Matrices are one of the most powerful tools in mathematics. This mathematical tool simplifies our work to a great extent when compared with other straight forward methods. The evolution of concept of matrices is the result of an attempt to obtain compact and simple methods of solving system of linear equations. Matrices are not only used as a representation of the coefficients in system of linear equations, but utility of matrices far exceeds that use. Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science like budgeting, sales projection, cost estimation, analysing the results of an experiment etc. Also, many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. Matrices are also used in cryptography. This mathematical tool is not only used in certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management.

In this chapter, we shall find it interesting to become acquainted with the fundamentals of matrix and matrix algebra.

3.2 Matrix

Suppose we wish to express the information that Radha has 15 notebooks. We may express it as [15] with the understanding that the number inside [ ] is the number of notebooks that Radha has. Now, if we have to express that Radha has 15 notebooks and 6 pens. We may express it as [15 6] with the understanding that first number inside [ ] is the number of notebooks while the other one is the number of pens possessed by Radha. Let us now suppose that we wish to express the information of possession of notebooks and pens by Radha and her two friends Fauzia and Simran which is as follows:

Radha	has	15	notebooks	and	6 pens,
Fauzia	has	10	notebooks	and	2 pens,
Simran	has	13	notebooks	and	5 pens.

Now this could be arranged in the tabular form as follows:

	Notebooks	**Pens**
Radha	15	6
Fauzia	10	2
Simran	13	5

and this can be expressed as

$$\begin{array}{cl} \begin{bmatrix} 15 & 6 \\ 10 & 2 \\ 13 & 5 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \\ \leftarrow \text{Third row} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} \end{array} & \end{array}$$

or

	Radha	**Fauzia**	**Simran**
Notebooks	15	10	13
Pens	6	2	5

which can be expressed as:

$$\begin{array}{cl} \begin{bmatrix} 15 & 10 & 13 \\ 6 & 2 & 5 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} & \text{Third Column} \end{array} & \end{array}$$

In the first arrangement the entries in the first column represent the number of note books possessed by Radha, Fauzia and Simran, respectively and the entries in the second column represent the number of pens possessed by Radha, Fauzia and Simran, respectively. Similarly, in the second arrangement, the entries in the first row represent the number of notebooks possessed by Radha, Fauzia and Simran, respectively. The entries in the second row represent the number of pens possessed by Radha, Fauzia and Simran, respectively. An arrangement or display of the above kind is called a *matrix*. Formally, we define matrix as:

Definition 1 A *matrix* is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

We denote matrices by capital letters. The following are some examples of matrices:

$$\mathrm{A}=\begin{bmatrix}-2 & 5\\ 0 & \sqrt{5}\\ 3 & 6\end{bmatrix},\ \mathrm{B}=\begin{bmatrix}2+i & 3 & -\frac{1}{2}\\ 3.5 & -1 & 2\\ \sqrt{3} & 5 & \frac{5}{7}\end{bmatrix},\ \mathrm{C}=\begin{bmatrix}1+x & x^3 & 3\\ \cos x & \sin x+2 & \tan x\end{bmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute, ***rows*** of the matrix and the vertical lines of elements are said to constitute, ***columns*** of the matrix. Thus A has 3 rows and 2 columns, B has 3 rows and 3 columns while C has 2 rows and 3 columns.

3.2.1 *Order of a matrix*

A matrix having m rows and n columns is called a matrix of *order* $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). So referring to the above examples of matrices, we have A as 3×2 matrix, B as 3×3 matrix and C as 2×3 matrix. We observe that A has $3 \times 2 = 6$ elements, B and C have 9 and 6 elements, respectively.

In general, an $m \times n$ matrix has the following rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots & a_{1j} & \ldots & a_{1n}\\ a_{21} & a_{22} & a_{23} \ldots & a_{2j} & \ldots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ a_{i1} & a_{i2} & a_{i3} \ldots & a_{ij} & \ldots & a_{in}\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & a_{m3} \ldots & a_{mj} & \ldots & a_{mn}\end{bmatrix}_{m \times n}$$

or $\mathrm{A} = [a_{ij}]_{m \times n}$, $1 \le i \le m$, $1 \le j \le n$ $i, j \in \mathrm{N}$

Thus the i^{th} row consists of the elements $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}$, while the j^{th} column consists of the elements $a_{1j}, a_{2j}, a_{3j}, \ldots, a_{mj}$,

In general a_{ij}, is an element lying in the i^{th} row and j^{th} column. We can also call it as the $(i, j)^{\text{th}}$ element of A. The number of elements in an $m \times n$ matrix will be equal to mn.

Note In this chapter

1. We shall follow the notation, namely $A = [a_{ij}]_{m \times n}$ to indicate that A is a matrix of order $m \times n$.
2. We shall consider only those matrices whose elements are real numbers or functions taking real values.

We can also represent any point (x, y) in a plane by a matrix (column or row) as $\begin{bmatrix} x \\ y \end{bmatrix}$ (or $[x, y]$). For example point P(0, 1) as a matrix representation may be given as

$$P = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } [0 \ 1].$$

Observe that in this way we can also express the vertices of a closed rectilinear figure in the form of a matrix. For example, consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (–1, 2).

Now, quadrilateral ABCD in the matrix form, can be represented as

$$X = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2 \times 4} \end{array} \quad \text{or} \quad Y = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4 \times 2}$$

Thus, matrices can be used as representation of vertices of geometrical figures in a plane.

Now, let us consider some examples.

Example 1 Consider the following information regarding the number of men and women workers in three factories I, II and III

	Men workers	**Women workers**
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent?

Solution The information is represented in the form of a 3 × 2 matrix as follows:

$$A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

Example 2 If a matrix has 8 elements, what are the possible orders it can have?

Solution We know that if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1 × 8, 8 ×1, 4 × 2, 2 × 4

Example 3 Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i-3j|$.

Solution In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Now $\qquad a_{ij} = \frac{1}{2}|i-3j|,\ i = 1, 2, 3$ and $j = 1, 2.$

Therefore

$$a_{11} = \frac{1}{2}|1-3\times1| = 1 \qquad a_{12} = \frac{1}{2}|1-3\times2| = \frac{5}{2}$$

$$a_{21} = \frac{1}{2}|2-3\times1| = \frac{1}{2} \qquad a_{22} = \frac{1}{2}|2-3\times2| = 2$$

$$a_{31} = \frac{1}{2}|3-3\times1| = 0 \qquad a_{32} = \frac{1}{2}|3-3\times2| = \frac{3}{2}$$

Hence the required matrix is given by $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$.

3.3 Types of Matrices

In this section, we shall discuss different types of matrices.

(i) **Column matrix**

A matrix is said to be a *column matrix* if it has only one column.

For example, $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$ is a column matrix of order 4 × 1.

In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(ii) **Row matrix**

A matrix is said to be a *row matrix* if it has only one row.

For example, $B = \begin{bmatrix} -\frac{1}{2} & \sqrt{5} & 2 & 3 \end{bmatrix}_{1\times 4}$ is a row matrix.

In general, $B = [b_{ij}]_{1 \times n}$ is a row matrix of order $1 \times n$.

(iii) **Square matrix**

A matrix in which the number of rows are equal to the number of columns, is said to be a *square matrix*. Thus an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'.

For example $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m.

Note If $A = [a_{ij}]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, ..., a_{nn}$ are said to constitute the *diagonal*, of the matrix A. Thus, if $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 4 & -1 \\ 3 & 5 & 6 \end{bmatrix}$.

Then the elements of the diagonal of A are 1, 4, 6.

(iv) **Diagonal matrix**

A square matrix B = $[b_{ij}]_{m \times m}$ is said to be a *diagonal matrix* if all its non diagonal elements are zero, that is a matrix B = $[b_{ij}]_{m \times m}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.

For example, $A = [4]$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, are diagonal matrices of order 1, 2, 3, respectively.

(v) **Scalar matrix**

A diagonal matrix is said to be a *scalar matrix* if its diagonal elements are equal, that is, a square matrix B = $[b_{ij}]_{n \times n}$ is said to be a scalar matrix if

$$b_{ij} = 0, \quad \text{when } i \neq j$$
$$b_{ij} = k, \quad \text{when } i = j, \text{ for some constant } k.$$

For example

$A = [3]$, $\quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\quad C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$

are scalar matrices of order 1, 2 and 3, respectively.

(vi) **Identity matrix**

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an *identity matrix*. In other words, the square matrix A = $[a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We denote the identity matrix of order n by I_n. When order is clear from the context, we simply write it as I.

For example [1], $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3, respectively.

Observe that a scalar matrix is an identity matrix when $k = 1$. But every identity matrix is clearly a scalar matrix.

(vii) **Zero matrix**

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero.

For example, [0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, [0, 0] are all zero matrices. We denote zero matrix by O. Its order will be clear from the context.

3.3.1 *Equality of matrices*

Definition 2 Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if

(i) they are of the same order

(ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices. Symbolically, if two matrices A and B are equal, we write A = B.

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$, then $x = -1.5$, $y = 0$, $z = 2$, $a = \sqrt{6}$, $b = 3$, $c = 2$

Example 4 If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

Find the values of a, b, c, x, y and z.

Solution As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$$x + 3 = 0, \qquad z + 4 = 6, \qquad 2y - 7 = 3y - 2$$

$$a - 1 = -3, \qquad 0 = 2c + 2 \qquad b - 3 = 2b + 4,$$

Simplifying, we get

$$a = -2,\ b = -7,\ c = -1,\ x = -3,\ y = -5,\ z = 2$$

Example 5 Find the values of a, b, c, and d from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Solution By equality of two matrices, equating the corresponding elements, we get

$$2a + b = 4 \qquad 5c - d = 11$$
$$a - 2b = -3 \qquad 4c + 3d = 24$$

Solving these equations, we get

$$a = 1, b = 2, c = 3 \text{ and } d = 4$$

EXERCISE 3.1

1. In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:

(i) The order of the matrix, (ii) The number of elements,

(iii) Write the elements a_{13}, a_{21}, a_{33}, a_{24}, a_{23}.

2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

4. Construct a 2×2 matrix, $A = [a_{ij}]$, whose elements are given by:

(i) $a_{ij} = \dfrac{(i+j)^2}{2}$ (ii) $a_{ij} = \dfrac{i}{j}$ (iii) $a_{ij} = \dfrac{(i+2j)^2}{2}$

5. Construct a 3×4 matrix, whose elements are given by:

(i) $a_{ij} = \dfrac{1}{2}|-3i + j|$ (ii) $a_{ij} = 2i - j$

6. Find the values of x, y and z from the following equations:

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

7. Find the value of a, b, c and d from the equation:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

8. $A = [a_{ij}]_{m \times n}$ is a square matrix, if

(A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these

9. Which of the given values of x and y make the following pair of matrices equal

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

(A) $x = \frac{-1}{3},\ y = 7$ (B) Not possible to find

(C) $y = 7,\ x = \frac{-2}{3}$ (D) $x = \frac{-1}{3},\ y = \frac{-2}{3}$

10. The number of all possible matrices of order 3×3 with each entry 0 or 1 is:

(A) 27 (B) 18 (C) 81 (D) 512

3.4 Operations on Matrices

In this section, we shall introduce certain operations on matrices, namely, addition of matrices, multiplication of a matrix by a scalar, difference and multiplication of matrices.

3.4.1 *Addition of matrices*

Suppose Fatima has two factories at places A and B. Each factory produces sport shoes for boys and girls in three different price categories labelled 1, 2 and 3. The quantities produced by each factory are represented as matrices given below:

$$\begin{array}{c} \text{Factory at A} \\ \begin{array}{cc} & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array} \end{array} \qquad \begin{array}{c} \text{Factory at B} \\ \begin{array}{cc} & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 90 & 50 \\ 70 & 55 \\ 75 & 75 \end{bmatrix} \end{array} \end{array}$$

Suppose Fatima wants to know the total production of sport shoes in each price category. Then the total production

In category 1 : for boys (80 + 90), for girls (60 + 50)

In category 2 : for boys (75 + 70), for girls (65 + 55)

In category 3 : for boys (90 + 75), for girls (85 + 75)

This can be represented in the matrix form as $\begin{bmatrix} 80+90 & 60+50 \\ 75+70 & 65+55 \\ 90+75 & 85+75 \end{bmatrix}$.

This new matrix is the **sum** of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is a 2×3 matrix and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ is another 2×3 matrix. Then, we define $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$.

In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$. Then, the sum of the two matrices A and B is *defined* as a matrix $C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$, for all possible values of i and j.

Example 6 Given $A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{bmatrix}$, find A + B

Since A, B are of the same order 2 × 3. Therefore, addition of A and B is defined and is given by

$$A + B = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 1-1 \\ 2-2 & 3+3 & 0+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 0 \\ 0 & 6 & \frac{1}{2} \end{bmatrix}$$

Note

1. We emphasise that if A and B are not of the same order, then A + B is not defined. For example if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then $A + B$ is not defined.
2. We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.

3.4.2 *Multiplication of a matrix by a scalar*

Now suppose that Fatima has doubled the production at a factory A in all categories (refer to 3.4.1).

Previously quantities (in standard units) produced by factory A were

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array}$$

Revised quantities produced by factory A are as given below:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{bmatrix} 2\times 80 & 2\times 60 \\ 2\times 75 & 2\times 65 \\ 2\times 90 & 2\times 85 \end{bmatrix} \end{array}$$

This can be represented in the matrix form as $\begin{bmatrix} 160 & 120 \\ 150 & 130 \\ 180 & 170 \end{bmatrix}$. We observe that the new matrix is obtained by multiplying each element of the previous matrix by 2.

In general, we may define *multiplication of a matrix* by a scalar as follows: if $A = [a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

In other words, $kA = k\,[a_{ij}]_{m \times n} = [k\,(a_{ij})]_{m \times n}$, that is, $(i, j)^{\text{th}}$ element of kA is ka_{ij} for all possible values of i and j.

For example, if $\quad A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then

$$3A = 3\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

Negative of a matrix The negative of a matrix is denoted by $-A$. We define $-A = (-1)\,A$.

For example, let $A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}$, then – A is given by

$$-A = (-1)A = (-1)\begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

Difference of matrices If $A = [a_{ij}]$, $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$, then difference A – B is defined as a matrix $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$, for all value of i and j. In other words, D = A – B = A + (–1) B, that is sum of the matrix A and the matrix – B.

Example 7 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find 2A – B.

Solution We have

$$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

3.4.3 *Properties of matrix addition*

The addition of matrices satisfy the following properties:

(i) **Commutative Law** If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, then A + B = B + A.

Now $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ (addition of numbers is commutative)

$= ([b_{ij}] + [a_{ij}]) = B + A$

(ii) **Associative Law** For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$, (A + B) + C = A + (B + C).

Now $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij} + (b_{ij} + c_{ij})]$ (Why?)

$= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C)$

(iii) **Existence of additive identity** Let A = $[a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

(iv) **The existence of additive inverse** Let A = $[a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as – A = $[-a_{ij}]_{m \times n}$ such that A + (– A) = (– A) + A= O. So – A is the additive inverse of A or negative of A.

3.4.4 *Properties of scalar multiplication of a matrix*

If A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order, say $m \times n$, and k and l are scalars, then

(i) $k(A + B) = kA + kB$, (ii) $(k + l)A = kA + lA$

(ii) $$k(A + B) = k([a_{ij}] + [b_{ij}])$$
$$= k[a_{ij} + b_{ij}] = [k(a_{ij} + b_{ij})] = [(k\,a_{ij}) + (k\,b_{ij})]$$
$$= [k\,a_{ij}] + [k\,b_{ij}] = k[a_{ij}] + k[b_{ij}] = kA + kB$$

(iii) $$(k + l)A = (k + l)[a_{ij}]$$
$$= [(k + l)a_{ij}] + [k\,a_{ij}] + [l\,a_{ij}] = k[a_{ij}] + l[a_{ij}] = kA + lA$$

Example 8 If $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B.

Solution We have 2A + 3X = 5B

or $\quad 2A + 3X - 2A = 5B - 2A$

or $\quad 2A - 2A + 3X = 5B - 2A$ (Matrix addition is commutative)

or $\quad O + 3X = 5B - 2A$ (– 2A is the additive inverse of 2A)

or $\quad 3X = 5B - 2A$ (O is the additive identity)

or $$X = \frac{1}{3}(5B - 2A)$$

or $$X = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Example 9 Find X and Y, if $X+Y=\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X-Y=\begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution We have $(X+Y)+(X-Y)=\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}+\begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

or $\qquad (X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$

or $\qquad X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Also $\qquad (X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or $\qquad (X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$

or $\qquad Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Example 10 Find the values of x and y from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Solution We have

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $\begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

or $2x + 3 = 7$ and $2y - 4 = 14$ (Why?)

or $2x = 7 - 3$ and $2y = 18$

or $x = \frac{4}{2}$ and $y = \frac{18}{2}$

i.e. $x = 2$ and $y = 9$.

Example 11 Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in Rupees)

$$\text{A} = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

October Sales (in Rupees)

$$\text{B} = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(i) Find the combined sales in September and October for each farmer in each variety.

(ii) Find the decrease in sales from September to October.

(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

Solution

(i) Combined sales in September and October for each farmer in each variety is given by

$$\text{A} + \text{B} = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 15,000 & 30,000 & 36,000 \\ 70,000 & 40,000 & 20,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(ii) Change in sales from September to October is given by

$$A - B = \begin{array}{c} \quad\; \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 5000 & 10{,}000 & 24{,}000 \\ 30{,}000 & 20{,}000 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(iii) 2% of B $= \dfrac{2}{100} \times B = 0.02 \times B$

$$= 0.02 \begin{array}{c} \quad\; \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 5000 & 10{,}000 & 6000 \\ 20{,}000 & 10{,}000 & 10{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

$$= \begin{array}{c} \quad\; \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 100 & 200 & 120 \\ 400 & 200 & 200 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

Thus, in October Ramkishan receives ₹ 100, ₹ 200 and ₹ 120 as profit in the sale of each variety of rice, respectively, and Grucharan Singh receives profit of ₹ 400, ₹ 200 and ₹ 200 in the sale of each variety of rice, respectively.

3.4.5 *Multiplication of matrices*

Suppose Meera and Nadeem are two friends. Meera wants to buy 2 pens and 5 story books, while Nadeem needs 8 pens and 10 story books. They both go to a shop to enquire about the rates which are quoted as follows:

Pen – ₹ 5 each, story book – ₹ 50 each.

How much money does each need to spend? Clearly, Meera needs ₹ $(5 \times 2 + 50 \times 5)$ that is ₹ 260, while Nadeem needs $(8 \times 5 + 50 \times 10)$ ₹, that is ₹ 540. In terms of matrix representation, we can write the above information as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 50 \end{bmatrix} \qquad \begin{bmatrix} 5 \times 2 + 5 \times 50 \\ 8 \times 5 + 10 \times 50 \end{bmatrix} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}$$

Suppose that they enquire about the rates from another shop, quoted as follows:

pen – ₹ 4 each, story book – ₹ 40 each.

Now, the money required by Meera and Nadeem to make purchases will be respectively ₹ $(4 \times 2 + 40 \times 5) =$ ₹ 208 and ₹ $(8 \times 4 + 10 \times 40) =$ ₹ 432

Again, the above information can be represented as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 40 \end{bmatrix} \qquad \begin{bmatrix} 4\times 2 + 40\times 5 \\ 8\times 4 + 10\times 40 \end{bmatrix} = \begin{bmatrix} 208 \\ 432 \end{bmatrix}$$

Now, the information in both the cases can be combined and expressed in terms of matrices as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 & 4 \\ 50 & 40 \end{bmatrix} \qquad \begin{bmatrix} 5\times 2 + 5\times 50 & 4\times 2 + 40\times 5 \\ 8\times 5 + 10\times 50 & 8\times 4 + 10\times 40 \end{bmatrix}$$

$$= \begin{bmatrix} 260 & 208 \\ 540 & 432 \end{bmatrix}$$

The above is an example of multiplication of matrices. We observe that, for multiplication of two matrices A and B, the number of columns in A should be equal to the number of rows in B. Furthermore for getting the elements of the product matrix, we take rows of A and columns of B, multiply them element-wise and take the sum. Formally, we define multiplication of matrices as follows:

The *product* of two matrices A and B is *defined* if the number of columns of A is equal to the number of rows of B. Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$. To get the $(i, k)^{th}$ element c_{ik} of the matrix C, we take the i^{th} row of A and k^{th} column of B, multiply them elementwise and take the sum of all these products. In other words, if $A = [a_{ij}]_{m \times n}$, $B = [b_{jk}]_{n \times p}$, then the i^{th} row of A is $[a_{i1}\ a_{i2}\ \ldots\ a_{in}]$ and the k^{th} column of

B is $\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$, then $c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + a_{i3}\, b_{3k} + \ldots + a_{in}\, b_{nk} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$.

The matrix $C = [c_{ik}]_{m \times p}$ is the product of A and B.

For example, if $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$, then the product CD is defined and is given by $CD = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$. This is a 2×2 matrix in which each entry is the sum of the products across some row of C with the corresponding entries down some column of D. These four computations are

Entry in first row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} (1)(2)+(-1)(-1)+(2)(5) & ? \\ ? & ? \end{bmatrix}$

Entry in first row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & (1)(7)+(-1)(1)+2(-4) \\ ? & ? \end{bmatrix}$

Entry in second row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 0(2)+3(-1)+4(5) & ? \end{bmatrix}$

Entry in second row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 17 & 0(7)+3(1)+4(-4) \end{bmatrix}$

Thus $CD = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}$

Example 12 Find AB, if $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$.

Solution The matrix A has 2 columns which is equal to the number of rows of B. Hence AB is defined. Now

$$AB = \begin{bmatrix} 6(2)+9(7) & 6(6)+9(9) & 6(0)+9(8) \\ 2(2)+3(7) & 2(6)+3(9) & 2(0)+3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 12+63 & 36+81 & 0+72 \\ 4+21 & 12+27 & 0+24 \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Remark If AB is defined, then BA need not be defined. In the above example, AB is defined but BA is not defined because B has 3 column while A has only 2 (and not 3) rows. If A, B are, respectively $m \times n$, $k \times l$ matrices, then both AB and BA are defined **if and only if** $n = k$ and $l = m$. In particular, if both A and B are square matrices of the same order, then both AB and BA are defined.

Non-commutativity of multiplication of matrices

Now, we shall see by an example that even if AB and BA are both defined, it is not necessary that AB = BA.

Example 13 If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB, BA. Show that AB $\neq$ BA.

Solution Since A is a 2×3 matrix and B is 3×2 matrix. Hence AB and BA are both defined and are matrices of order 2×2 and 3×3, respectively. Note that

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

and $$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly AB $\neq$ BA

In the above example both AB and BA are of different order and so AB $\neq$ BA. But one may think that perhaps AB and BA could be the same if they were of the same order. But it is not so, here we give an example to show that even if AB and BA are of same order they may not be same.

Example 14 If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

and $$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ Clearly AB $\neq$ BA.

Thus matrix multiplication is not commutative.

Note This does not mean that AB ≠ BA for every pair of matrices A, B for which AB and BA, are defined. For instance,

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, then AB = BA = $\begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$

Observe that multiplication of diagonal matrices of same order will be commutative.

Zero matrix as the product of two non zero matrices

We know that, for real numbers a, b if $ab = 0$, then either $a = 0$ or $b = 0$. This need not be true for matrices, we will observe this through an example.

Example 15 Find AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

Solution We have $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

3.4.6 *Properties of multiplication of matrices*

The multiplication of matrices possesses the following properties, which we state without proof.

1. **The associative law** For any three matrices A, B and C. We have (AB) C = A (BC), whenever both sides of the equality are defined.
2. **The distributive law** For three matrices A, B and C.
   (i) A (B+C) = AB + AC
   (ii) (A+B) C = AC + BC, whenever both sides of equality are defined.
3. **The existence of multiplicative identity** For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

Now, we shall verify these properties by examples.

Example 16 If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, find A(BC), (AB)C and show that (AB)C = A(BC).

Solution We have $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 3+2-4 \\ 2+0-3 & 6+0+12 \\ 3+0-2 & 9-2+8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$

$$(AB)\ (C) = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+0 & 6-2 & -8+1 \\ -1+36 & -2+0 & -3-36 & 4+18 \\ 1+30 & 2+0 & 3-30 & -4+15 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

Now $\quad BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ -1+8 & -2+0 & -3-8 & 4+4 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

Therefore $\quad A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 7+4-7 & 2+0+2 & -3-4+11 & -1+2-8 \\ 14+0+21 & 4+0-6 & -6+0-33 & -2+0+24 \\ 21-4+14 & 6+0-4 & -9+4-22 & -3-2+16 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}. \text{ Clearly, } (AB)\,C = A\,(BC)$$

Example 17 If $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

Calculate AC, BC and (A + B)C. Also, verify that (A + B)C = AC + BC

Solution Now, $A + B = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix}$

So $$(A + B)\,C = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-14+24 \\ -10+0+30 \\ 16+12+0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

Further $$AC = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-12+21 \\ -12+0+24 \\ 14+16+0 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}$$

and $$BC = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-2+3 \\ 2+0+6 \\ 2-4+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}$$

So $$AC + BC = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

Clearly, $(A + B)\,C = AC + BC$

Example 18 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40\,I = O$

Solution We have $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

So $$A^3 = A\,A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Now

$$A^3 - 23A - 40I = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Example 19 In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as

$$A = \begin{array}{c} \textit{Cost per contact} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{array}{c} \begin{array}{ccc} \text{Telephone} & \text{Housecall} & \text{Letter} \end{array} \\ \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \to X \\ \to Y \end{array}$$. Find the total amount spent by the group in the two cities X and Y.

Solution We have

$$BA = \begin{bmatrix} 40{,}000 + 50{,}000 + 250{,}000 \\ 120{,}000 + 100{,}000 + 500{,}000 \end{bmatrix} \begin{matrix} \rightarrow X \\ \rightarrow Y \end{matrix}$$

$$= \begin{bmatrix} 340{,}000 \\ 720{,}000 \end{bmatrix} \begin{matrix} \rightarrow X \\ \rightarrow Y \end{matrix}$$

So the total amount spent by the group in the two cities is 340,000 paise and 720,000 paise, i.e., ₹3400 and ₹7200, respectively.

EXERCISE 3.2

1. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find each of the following:

(i) A + B (ii) A – B (iii) 3A – C

(iv) AB (v) BA

2. Compute the following:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

3. Compute the indicated products.

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, then compute (A+B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

5. If $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$, then compute 3A – 5B.

6. Simplify $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

7. Find X and Y, if

(i) $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$

8. Find X, if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

9. Find x and y, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

10. Solve the equation for x, y, z and t, if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

11. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y.

12. Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of x, y, z and w.

13. If $F(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$, show that F($x$) F($y$) = F($x$ + y).

14. Show that

(i) $\begin{bmatrix}5 & -1\\ 6 & 7\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix} \neq \begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & -1\\ 6 & 7\end{bmatrix}$

(ii) $\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 2 & 3 & 4\end{bmatrix} \neq \begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 2 & 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}$

15. Find $A^2 - 5A + 6I$, if $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$

16. If $A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$

17. If $A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, find k so that $A^2 = kA - 2I$

18. If $A=\begin{bmatrix}0 & -\tan\dfrac{\alpha}{2}\\ \tan\dfrac{\alpha}{2} & 0\end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

19. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

(a) ₹ 1800 (b) ₹ 2000

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹ 80, ₹ 60 and ₹ 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. Choose the correct answer in Exercises 21 and 22.

21. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$

(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

22. If $n = p$, then the order of the matrix 7X – 5Z is:

(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

3.5. Transpose of a Matrix

In this section, we shall learn about transpose of a matrix and special types of matrices such as symmetric and skew symmetric matrices.

Definition 3 If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose of the matrix A is denoted by A' or (A^T). In other words, if $A = [a_{ij}]_{m \times n}$, then $A' = [a_{ji}]_{n \times m}$. For example,

$$\text{if } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3 \times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2 \times 3}$$

3.5.1 Properties of transpose of the matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any matrices A and B of suitable orders, we have

(i) $(A')' = A$, (ii) $(kA)' = kA'$ (where k is any constant)

(iii) $(A + B)' = A' + B'$ (iv) $(A\,B)' = B'\,A'$

Example 20 If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, verify that

(i) $(A')' = A$, (ii) $(A + B)' = A' + B'$,

(iii) $(kB)' = kB'$, where k is any constant.

Solution

(i) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow (A')' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A$$

Thus $(A')' = A$

(ii) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 5 & \sqrt{3}-1 & 4 \\ 5 & 4 & 4 \end{bmatrix}$$

Therefore $$(A + B)' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Now $$A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}, \; B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix},$$

So $$A' + B' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Thus $(A + B)' = A' + B'$

(iii) We have

$$kB = k\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}$$

Then $$(kB)' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k\begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'$$

Thus $(kB)' = kB'$

Example 21 If $A=\begin{bmatrix}-2\\4\\5\end{bmatrix}, B=\begin{bmatrix}1 & 3 & -6\end{bmatrix}$, verify that $(AB)' = B'A'$.

Solution We have

$$A = \begin{bmatrix}-2\\4\\5\end{bmatrix}, B=\begin{bmatrix}1 & 3 & -6\end{bmatrix}$$

then
$$AB = \begin{bmatrix}-2\\4\\5\end{bmatrix}\begin{bmatrix}1 & 3 & -6\end{bmatrix} = \begin{bmatrix}-2 & -6 & 12\\4 & 12 & -24\\5 & 15 & -30\end{bmatrix}$$

Now
$$A' = [-2\ 4\ 5]\ ,\ B'=\begin{bmatrix}1\\3\\-6\end{bmatrix}$$

$$B'A' = \begin{bmatrix}1\\3\\-6\end{bmatrix}\begin{bmatrix}-2 & 4 & 5\end{bmatrix}=\begin{bmatrix}-2 & 4 & 5\\-6 & 12 & 15\\12 & -24 & -30\end{bmatrix}=(AB)'$$

Clearly $(AB)' = B'A'$

3.6 Symmetric and Skew Symmetric Matrices

Definition 4 A square matrix $A = [a_{ij}]$ is said to be *symmetric* if $A' = A$, that is, $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

For example $A=\begin{bmatrix}\sqrt{3} & 2 & 3\\2 & -1.5 & -1\\3 & -1 & 1\end{bmatrix}$ is a symmetric matrix as $A' = A$

Definition 5 A square matrix $A = [a_{ij}]$ is said to be *skew symmetric* matrix if $A' = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j. Now, if we put $i = j$, we have $a_{ii} = -a_{ii}$. Therefore $2a_{ii} = 0$ or $a_{ii} = 0$ for all i's.

This means that all the diagonal elements of a skew symmetric matrix are zero.

For example, the matrix $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$ is a skew symmetric matrix as B′= –B

Now, we are going to prove some results of symmetric and skew-symmetric matrices.

Theorem 1 For any square matrix A with real number entries, A + A′ is a symmetric matrix and A – A′ is a skew symmetric matrix.

Proof Let B = A + A′, then

$$
\begin{aligned}
B' &= (A + A')' \\
&= A' + (A')' \quad (\text{as } (A + B)' = A' + B') \\
&= A' + A \quad (\text{as } (A')' = A) \\
&= A + A' \quad (\text{as } A + B = B + A) \\
&= B
\end{aligned}
$$

Therefore B = A + A′ is a symmetric matrix

Now let C = A – A′

$$
\begin{aligned}
C' &= (A - A')' = A' - (A')' \quad \text{(Why?)} \\
&= A' - A \quad \text{(Why?)} \\
&= -(A - A') = -C
\end{aligned}
$$

Therefore C = A – A′ is a skew symmetric matrix.

Theorem 2 Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Proof Let A be a square matrix, then we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

From the Theorem 1, we know that (A + A′) is a symmetric matrix and (A – A′) is a skew symmetric matrix. Since for any matrix A, $(kA)' = kA'$, it follows that $\frac{1}{2}(A + A')$ is symmetric matrix and $\frac{1}{2}(A - A')$ is skew symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Example 22 Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution Here

$$B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

Let
$$P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix},$$

Now
$$P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, let
$$Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then
$$Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q$$

Thus $\quad Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now
$$P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

EXERCISE 3.3

1. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

2. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$, (ii) $(A - B)' = A' - B'$

3. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$ (ii) $(A - B)' = A' - B'$

4. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A + 2B)'$

5. For the matrices A and B, verify that $(AB)' = B'A'$, where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

6. If (i) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that $A'A = I$

(ii) If $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, then verify that $A'A = I$

7. (i) Show that the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix.

(ii) Show that the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix.

8. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that

(i) $(A + A')$ is a symmetric matrix

(ii) $(A - A')$ is a skew symmetric matrix

9. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

10. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$

Choose the correct answer in the Exercises 11 and 12.

11. If A, B are symmetric matrices of same order, then AB – BA is a

(A) Skew symmetric matrix (B) Symmetric matrix

(C) Zero matrix (D) Identity matrix

12. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, and $A + A' = I$, then the value of α is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) π (D) $\frac{3\pi}{2}$

3.7 Elementary Operation (Transformation) of a Matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as *elementary operations* or *transformations*.

(i) *The interchange of any two rows or two columns*. Symbolically the interchange of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

For example, applying $R_1 \leftrightarrow R_2$ to $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \end{bmatrix}$, we get $\begin{bmatrix} -1 & \sqrt{3} & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 7 \end{bmatrix}$.

(ii) *The multiplication of the elements of any row or column by a non zero number*. Symbolically, the multiplication of each element of the i^{th} row by k, where $k \neq 0$ is denoted by $R_i \to kR_i$.

The corresponding column operation is denoted by $C_i \to kC_i$

For example, applying $C_3 \to \frac{1}{7}C_3$, to $B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 & \frac{1}{7} \\ -1 & \sqrt{3} & \frac{1}{7} \end{bmatrix}$

(iii) *The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.*

Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by k is denoted by $R_i \to R_i + kR_j$.

The corresponding column operation is denoted by $C_i \to C_i + kC_j$.

For example, applying $R_2 \to R_2 - 2R_1$, to $C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$.

3.8 Invertible Matrices

Definition 6 If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the *inverse* matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

For example, let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

Now $$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

Note

1. A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.
2. If B is the inverse of A, then A is also the inverse of B.

Theorem 3 (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.

Proof Let $A = [a_{ij}]$ be a square matrix of order m. If possible, let B and C be two inverses of A. We shall show that B = C.

Since B is the inverse of A

$$AB = BA = I \qquad \text{... (1)}$$

Since C is also the inverse of A

$$AC = CA = I \qquad \text{... (2)}$$

Thus $$B = BI = B\,(AC) = (BA)\,C = IC = C$$

Theorem 4 If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof From the definition of inverse of a matrix, we have

$$(AB)\,(AB)^{-1} = 1$$

or $\quad A^{-1}\,(AB)\,(AB)^{-1} = A^{-1}I \quad$ (Pre multiplying both sides by A^{-1})

or $\quad (A^{-1}A)\,B\,(AB)^{-1} = A^{-1} \quad$ (Since $A^{-1}\,I = A^{-1}$)

or $\quad IB\,(AB)^{-1} = A^{-1}$

or $\quad B\,(AB)^{-1} = A^{-1}$

or $\quad B^{-1}\,B\,(AB)^{-1} = B^{-1}\,A^{-1}$

or $\quad I\,(AB)^{-1} = B^{-1}\,A^{-1}$

Hence $\quad (AB)^{-1} = B^{-1}\,A^{-1}$

3.8.1 *Inverse of a matrix by elementary operations*

Let X, A and B be matrices of, the same order such that X = AB. In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the first matrix A of the product AB on RHS.

Similarly, in order to apply a sequence of elementary column operations on the matrix equation X = AB, we will apply, these operations simultaneously on X and on the second matrix B of the product AB on RHS.

In view of the above discussion, we conclude that if A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations, write A = IA and apply a sequence of row operation on A = IA till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then, write A = AI and apply a sequence of column operations on A = AI till we get, I = AB.

Remark In case, after applying one or more elementary row (column) operations on A = IA (A = AI), if we obtain all zeros in one or more rows of the matrix A on L.H.S., then A^{-1} does not exist.

Example 23 By using elementary operations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Solution In order to use elementary row operations we may write A = IA.

or $\quad \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$, then $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$ (applying $R_2 \to R_2 - 2R_1$)

or $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} A$ (applying $R_2 \rightarrow -\frac{1}{5} R_2$)

or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} A$ (applying $R_1 \rightarrow R_1 - 2R_2$)

Thus $A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$

Alternatively, in order to use elementary column operations, we write A = AI, i.e.,

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying $C_2 \rightarrow C_2 - 2C_1$, we get

$$\begin{bmatrix} 1 & 0 \\ 2 & -5 \end{bmatrix} = A \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Now applying $C_2 \rightarrow -\frac{1}{5} C_2$, we have

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & \frac{-1}{5} \end{bmatrix}$$

Finally, applying $C_1 \rightarrow C_1 - 2C_2$, we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$

Example 24 Obtain the inverse of the following matrix using elementary operations $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution Write A = I A, i.e., $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \text{ (applying } R_1 \leftrightarrow R_2)$$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \text{ (applying } R_3 \to R_3 - 3R_1)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \text{ (applying } R_1 \to R_1 - 2R_2)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \text{ (applying } R_3 \to R_3 + 5R_2)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \text{ (applying } R_3 \to \frac{1}{2} R_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \text{ (applying } R_1 \to R_1 + R_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{applying } R_2 \to R_2 - 2R_3)$$

Hence $$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Alternatively, write A = AI, i.e.,

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (C_1 \leftrightarrow C_2)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad (C_3 \to C_3 - 2C_1)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad (C_3 \to C_3 + C_2)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (C_3 \to \frac{1}{2} C_3)$$

$$\text{or} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (C_1 \to C_1 - 2C_2)$$

$$\text{or} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -4 & 0 & -1 \\ \frac{5}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad (C_1 \to C_1 + 5C_3)$$

$$\text{or} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} \quad (C_2 \to C_2 - 3C_3)$$

$$\text{Hence} \qquad A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Example 25 Find P^{-1}, if it exists, given $P = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$.

Solution We have P = I P, i.e., $\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P$.

$$\text{or} \qquad \begin{bmatrix} 1 & \frac{-1}{5} \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix} P \quad \left(\text{applying } R_1 \to \frac{1}{10} R_1\right)$$

or $$\begin{bmatrix} 1 & \frac{-1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \text{P} \quad (\text{applying } R_2 \to R_2 + 5R_1)$$

We have all zeros in the second row of the left hand side matrix of the above equation. Therefore, P^{-1} does not exist.

EXERCISE 3.4

Using elementary transformations, find the inverse of each of the matrices, if it exists in Exercises 1 to 17.

1. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ **8.** $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$ **11.** $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$ **12.** $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ **14.** $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. **15.** $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$ **17.** $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

18. Matrices A and B will be inverse of each other only if

(A) AB = BA (B) AB = BA = 0

(C) AB = 0, BA = I (D) AB = BA = I

Miscellaneous Examples

Example 26 If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$.

Solution We shall prove the result by using principle of mathematical induction.

We have $\quad P(n) : \text{If } A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$

$$P(1) : A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ so } A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k) : A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k + 1$

Now $\quad A^{k+1} = A \cdot A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & \cos\theta\sin k\theta + \sin\theta\cos k\theta \\ -\sin\theta\cos k\theta + \cos\theta\sin k\theta & -\sin\theta\sin k\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$. Thus by principle of mathematical induction, we have $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, holds for all natural numbers.

Example 27 If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is AB = BA.

Solution Since A and B are both symmetric matrices, therefore $A' = A$ and $B' = B$.

Let $\quad$ AB be symmetric, then $(AB)' = AB$

But $\quad (AB)' = B'A' = BA$ (Why?)

Therefore $\quad BA = AB$

Conversely, if AB = BA, then we shall show that AB is symmetric.

Now
$$(AB)' = B'A'$$
$$= B\,A \text{ (as A and B are symmetric)}$$
$$= AB$$

Hence AB is symmetric.

Example 28 Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that CD – AB = O.

Solution Since A, B, C are all square matrices of order 2, and CD – AB is well defined, D must be a square matrix of order 2.

Let $\quad D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then CD – AB = 0 gives

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

or
$$\begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or
$$\begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$2a + 5c - 3 = 0 \quad \text{... (1)}$$
$$3a + 8c - 43 = 0 \quad \text{... (2)}$$
$$2b + 5d = 0 \quad \text{... (3)}$$
and
$$3b + 8d - 22 = 0 \quad \text{... (4)}$$

Solving (1) and (2), we get $a = -191$, $c = 77$. Solving (3) and (4), we get $b = -110$, $d = 44$.

Therefore
$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

Miscellaneous Exercise on Chapter 3

1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1} bA$, where I is the identity matrix of order 2 and $n \in \mathbf{N}$.

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$, $n \in \mathbf{N}$.

3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n is any positive integer.

4. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

5. Show that the matrix $B'AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.

6. Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A'A = I$.

7. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

8. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

9. Find x, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

10. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:

Market	**Products**		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x, y and z are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectively. Find the gross profit.

11. Find the matrix X so that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

12. If A and B are square matrices of the same order such that AB = BA, then prove by induction that $AB^n = B^nA$. Further, prove that $(AB)^n = A^nB^n$ for all $n \in$ N.

Choose the correct answer in the following questions:

13. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2 = I$, then

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$

14. If the matrix A is both symmetric and skew symmetric, then

(A) A is a diagonal matrix (B) A is a zero matrix

(C) A is a square matrix (D) None of these

15. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(A) A (B) I – A (C) I (D) 3A

Summary

- A matrix is an ordered rectangular array of numbers or functions.
- A matrix having m rows and n columns is called a matrix of order $m \times n$.
- $[a_{ij}]_{m \times 1}$ is a column matrix.
- $[a_{ij}]_{1 \times n}$ is a row matrix.
- An $m \times n$ matrix is a square matrix if $m = n$.
- $A = [a_{ij}]_{m \times m}$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

- $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = 0$, when $i \neq j$, $a_{ij} = k$, (k is some constant), when $i = j$.
- $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$, $a_{ij} = 0$, when $i \neq j$.
- A zero matrix has all its elements as zero.
- $A = [a_{ij}] = [b_{ij}] = B$ if (i) A and B are of same order, (ii) $a_{ij} = b_{ij}$ for all possible values of i and j.
- $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$
- $-A = (-1)A$
- $A - B = A + (-1) B$
- $A + B = B + A$
- $(A + B) + C = A + (B + C)$, where A, B and C are of same order.
- $k(A + B) = kA + kB$, where A and B are of same order, k is constant.
- $(k + l) A = kA + lA$, where k and l are constant.
- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$, then $AB = C = [c_{ik}]_{m \times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$
- (i) $A(BC) = (AB)C$, (ii) $A(B + C) = AB + AC$, (iii) $(A + B)C = AC + BC$
- If $A = [a_{ij}]_{m \times n}$, then A' or $A^T = [a_{ji}]_{n \times m}$
- (i) $(A')' = A$, (ii) $(kA)' = kA'$, (iii) $(A + B)' = A' + B'$, (iv) $(AB)' = B'A'$
- A is a symmetric matrix if $A' = A$.
- A is a skew symmetric matrix if $A' = -A$.
- Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- Elementary operations of a matrix are as follows:
  (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
  (ii) $R_i \to kR_i$ or $C_i \to kC_i$
  (iii) $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$
- If A and B are two square matrices such that $AB = BA = I$, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B.
- Inverse of a square matrix, if it exists, is unique.

Chapter 4

DETERMINANTS

❖ *All Mathematical truths are relative and conditional. — C.P. STEINMETZ* ❖

4.1 Introduction

P.S. Laplace
(1749-1827)

In the previous chapter, we have studied about matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Now, this system of equations has a unique solution or not, is determined by the number $a_1 b_2 - a_2 b_1$. (Recall that if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or, $a_1 b_2 - a_2 b_1 \neq 0$, then the system of linear equations has a unique solution). The number $a_1 b_2 - a_2 b_1$ which determines uniqueness of solution is associated with the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called the determinant of A or det A. Determinants have wide applications in Engineering, Science, Economics, Social Science, etc.

In this chapter, we shall study determinants up to order three only with real entries. Also, we will study various properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and solution of linear equations in two or three variables using inverse of a matrix.

4.2 Determinant

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

This may be thought of as a function which associates each square matrix with a unique number (real or complex). If M is the set of square matrices, K is the set of numbers (real or complex) and $f : M \to K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of A. It is also denoted by $|A|$ or det A or Δ.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

Remarks

(i) For matrix A, $|A|$ is read as determinant of A and not modulus of A.

(ii) Only square matrices have determinants.

4.2.1 *Determinant of a matrix of order one*

Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a

4.2.2 *Determinant of a matrix of order two*

Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ be a matrix of order 2×2,

then the determinant of A is defined as:

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 1 Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$.

Solution We have $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1) = 4 + 4 = 8.$

Example 2 Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

Solution We have

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x(x) - (x+1)(x-1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$$

4.2.3 *Determinant of a matrix of order 3 × 3*

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order

3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shown below.

Consider the determinant of square matrix A = $[a_{ij}]_{3 \times 3}$

i.e.,
$$|A| = \begin{vmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion along first Row (R_1)

Step 1 Multiply first element a_{11} of R_1 by $(-1)^{(1+1)}$ $[(-1)^{\text{sum of suffixes in } a_{11}}]$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) of | A | as a_{11} lies in R_1 and C_1,

i.e.,
$$(-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Step 2 Multiply 2nd element a_{12} of R_1 by $(-1)^{1+2}$ $[(-1)^{\text{sum of suffixes in } a_{12}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and 2nd column (C_2) of | A | as a_{12} lies in R_1 and C_2,

i.e.,
$$(-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Step 3 Multiply third element a_{13} of R_1 by $(-1)^{1+3}$ $[(-1)^{\text{sum of suffixes in } a_{13}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and third column (C_3) of | A | as a_{13} lies in R_1 and C_3,

i.e.,
$$(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Step 4 Now the expansion of determinant of A, that is, | A | written as sum of all three terms obtained in steps 1, 2 and 3 above is given by

$$\det A = |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or
$$|A| = a_{11} (a_{22}\, a_{33} - a_{32}\, a_{23}) - a_{12} (a_{21}\, a_{33} - a_{31}\, a_{23}) + a_{13} (a_{21}\, a_{32} - a_{31}\, a_{22})$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{32}\, a_{23} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{31}\, a_{23} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \quad \ldots (1)$$

Note We shall apply all four steps together.

Expansion along second row (R_2)

$$|\,A\,| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_2, we get

$$|\,A\,| = (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{2+3} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -\, a_{21}\,(a_{12}\, a_{33} - a_{32}\, a_{13}) + a_{22}\,(a_{11}\, a_{33} - a_{31}\, a_{13}) - a_{23}\,(a_{11}\, a_{32} - a_{31}\, a_{12})$$

$$|\,A\,| = -\, a_{21}\, a_{12}\, a_{33} + a_{21}\, a_{32}\, a_{13} + a_{22}\, a_{11}\, a_{33} - a_{22}\, a_{31}\, a_{13} - a_{23}\, a_{11}\, a_{32} + a_{23}\, a_{31}\, a_{12}$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \quad \ldots (2)$$

Expansion along first Column (C_1)

$$|\,A\,| = \begin{vmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & a_{22} & a_{23} \\ \boldsymbol{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

By expanding along C_1, we get

$$|\,A\,| = a_{11}\,(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ a_{31}\,(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{21}\,(a_{12}\, a_{33} - a_{13}\, a_{32}) + a_{31}\,(a_{12}\, a_{23} - a_{13}\, a_{22})$$

$$|A| = a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{21}\, a_{12}\, a_{33} + a_{21}\, a_{13}\, a_{32} + a_{31}\, a_{12}\, a_{23} - a_{31}\, a_{13}\, a_{22}$$
$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \qquad \ldots (3)$$

Clearly, values of $|A|$ in (1), (2) and (3) are equal. It is left as an exercise to the reader to verify that the values of $|A|$ by expanding along R_3, C_2 and C_3 are equal to the value of $|A|$ obtained in (1), (2) or (3).

Hence, expanding a determinant along any row or column gives same value.

Remarks

(i) For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.

(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or –1 according as $(i + j)$ is even or odd.

(iii) Let $A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then, it is easy to verify that A = 2B. Also $|A| = 0 - 8 = -8$ and $|B| = 0 - 2 = -2$.

Observe that, $|A| = 4(-2) = 2^2|B|$ or $|A| = 2^n|B|$, where $n = 2$ is the order of square matrices A and B.

In general, if $A = kB$ where A and B are square matrices of order n, then $|A| = k^n |B|$, where $n = 1, 2, 3$

Example 3 Evaluate the determinant $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$.

Solution Note that in the third column, two entries are zero. So expanding along third column (C_3), we get

$$\Delta = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$
$$= 4(-1 - 12) - 0 + 0 = -52$$

Example 4 Evaluate $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$.

Solution Expanding along R_1, we get

$$\Delta = 0\begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} - \cos\alpha\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

$$= 0 - \sin\alpha\,(0 - \sin\beta\cos\alpha) - \cos\alpha\,(\sin\alpha\sin\beta - 0)$$

$$= \sin\alpha\sin\beta\cos\alpha - \cos\alpha\sin\alpha\sin\beta = 0$$

Example 5 Find values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

Solution We have $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

i.e. $\quad 3 - x^2 = 3 - 8$

i.e. $\quad x^2 = 8$

Hence $\quad x = \pm 2\sqrt{2}$

EXERCISE 4.1

Evaluate the determinants in Exercises 1 and 2.

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

5. Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

7. Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ± 6 (C) – 6 (D) 0

4.3 Properties of Determinants

In the previous section, we have learnt how to expand the determinants. In this section, we will study some properties of determinants which simplifies its evaluation by obtaining maximum number of zeros in a row or a column. These properties are true for determinants of any order. However, we shall restrict ourselves upto determinants of order 3 only.

Property 1 The value of the determinant remains unchanged if its rows and columns are interchanged.

Verification Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

By interchanging the rows and columns of Δ, we get the determinant

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding Δ_1 along first column, we get

$$\Delta_1 = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Hence $\Delta = \Delta_1$

Remark It follows from above property that if A is a square matrix, then det (A) = det (A′), where A′ = transpose of A.

Note If R_i = *i*th row and C_i = *i*th column, then for interchange of row and columns, we will symbolically write $C_i \leftrightarrow R_i$

Let us verify the above property by example.

Example 6 Verify Property 1 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

Solution Expanding the determinant along first row, we have

$$\Delta = 2\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$$

$$= 2 (0 - 20) + 3 (-42 - 4) + 5 (30 - 0)$$

$$= -40 - 138 + 150 = -28$$

By interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix} \quad \text{(Expanding along first column)}$$

$$= 2\begin{vmatrix} 0 & 5 \\ 4 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 1 \\ 4 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 1 \\ 0 & 5 \end{vmatrix}$$

$$= 2 (0 - 20) + 3 (-42 - 4) + 5 (30 - 0)$$

$$= -40 - 138 + 150 = -28$$

Clearly $\Delta = \Delta_1$

Hence, Property 1 is verified.

Property 2 If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

Verification Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Interchanging first and third rows, the new determinant obtained is given by

$$\Delta_1 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Expanding along third row, we get

$$\Delta_1 = a_1 (c_2 b_3 - b_2 c_3) - a_2 (c_1 b_3 - c_3 b_1) + a_3 (b_2 c_1 - b_1 c_2)$$
$$= - [a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)]$$

Clearly $\Delta_1 = - \Delta$

Similarly, we can verify the result by interchanging any two columns.

Note We can denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$.

Example 7 Verify Property 2 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$.

Solution $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = -28$ (See Example 6)

Interchanging rows R_2 and R_3 i.e., $R_2 \leftrightarrow R_3$, we have

$$\Delta_1 = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 5 & -7 \\ 6 & 0 & 4 \end{vmatrix}$$

Expanding the determinant Δ_1 along first row, we have

$$\Delta_1 = 2\begin{vmatrix} 5 & -7 \\ 0 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -7 \\ 6 & 4 \end{vmatrix} + 5\begin{vmatrix} 1 & 5 \\ 6 & 0 \end{vmatrix}$$

$$= 2 (20 - 0) + 3 (4 + 42) + 5 (0 - 30)$$
$$= 40 + 138 - 150 = 28$$

Clearly $\Delta_1 = -\Delta$

Hence, Property 2 is verified.

Property 3 If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

Proof If we interchange the identical rows (or columns) of the determinant Δ, then Δ does not change. However, by Property 2, it follows that Δ has changed its sign

Therefore $\Delta = -\Delta$

or $\Delta = 0$

Let us verify the above property by an example.

Example 8 Evaluate $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

Solution Expanding along first row, we get

$$\Delta = 3\,(6-6) - 2\,(6-9) + 3\,(4-6)$$
$$= 0 - 2\,(-3) + 3\,(-2) = 6 - 6 = 0$$

Here R_1 and R_3 are identical.

Property 4 If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

Verification Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and Δ_1 be the determinant obtained by multiplying the elements of the first row by k. Then

$$\Delta_1 = \begin{vmatrix} k\,a_1 & k\,b_1 & k\,c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along first row, we get

$$\Delta_1 = k\,a_1\,(b_2\,c_3 - b_3\,c_2) - k\,b_1\,(a_2\,c_3 - c_2\,a_3) + k\,c_1\,(a_2\,b_3 - b_2\,a_3)$$
$$= k\,[a_1\,(b_2\,c_3 - b_3\,c_2) - b_1\,(a_2\,c_3 - c_2\,a_3) + c_1\,(a_2\,b_3 - b_2\,a_3)]$$
$$= k\,\Delta$$

Hence $$\begin{vmatrix} k\,a_1 & k\,b_1 & k\,c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Remarks

(i) By this property, we can take out any common factor from any one row or any one column of a given determinant.

(ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero. For example

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ k\,a_1 & k\,a_2 & k\,a_3 \end{vmatrix} = 0 \text{ (rows } R_1 \text{ and } R_2 \text{ are proportional)}$$

Example 9 Evaluate $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Solution Note that $$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$$

(Using Properties 3 and 4)

Property 5 If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

For example, $$\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Verification L.H.S. = $$\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expanding the determinants along the first row, we get

$$\begin{aligned}\Delta &= (a_1 + \lambda_1)\,(b_2\, c_3 - c_2\, b_3) - (a_2 + \lambda_2)\,(b_1\, c_3 - b_3\, c_1)\\ &\quad + (a_3 + \lambda_3)\,(b_1\, c_2 - b_2\, c_1)\\ &= a_1\,(b_2\, c_3 - c_2\, b_3) - a_2\,(b_1\, c_3 - b_3\, c_1) + a_3\,(b_1\, c_2 - b_2\, c_1)\\ &\quad + \lambda_1\,(b_2\, c_3 - c_2\, b_3) - \lambda_2\,(b_1\, c_3 - b_3\, c_1) + \lambda_3\,(b_1\, c_2 - b_2\, c_1)\end{aligned}$$

(by rearranging terms)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{R.H.S.}$$

Similarly, we may verify Property 5 for other rows or columns.

Example 10 Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

Solution We have $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

(by Property 5)

$= 0 + 0 = 0$ (Using Property 3 and Property 4)

Property 6 If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + k\,C_j$.

Verification

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a_1 + k\,c_1 & a_2 + k\,c_2 & a_3 + k\,c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

where Δ_1 is obtained by the operation $R_1 \to R_1 + kR_3$.

Here, we have multiplied the elements of the third row (R_3) by a constant k and added them to the corresponding elements of the first row (R_1).

Symbolically, we write this operation as $R_1 \to R_1 + k\,R_3$.

Now, again

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k\,c_1 & k\,c_2 & k\,c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{(Using Property 5)}$$

$$= \Delta + 0 \qquad \text{(since } R_1 \text{ and } R_3 \text{ are proportional)}$$

Hence $\Delta = \Delta_1$

Remarks

(i) If Δ_1 is the determinant obtained by applying $R_i \to kR_i$ or $C_i \to kC_i$ to the determinant Δ, then $\Delta_1 = k\Delta$.

(ii) If more than one operation like $R_i \to R_i + kR_j$ is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operations.

Example 11 Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$.

Solution Applying operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ to the given determinant Δ, we have

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying $R_3 \to R_3 - 3R_2$, we get

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we obtain

$$\Delta = a\begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0 + 0$$

$$= a\,(a^2 - 0) = a\,(a^2) = a^3$$

Example 12 Without expanding, prove that

$$\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Solution Applying $R_1 \to R_1 + R_2$ to Δ, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

Since the elements of R_1 and R_3 are proportional, $\Delta = 0$.

Example 13 Evaluate

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Solution Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c\,(a-b) \\ 0 & c-a & b\,(a-c) \end{vmatrix}$$

Taking factors $(b-a)$ and $(c-a)$ common from R_2 and R_3, respectively, we get

$$\Delta = (b-a)\,(c-a) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$= (b-a)\,(c-a)\,[(-b+c)]$ (Expanding along first column)

$= (a-b)\,(b-c)\,(c-a)$

Example 14 Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Solution Let $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$ to Δ, we get

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expanding along R_1, we obtain

$$\Delta = 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2\,c\,(a\,b + b^2 - bc) - 2\,b\,(b\,c - c^2 - ac)$$

$$= 2\,a\,b\,c + 2\,cb^2 - 2\,bc^2 - 2\,b^2c + 2\,bc^2 + 2\,abc$$

$$= 4\,abc$$

Example 15 If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then show that $1 + xyz = 0$

Solution We have

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \quad \text{(Using Property 5)}$$

$$= (-1)^2\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad \text{(Using } C_3 \leftrightarrow C_2 \text{ and then } C_1 \leftrightarrow C_2\text{)}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz)$$

$$= (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \qquad \text{(Using } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

Taking out common factor $(y - x)$ from R_2 and $(z - x)$ from R_3, we get

$$\Delta = (1+xyz)\,(y-x)\,(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

$$= (1 + xyz)\,(y - x)\,(z - x)\,(z - y) \quad \text{(on expanding along } C_1\text{)}$$

Since $\Delta = 0$ and x, y, z are all different, i.e., $x - y \neq 0$, $y - z \neq 0$, $z - x \neq 0$, we get $1 + xyz = 0$

Example 16 Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab$$

Solution Taking out factors a,b,c common from R_1, R_2 and R_3, we get

$$\text{L.H.S.} = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Now applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left[1(1-0)\right]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab = \text{R.H.S.}$$

Note Alternately try by applying $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - C_2$, then apply $C_1 \to C_1 - a\,C_3$.

EXERCISE 4.2

Using the property of determinants and without expanding in Exercises 1 to 7, prove that:

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

2. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

4. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

5. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

6. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

7. $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

By using properties of determinants, in Exercises 8 to 14, show that:

8. (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

9. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

10. (i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

11. (i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

12. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \left(1-x^3\right)^2$

13. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$

14. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

Choose the correct answer in Exercises 15 and 16.

15. Let A be a square matrix of order 3 × 3, then $|kA|$ is equal to

(A) $k|A|$ (B) $k^2|A|$ (C) $k^3|A|$ (D) $3k|A|$

16. Which of the following is correct

(A) Determinant is a square matrix.

(B) Determinant is a number associated to a matrix.

(C) Determinant is a number associated to a square matrix.

(D) None of these

4.4 Area of a Triangle

In earlier classes, we have studied that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by the expression $\frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]$. Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \ldots (1)$$

Remarks

(i) Since area is a positive quantity, we always take the absolute value of the determinant in (1).

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

Example 17 Find the area of the triangle whose vertices are (3, 8), (– 4, 2) and (5, 1).

Solution The area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3(2-1)-8(-4-5)+1(-4-10)\right]$$

$$= \frac{1}{2}\,(3+72-14) = \frac{61}{2}$$

Example 18 Find the equation of the line joining A(1, 3) and B (0, 0) using determinants and find k if D(k, 0) is a point such that area of triangle ABD is 3sq units.

Solution Let P (x, y) be any point on AB. Then, area of triangle ABP is zero (Why?). So

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

This gives $\frac{1}{2}(y-3x) = 0$ or $y = 3x$,

which is the equation of required line AB.

Also, since the area of the triangle ABD is 3 sq. units, we have

$$\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

This gives, $\frac{-3k}{2} = \pm 3$, i.e., $k = \mp 2$.

EXERCISE 4.3

1. Find area of the triangle with vertices at the point given in each of the following :

   (i) (1, 0), (6, 0), (4, 3) (ii) (2, 7), (1, 1), (10, 8)

   (iii) (–2, –3), (3, 2), (–1, –8)

2. Show that points

   A $(a, b + c)$, B $(b, c + a)$, C $(c, a + b)$ are collinear.

3. Find values of k if area of triangle is 4 sq. units and vertices are

   (i) $(k, 0), (4, 0), (0, 2)$ (ii) $(-2, 0), (0, 4), (0, k)$

4. (i) Find equation of line joining (1, 2) and (3, 6) using determinants.

   (ii) Find equation of line joining (3, 1) and (9, 3) using determinants.

5. If area of triangle is 35 sq units with vertices $(2, -6)$, $(5, 4)$ and $(k, 4)$. Then k is

   (A) 12 (B) -2 (C) $-12, -2$ (D) $12, -2$

4.5 Minors and Cofactors

In this section, we will learn to write the expansion of a determinant in compact form using minors and cofactors.

Definition 1 Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Remark Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n - 1$.

Example 19 Find the minor of element 6 in the determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution Since 6 lies in the second row and third column, its minor M_{23} is given by

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6 \text{ (obtained by deleting } R_2 \text{ and } C_3 \text{ in } \Delta).$$

Definition 2 Cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij} \text{ is minor of } a_{ij}.$$

Example 20 Find minors and cofactors of all the elements of the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Solution Minor of the element a_{ij} is M_{ij}

Here $a_{11} = 1$. So M_{11} = Minor of $a_{11} = 3$

M_{12} = Minor of the element $a_{12} = 4$

M_{21} = Minor of the element $a_{21} = -2$

M_{22} = Minor of the element $a_{22} = 1$

Now, cofactor of a_{ij} is A_{ij}. So

$A_{11} = (-1)^{1+1}\ M_{11} = (-1)^2\ (3) = 3$

$A_{12} = (-1)^{1+2}\ M_{12} = (-1)^3\ (4) = -4$

$A_{21} = (-1)^{2+1}\ M_{21} = (-1)^3\ (-2) = 2$

$A_{22} = (-1)^{2+2}\ M_{22} = (-1)^4\ (1) = 1$

Example 21 Find minors and cofactors of the elements a_{11}, a_{21} in the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution By definition of minors and cofactors, we have

Minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\,a_{33} - a_{23}\,a_{32}$

Cofactor of $a_{11} = A_{11} = (-1)^{1+1}\ M_{11} = a_{22}\,a_{33} - a_{23}\,a_{32}$

Minor of $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\,a_{33} - a_{13}\,a_{32}$

Cofactor of $a_{21} = A_{21} = (-1)^{2+1}\ M_{21} = (-1)\ (a_{12}\,a_{33} - a_{13}\,a_{32}) = -\,a_{12}\,a_{33} + a_{13}\,a_{32}$

Remark Expanding the determinant Δ, in Example 21, along R_1, we have

$$\Delta = (-1)^{1+1}\,a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}\,a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}\ a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}\,A_{11} + a_{12}\,A_{12} + a_{13}\,A_{13}$, where A_{ij} is cofactor of a_{ij}

= sum of product of elements of R_1 with their corresponding cofactors

Similarly, Δ can be calculated by other five ways of expansion that is along R_2, R_3, C_1, C_2 and C_3.

Hence Δ = sum of the product of elements of any row (or column) with their corresponding cofactors.

Note If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example,

$$\Delta = a_{11}\,A_{21} + a_{12}\,A_{22} + a_{13}\,A_{23}$$

$$= a_{11}\,(-1)^{1+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\,(-1)^{1+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\,(-1)^{1+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ (since } R_1 \text{ and } R_2 \text{ are identical)}$$

Similarly, we can try for other rows and columns.

Example 22 Find minors and cofactors of the elements of the determinant

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} \text{ and verify that } a_{11}\,A_{31} + a_{12}\,A_{32} + a_{13}\,A_{33} = 0$$

Solution We have $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$; $A_{11} = (-1)^{1+1}(-20) = -20$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46; \qquad A_{12} = (-1)^{1+2}(-46) = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30; \qquad A_{13} = (-1)^{1+3}(30) = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4; \qquad A_{21} = (-1)^{2+1}(-4) = 4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19; \qquad A_{22} = (-1)^{2+2}(-19) = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13; \qquad A_{23} = (-1)^{2+3}(13) = -13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12; \qquad A_{31} = (-1)^{3+1}(-12) = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22; \qquad A_{32} = (-1)^{3+2}(-22) = 22$$

and $$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18; \qquad A_{33} = (-1)^{3+3}(18) = 18$$

Now $a_{11} = 2$, $a_{12} = -3$, $a_{13} = 5$; $A_{31} = -12$, $A_{32} = 22$, $A_{33} = 18$

So $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

$= 2(-12) + (-3)(22) + 5(18) = -24 - 66 + 90 = 0$

EXERCISE 4.4

Write Minors and Cofactors of the elements of following determinants:

1. (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

2. (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

3. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij}, then value of Δ is given by

(A) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ (B) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(C) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ (D) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

4.6 Adjoint and Inverse of a Matrix

In the previous chapter, we have studied inverse of a matrix. In this section, we shall discuss the condition for existence of inverse of a matrix.

To find inverse of a matrix A, i.e., A^{-1} we shall first define adjoint of a matrix.

4.6.1 *Adjoint of a matrix*

Definition 3 The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by *adj* A.

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then
$$adj\,A = \text{Transpose of} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example 23 Find *adj* A for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution We have $A_{11} = 4$, $A_{12} = -1$, $A_{21} = -3$, $A_{22} = 2$

Hence
$$adj\,A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Remark For a square matrix of order 2, given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The *adj* A can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}, i.e.,

$$adj\,A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Change sign Interchange

We state the following theorem without proof.

Theorem 1 If A be any given square matrix of order n, then

$$A(adj\,A) = (adj\,A)\,A = |A|\,I,$$

where I is the identity matrix of order n

Verification

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to $|A|$ and otherwise zero, we have

$$A\,(adj\ A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I$$

Similarly, we can show $(adj\ A)\ A = |A|\ I$

Hence $A\,(adj\ A) = (adj\ A)\ A = |A|\ I$

Definition 4 A square matrix A is said to be singular if $|A| = 0$.

For example, the determinant of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is zero

Hence A is a singular matrix.

Definition 5 A square matrix A is said to be non-singular if $|A| \neq 0$

Let $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$.

Hence A is a nonsingular matrix

We state the following theorems without proof.

Theorem 2 If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

Theorem 3 The determinant of the product of matrices is equal to product of their respective determinants, that is, $|AB| = |A|\ |B|$, where A and B are square matrices of the same order

Remark We know that $(adj\ A)\ A = |A|\ I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}, |A| \neq 0$

Writing determinants of matrices on both sides, we have

$$|(adj\,A)\,A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

i.e. $$|(adj\ A)|\ |A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{(Why?)}$$

i.e. $$|(adj\ A)|\ |A| = |A|^3 \quad (1)$$

i.e. $$|(adj\ A)| = |A|^2$$

In general, if A is *a* square matrix of order *n*, then $|adj(A)| = |A|^{n-1}$.

Theorem 4 A square matrix A is invertible if and only if A is nonsingular matrix.

Proof Let A be invertible matrix of order *n* and I be the identity matrix of order *n*. Then, there exists a square matrix B of order *n* such that AB = BA = I

Now $\quad AB = I.\ \text{So}\ |AB| = |I| \quad \text{or} \quad |A|\ |B| = 1 \quad (\text{since } |I|=1, |AB|=|A||B|)$

This gives $\quad |A| \neq 0$. Hence A is nonsingular.

Conversely, let A be nonsingular. Then $|A| \neq 0$

Now $\quad A\,(adj\ A) = (adj\ A)\ A = |A|\,I \qquad$ (Theorem 1)

or $$A\left(\frac{1}{|A|}adj\,A\right) = \left(\frac{1}{|A|}adj\,A\right)A = I$$

or $$AB = BA = I, \text{ where } B = \frac{1}{|A|}adj\,A$$

Thus $$A \text{ is invertible and } A^{-1} = \frac{1}{|A|}adj\,A$$

Example 24 If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that $A\ adj\ A = |A|\ I$. Also find A^{-1}.

Solution We have $|A| = 1\,(16 - 9) - 3\,(4 - 3) + 3\,(3 - 4) = 1 \neq 0$

Now $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$, $A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, $A_{33} = 1$

Therefore $$adj\ A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $$A\,(adj\ A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,.\,I$$

Also $$A^{-1} = \frac{1}{|A|}\,adj\ A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Example 25 If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution We have $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

Since, $|AB| = -11 \neq 0$, $(AB)^{-1}$ exists and is given by

$$(AB)^{-1} = \frac{1}{|AB|}\,adj\,(AB) = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Further, $|A| = -11 \neq 0$ and $|B| = 1 \neq 0$. Therefore, A^{-1} and B^{-1} both exist and are given by

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore $B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

Hence $(AB)^{-1} = B^{-1} A^{-1}$

Example 26 Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using this equation, find A^{-1}.

Solution We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now $A^2 - 4A + I = O$

Therefore $A\,A - 4A = -I$

or $A\,A\,(A^{-1}) - 4\,A\,A^{-1} = -I\,A^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A\,(A\,A^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

EXERCISE 4.5

Find adjoint of each of the matrices in Exercises 1 and 2.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Verify A (*adj* A) = (*adj* A) A = |A| I in Exercises 3 and 4

3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

5. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1} A^{-1}$.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Show that $A^3 - 6A^2 + 5A + 11\, I = O$. Hence, find A^{-1}.

16. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

17. Let A be a nonsingular square matrix of order 3 × 3. Then $|adj\, A|$ is equal to

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3|A|$

18. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

(A) det (A) (B) $\dfrac{1}{\det(A)}$ (C) 1 (D) 0

4.7 Applications of Determinants and Matrices

In this section, we shall discuss application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

Consistent system A system of equations is said to be *consistent* if its solution (one or more) exists.

Inconsistent system A system of equations is said to be *inconsistent* if its solution does not exist.

Note In this chapter, we restrict ourselves to the system of linear equations having unique solutions only.

4.7.1 *Solution of system of linear equations using inverse of a matrix*

Let us express the system of linear equations as matrix equations and solve them using inverse of the coefficient matrix.

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then, the system of equations can be written as, AX = B, i.e.,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I If A is a nonsingular matrix, then its inverse exists. Now

$$AX = B$$

or $$A^{-1}(AX) = A^{-1}B \qquad \text{(premultiplying by } A^{-1}\text{)}$$

or $$(A^{-1}A)X = A^{-1}B \qquad \text{(by associative property)}$$

or $$IX = A^{-1}B$$

or $$X = A^{-1}B$$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

Case II If A is a singular matrix, then $|A| = 0$.

In this case, we calculate (*adj* A) B.

If (*adj* A) B $\neq$ O, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If (*adj* A) B = O, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Example 27 Solve the system of equations

$$2x + 5y = 1$$
$$3x + 2y = 7$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now, $|A| = -11 \neq 0$, Hence, A is nonsingular matrix and so has a unique solution.

Note that $$A^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Therefore $$X = A^{-1}B = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence $x = 3, y = -1$

Example 28 Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

We see that

$$|A| = 3\,(2 - 3) + 2(4 + 4) + 3\,(-6 - 4) = -17 \neq 0$$

Hence, A is nonsingular and so its inverse exists. Now

$A_{11} = -1,\quad A_{12} = -8,\quad A_{13} = -10$
$A_{21} = -5,\quad A_{22} = -6,\quad A_{23} = 1$
$A_{31} = -1,\quad A_{32} = 9,\quad A_{33} = 7$

Therefore $$A^{-1} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

So $$X = A^{-1}B = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence $x = 1, y = 2$ and $z = 3$.

Example 29 The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

Solution Let first, second and third numbers be denoted by x, y and z, respectively. Then, according to given conditions, we have

$$x + y + z = 6$$
$$y + 3z = 11$$
$$x + z = 2y \text{ or } x - 2y + z = 0$$

This system can be written as A X = B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Here $|A| = 1(1+6) - (0-3) + (0-1) = 9 \neq 0$. Now we find *adj* A

$A_{11} = 1\,(1 + 6) = 7,\quad A_{12} = -(0 - 3) = 3,\quad A_{13} = -1$
$A_{21} = -(1 + 2) = -3,\quad A_{22} = 0,\quad A_{23} = -(-2 - 1) = 3$
$A_{31} = (3 - 1) = 2,\quad A_{32} = -(3 - 0) = -3,\quad A_{33} = (1 - 0) = 1$

Hence $$adj\ A = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Thus $$A^{-1} = \frac{1}{|A|}\ adj\ (A) = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Since $$X = A^{-1} B$$

$$X = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 42-33+0 \\ 18+\ 0+0 \\ -6+33+0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus $x = 1, y = 2, z = 3$

EXERCISE 4.6

Examine the consistency of the system of equations in Exercises 1 to 6.

1. $x + 2y = 2$
$2x + 3y = 3$

2. $2x - y = 5$
$x + y = 4$

3. $x + 3y = 5$
$2x + 6y = 8$

4. $x + y + z = 1$
$2x + 3y + 2z = 2$
$ax + ay + 2az = 4$

5. $3x-y - 2z = 2$
$2y - z = -1$
$3x - 5y = 3$

6. $5x - y + 4z = 5$
$2x + 3y + 5z = 2$
$5x - 2y + 6z = -1$

Solve system of linear equations, using matrix method, in Exercises 7 to 14.

7. $5x + 2y = 4$
$7x + 3y = 5$

8. $2x - y = -2$
$3x + 4y = 3$

9. $4x - 3y = 3$
$3x - 5y = 7$

10. $5x + 2y = 3$
$3x + 2y = 5$

11. $2x + y + z = 1$
$x - 2y - z = \frac{3}{2}$
$3y - 5z = 9$

12. $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

13. $2x + 3y + 3z = 5$
$x - 2y + z = -4$
$3x - y - 2z = 3$

14. $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹ 70. Find cost of each item per kg by matrix method.

Miscellaneous Examples

Example 30 If *a, b, c* are positive and unequal, show that value of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ is negative.}$$

Solution Applying $C_1 \to C_1 + C_2 + C_3$ to the given determinant, we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \text{(Applying } R_2 \to R_2 - R_1, \text{and } R_3 \to R_3 - R_1)$$

$$= (a+b+c)\,[(c-b)(b-c) - (a-c)(a-b)] \quad \text{(Expanding along } C_1)$$

$$= (a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= \frac{-1}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= \frac{-1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

which is negative (since $a + b + c > 0$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 > 0$)

Example 31 If a, b, c, are in A.P, find value of

$$\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

Solution Applying $R_1 \to R_1 + R_3 - 2R_2$ to the given determinant, we obtain

$$\begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} = 0 \qquad (\text{Since } 2b = a + c)$$

Example 32 Show that

$$\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz\,(x + y + z)^3$$

Solution Applying $R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3$ to Δ and dividing by xyz, we get

$$\Delta = \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & x^2z \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking common factors x, y, z from C_1 C_2 and C_3, respectively, we get

$$\Delta = \frac{xyz}{xyz}\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have

$$\Delta = \begin{vmatrix} (y+z)^2 & x^2-(y+z)^2 & x^2-(y+z)^2 \\ y^2 & (x+z)^2-y^2 & 0 \\ z^2 & 0 & (x+y)^2-z^2 \end{vmatrix}$$

Taking common factor $(x + y + z)$ from C_2 and C_3, we have

$$\Delta = (x+y+z)^2 \begin{vmatrix} (y+z)^2 & x-(y+z) & x-(y+z) \\ y^2 & (x+z)-y & 0 \\ z^2 & 0 & (x+y)-z \end{vmatrix}$$

Applying $R_1 \to R_1 - (R_2 + R_3)$, we have

$$\Delta = (x+y+z)^2 \begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x-y+z & 0 \\ z^2 & 0 & x+y-z \end{vmatrix}$$

Applying $C_2 \to (C_2 + \frac{1}{y} C_1)$ and $C_3 \to C_3 + \frac{1}{z}C_1$, we get

$$\Delta = (x+y+z)^2 \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & x+z & \frac{y^2}{z} \\ z^2 & \frac{z^2}{y} & x+y \end{vmatrix}$$

Finally expanding along R_1, we have

$$\Delta = (x+y+z)^2 (2yz) [(x+z)(x+y) - yz] = (x+y+z)^2 (2yz) (x^2 + xy + xz)$$

$$= (x+y+z)^3 (2xyz)$$

Example 33 Use product $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix}$ to solve the system of equations

$$x - y + 2z = 1$$
$$2y - 3z = 1$$
$$3x - 2y + 4z = 2$$

Solution Consider the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Now, given system of equations can be written, in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

or $$\begin{matrix} x \\ y \\ z \end{matrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence $x = 0$, $y = 5$ and $z = 3$

Example 34 Prove that

$$\Delta = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

Solution Applying $R_1 \to R_1 - x\,R_2$ to Δ, we get

$$\Delta = \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

$$= (1-x^2)\begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

Applying $R_2 \to R_2 - x\,R_1$, we get

$$\Delta = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

Miscellaneous Exercises on Chapter 4

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

2. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

4. If a, b and c are real numbers, and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0,$$

Show that either $a + b + c = 0$ or $a = b = c$.

5. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0,\ a \neq 0$

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

8. Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

(i) $[adj\ \mathrm{A}]^{-1} = adj\ (\mathrm{A}^{-1})$ (ii) $(\mathrm{A}^{-1})^{-1} = \mathrm{A}$

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

10. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Using properties of determinants in Exercises 11 to 15, prove that:

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)\ (\gamma-\alpha)\ (\alpha-\beta)\ (\alpha+\beta+\gamma)$

12. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)\ (x-y)\ (y-z)\ (z-x)$, where p is any scalar.

13. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)\ (ab+bc+ca)$

14. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$ **15.** $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$

16. Solve the system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Choose the correct answer in Exercise 17 to 19.

17. If a, b, c, are in A.P, then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} \text{ is}$$

(A) 0 (B) 1 (C) x (D) $2x$

18. If x, y, z are nonzero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ (D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then

(A) $\text{Det}(A) = 0$ (B) $\text{Det}(A) \in (2, \infty)$

(C) $\text{Det}(A) \in (2, 4)$ (D) $\text{Det}(A) \in [2, 4]$

Summary

- Determinant of a matrix $A = [a_{11}]_{1 \times 1}$ is given by $|a_{11}| = a_{11}$
- Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

- Determinant of a matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by (expanding along R_1)

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

For any square matrix A, the |A| satisfy following properties.

- $|A'| = |A|$, where A' = transpose of A.
- If we interchange any two rows (or columns), then sign of determinant changes.
- If any two rows or any two columns are identical or proportional, then value of determinant is zero.
- If we multiply each element of a row or a column of a determinant by constant k, then value of determinant is multiplied by k.
- Multiplying a determinant by k means multiply elements of only one row (or one column) by k.
- If $A = [a_{ij}]_{3\times3}$, then $|k.A| = k^3\,|A|$
- If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.
- If to each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows or columns are added, then value of determinant remains same.

- Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and denoted by M_{ij}.
- Cofactor of a_{ij} of given by $A_{ij} = (-1)^{i+j} M_{ij}$
- Value of determinant of a matrix A is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example,

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For example, $a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0$
- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is cofactor of a_{ij}
- $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$, where A is square matrix of order n.
- A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$.
- If AB = BA = I, where B is square matrix, then B is called inverse of A. Also $A^{-1} = B$ or $B^{-1} = A$ and hence $(A^{-1})^{-1} = A$.
- A square matrix A has inverse if and only if A is non-singular.
- $A^{-1} = \dfrac{1}{|A|}(adj\ A)$
- If

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3,$$

then these equations can be written as A X = B, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- Unique solution of equation AX = B is given by $X = A^{-1}B$, where $|A| \neq 0$.
- A system of equation is consistent or inconsistent according as its solution exists or not.
- For a square matrix A in matrix equation AX = B
  - (i) $|A| \neq 0$, there exists unique solution
  - (ii) $|A| = 0$ and $(adj\ A)\ B \neq 0$, then there exists no solution
  - (iii) $|A| = 0$ and $(adj\ A)\ B = 0$, then system may or may not be consistent.

Historical Note

The Chinese method of representing the coefficients of the unknowns of several linear equations by using rods on a calculating board naturally led to the discovery of simple method of elimination. The arrangement of rods was precisely that of the numbers in a determinant. The Chinese, therefore, early developed the idea of subtracting columns and rows as in simplification of a determinant *Mikami, China*, *pp* 30, 93.

Seki Kowa, the greatest of the Japanese Mathematicians of seventeenth century in his work '*Kai Fukudai no Ho*' in 1683 showed that he had the idea of determinants and of their expansion. But he used this device only in eliminating a quantity from two equations and not directly in the solution of a set of simultaneous linear equations. T. Hayashi, "*The Fakudoi and Determinants in Japanese Mathematics*," in the proc. of the Tokyo Math. Soc., V.

Vendermonde was the first to recognise determinants as independent functions. He may be called the formal founder. Laplace (1772), gave general method of expanding a determinant in terms of its complementary minors. In 1773 Lagrange treated determinants of the second and third orders and used them for purpose other than the solution of equations. In 1801, Gauss used determinants in his theory of numbers.

The next great contributor was Jacques - Philippe - Marie Binet, (1812) who stated the theorem relating to the product of two matrices of *m*-columns and *n*-rows, which for the special case of $m = n$ reduces to the multiplication theorem.

Also on the same day, Cauchy (1812) presented one on the same subject. He used the word 'determinant' in its present sense. He gave the proof of multiplication theorem more satisfactory than Binet's.

The greatest contributor to the theory was Carl Gustav Jacob Jacobi, after this the word determinant received its final acceptance.

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Chapter 5

CONTINUITY AND DIFFERENTIABILITY

❖ *The whole of science is nothing more than a refinement of everyday thinking." — ALBERT EINSTEIN* ❖

5.1 Introduction

This chapter is essentially a continuation of our study of differentiation of functions in Class XI. We had learnt to differentiate certain functions like polynomial functions and trigonometric functions. In this chapter, we introduce the very important concepts of continuity, differentiability and relations between them. We will also learn differentiation of inverse trigonometric functions. Further, we introduce a new class of functions called exponential and logarithmic functions. These functions lead to powerful techniques of differentiation. We illustrate certain geometrically obvious conditions through differential calculus. In the process, we will learn some fundamental theorems in this area.

Sir Issac Newton (1642-1727)

5.2 Continuity

We start the section with two informal examples to get a feel of continuity. Consider the function

$$f(x) = \begin{cases} 1, \text{ if } x \le 0 \\ 2, \text{ if } x > 0 \end{cases}$$

This function is of course defined at every point of the real line. Graph of this function is given in the Fig 5.1. One can deduce from the graph that the value of the function at *nearby* points on x-axis remain *close* to each other except at $x = 0$. At the points near and to the left of 0, i.e., at points like – 0.1, – 0.01, – 0.001, the value of the function is 1. At the points near and to the right of 0, i.e., at points like 0.1, 0.01,

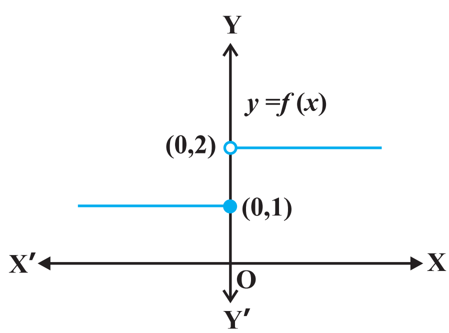

Fig 5.1

0.001, the value of the function is 2. Using the language of left and right hand limits, we may say that the left (respectively right) hand limit of f at 0 is 1 (respectively 2). In particular the left and right hand limits do not coincide. We also observe that the value of the function at $x = 0$ concides with the left hand limit. Note that when we try to draw the graph, we cannot draw it in one stroke, i.e., without lifting pen from the plane of the paper, we can not draw the graph of this function. In fact, we need to lift the pen when we come to 0 from left. This is one instance of function being not continuous at $x = 0$.

Now, consider the function defined as

$$f(x) = \begin{cases} 1, \text{ if } x \neq 0 \\ 2, \text{ if } x = 0 \end{cases}$$

This function is also defined at every point. Left and the right hand limits at $x = 0$ are both equal to 1. But the value of the function at $x = 0$ equals 2 which does not coincide with the common value of the left and right hand limits. Again, we note that we cannot draw the graph of the function without lifting the pen. This is yet another instance of a function being not continuous at $x = 0$.

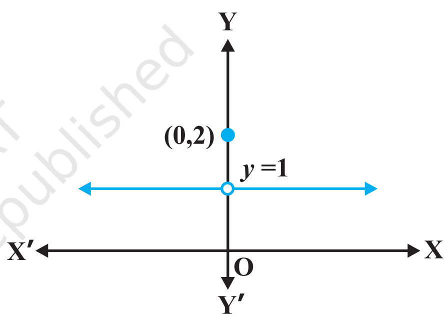

Fig 5.2

Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function *around* that point without lifting the pen from the plane of the paper.

Mathematically, it may be phrased precisely as follows:

Definition 1 Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and equal to each other, then f is said to be continuous at $x = c$. Recall that if the right hand and left hand limits at $x = c$ coincide, then we say that the common value is the limit of the function at $x = c$. Hence we may also rephrase the definition of continuity as follows: *a function is continuous at $x = c$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$.* If f is not continuous at c, we say f is *discontinuous* at c and c is called a *point of discontinuity* of f.

Example 1 Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.

Solution First note that the function is defined at the given point $x = 1$ and its value is 5. Then find the limit of the function at $x = 1$. Clearly

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (2x + 3) = 2(1) + 3 = 5$$

Thus $$\lim_{x \to 1} f(x) = 5 = f(1)$$

Hence, f is continuous at $x = 1$.

Example 2 Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.

Solution First note that the function is defined at the given point $x = 0$ and its value is 0. Then find the limit of the function at $x = 0$. Clearly

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0^2 = 0$$

Thus $$\lim_{x \to 0} f(x) = 0 = f(0)$$

Hence, f is continuous at $x = 0$.

Example 3 Discuss the continuity of the function f given by $f(x) = |\, x \,|$ at $x = 0$.

Solution By definition

$$f(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \quad \text{if } x \geq 0 \end{cases}$$

Clearly the function is defined at 0 and $f(0) = 0$. Left hand limit of f at 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$$

Similarly, the right hand limit of f at 0 is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

Thus, the left hand limit, right hand limit and the value of the function coincide at $x = 0$. Hence, f is continuous at $x = 0$.

Example 4 Show that the function f given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

Solution The function is defined at $x = 0$ and its value at $x = 0$ is 1. When $x \neq 0$, the function is given by a polynomial. Hence,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^3 + 3) = 0^3 + 3 = 3$$

Since the limit of f at $x = 0$ does not coincide with $f(0)$, the function is not continuous at $x = 0$. It may be noted that $x = 0$ is the only point of discontinuity for this function.

Example 5 Check the points where the constant function $f(x) = k$ is continuous.

Solution The function is defined at all real numbers and by definition, its value at any real number equals k. Let c be any real number. Then

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k$$

Since $f(c) = k = \lim_{x \to c} f(x)$ for any real number c, the function f is continuous at every real number.

Example 6 Prove that the identity function on real numbers given by $f(x) = x$ is continuous at every real number.

Solution The function is clearly defined at every point and $f(c) = c$ for every real number c. Also,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

Thus, $\lim_{x \to c} f(x) = c = f(c)$ and hence the function is continuous at every real number.

Having defined continuity of a function at a given point, now we make a natural extension of this definition to discuss continuity of a function.

Definition 2 A real function f is said to be continuous if it is continuous at every point in the domain of f.

This definition requires a bit of elaboration. Suppose f is a function defined on a closed interval $[a, b]$, then for f to be continuous, it needs to be continuous at every point in $[a, b]$ including the end points a and b. Continuity of f at a means

$$\lim_{x \to a^+} f(x) = f(a)$$

and continuity of f at b means

$$\lim_{x \to b^-} f(x) = f(b)$$

Observe that $\lim_{x \to a^-} f(x)$ *and* $\lim_{x \to b^+} f(x)$ *do not make sense. As a consequence of this definition, if f is defined only at one point, it is continuous there, i.e., if the domain of f is a singleton, f is a continuous function.*

Example 7 Is the function defined by $f(x) = |x|$, a continuous function?

Solution We may rewrite f as

$$f(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \quad \text{if } x \geq 0 \end{cases}$$

By Example 3, we know that f is continuous at $x = 0$.

Let c be a real number such that $c < 0$. Then $f(c) = -c$. Also

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-x) = -c \qquad \text{(Why?)}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at all negative real numbers.

Now, let c be a real number such that $c > 0$. Then $f(c) = c$. Also

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c \qquad \text{(Why?)}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at all positive real numbers. Hence, f is continuous at all points.

Example 8 Discuss the continuity of the function f given by $f(x) = x^3 + x^2 - 1$.

Solution Clearly f is defined at every real number c and its value at c is $c^3 + c^2 - 1$. We also know that

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 + x^2 - 1) = c^3 + c^2 - 1$$

Thus $\lim_{x \to c} f(x) = f(c)$, and hence f is continuous at every real number. This means f is a continuous function.

Example 9 Discuss the continuity of the function f defined by $f(x) = \frac{1}{x}$, $x \neq 0$.

Solution Fix any non zero real number c, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

Also, since for $c \neq 0$, $f(c) = \frac{1}{c}$, we have $\lim_{x \to c} f(x) = f(c)$ and hence, f is continuous at every point in the domain of f. Thus f is a continuous function.

We take this opportunity to explain the concept of *infinity*. This we do by analysing the function $f(x) = \dfrac{1}{x}$ near $x = 0$. To carry out this analysis we follow the usual trick of finding the value of the function at real numbers *close* to 0. Essentially we are trying to find the right hand limit of f at 0. We tabulate this in the following (Table 5.1).

Table 5.1

x	1	0.3	0.2	$0.1 = 10^{-1}$	$0.01 = 10^{-2}$	$0.001 = 10^{-3}$	10^{-n}
$f(x)$	1	3.333...	5	10	$100 = 10^2$	$1000 = 10^3$	10^n

We observe that as x gets closer to 0 from the right, the value of $f(x)$ shoots up higher. This may be rephrased as: the value of $f(x)$ may be made larger than any given number by choosing a positive real number *very close* to 0. In symbols, we write

$$\lim_{x \to 0^+} f(x) = +\infty$$

(to be read as: the right hand limit of $f(x)$ at 0 is plus infinity). We wish to emphasise that $+\infty$ is NOT a real number and hence the right hand limit of f at 0 does not exist (as a real number).

Similarly, the left hand limit of f at 0 may be found. The following table is self explanatory.

Table 5.2

x	-1	-0.3	-0.2	-10^{-1}	-10^{-2}	-10^{-3}	-10^{-n}
$f(x)$	-1	$-3.333...$	-5	-10	-10^2	-10^3	-10^n

From the Table 5.2, we deduce that the value of $f(x)$ may be made smaller than any given number by choosing a negative real number *very close* to 0. In symbols, we write

$$\lim_{x \to 0^-} f(x) = -\infty$$

(to be read as: the left hand limit of $f(x)$ at 0 is minus infinity). Again, we wish to emphasise that $-\infty$ is NOT a real number and hence the left hand limit of f at 0 does not exist (as a real number). The graph of the reciprocal function given in Fig 5.3 is a geometric representation of the above mentioned facts.

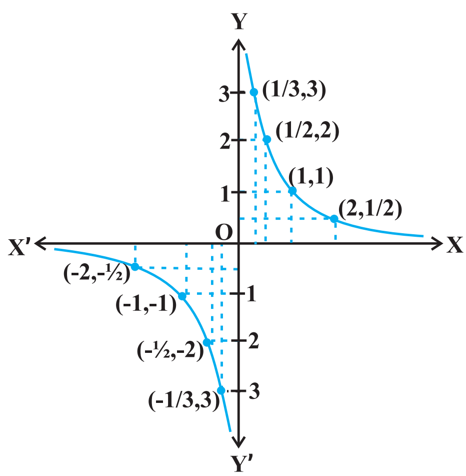

Fig 5.3

Example 10 Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x+2, \text{ if } x \le 1 \\ x-2, \text{ if } x > 1 \end{cases}$$

Solution The function f is defined at all points of the real line.

Case 1 If $c < 1$, then $f(c) = c + 2$. Therefore, $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x+2) = c+2$

Thus, f is continuous at all real numbers less than 1.

Case 2 If $c > 1$, then $f(c) = c - 2$. Therefore,

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x-2) = c - 2 = f(c)$$

Thus, f is continuous at all points $x > 1$.

Case 3 If $c = 1$, then the left hand limit of f at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

Since the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f. The graph of the function is given in Fig 5.4.

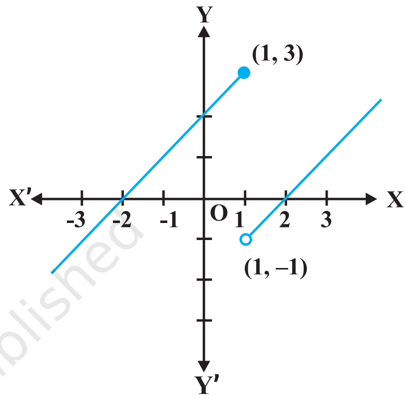

Fig 5.4

Example 11 Find all the points of discontinuity of the function f defined by

$$f(x) = \begin{cases} x+2, \text{ if } x < 1 \\ 0, \quad \text{if } x = 1 \\ x-2, \text{ if } x > 1 \end{cases}$$

Solution As in the previous example we find that f is continuous at all real numbers $x \ne 1$. The left hand limit of f at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

Since, the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f. The graph of the function is given in the Fig 5.5.

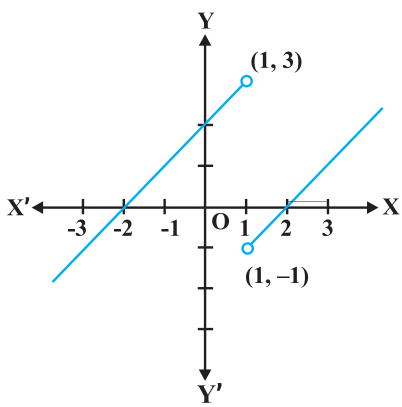

Fig 5.5

Example 12 Discuss the continuity of the function defined by

$$f(x) = \begin{cases} x+2, \text{ if } x<0 \\ -x+2, \text{ if } x>0 \end{cases}$$

Solution Observe that the function is defined at all real numbers except at 0. Domain of definition of this function is

$D_1 \cup D_2$ where $D_1 = \{x \in \mathbf{R} : x < 0\}$ and
$D_2 = \{x \in \mathbf{R} : x > 0\}$

Case 1 If $c \in D_1$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} (x + 2)$ $= c + 2 = f(c)$ and hence f is continuous in D_1.

Case 2 If $c \in D_2$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} (-x + 2)$ $= -c + 2 = f(c)$ and hence f is continuous in D_2.

Since f is continuous at all points in the domain of f, we deduce that f is continuous. Graph of this function is given in the Fig 5.6. Note that to graph this function we need to lift the pen from the plane of the paper, but we need to do that only for those points where the function is not defined.

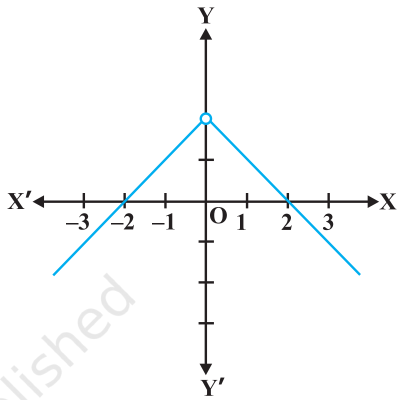

Fig 5.6

Example 13 Discuss the continuity of the function f given by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x^2, & \text{if } x < 0 \end{cases}$$

Solution Clearly the function is defined at every real number. Graph of the function is given in Fig 5.7. By inspection, it seems prudent to partition the domain of definition of f into three disjoint subsets of the real line.

Let $D_1 = \{x \in \mathbf{R} : x < 0\}$, $D_2 = \{0\}$ and
$D_3 = \{x \in \mathbf{R} : x > 0\}$

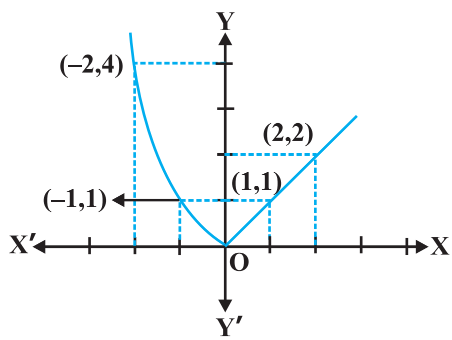

Fig 5.7

Case 1 At any point in D_1, we have $f(x) = x^2$ and it is easy to see that it is continuous there (see Example 2).

Case 2 At any point in D_3, we have $f(x) = x$ and it is easy to see that it is continuous there (see Example 6).

Case 3 Now we analyse the function at $x = 0$. The value of the function at 0 is $f(0) = 0$. The left hand limit of f at 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$

The right hand limit of f at 0 is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$ and hence f is continuous at 0. This means that f is continuous at every point in its domain and hence, f is a continuous function.

Example 14 Show that every polynomial function is continuous.

Solution Recall that a function p is a polynomial function if it is defined by $p(x) = a_0 + a_1 x + ... + a_n x^n$ for some natural number n, $a_n \neq 0$ and $a_i \in \mathbf{R}$. Clearly this function is defined for every real number. For a fixed real number c, we have

$$\lim_{x \to c} p(x) = p(c)$$

By definition, p is continuous at c. Since c is any real number, p is continuous at every real number and hence p is a continuous function.

Example 15 Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x.

Solution First observe that f is defined for all real numbers. Graph of the function is given in Fig 5.8. From the graph it looks like that f is discontinuous at every integral point. Below we explore, if this is true.

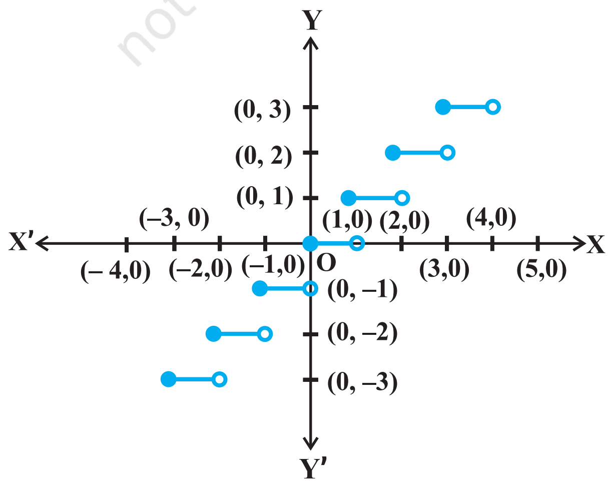

Fig 5.8

Case 1 Let c be a real number which is not equal to any integer. It is evident from the graph that for all real numbers *close* to c the value of the function is equal to $[c]$; i.e., $\lim_{x \to c} f(x) = \lim_{x \to c} [x] = [c]$. Also $f(c) = [c]$ and hence the function is continuous at all real numbers not equal to integers.

Case 2 Let c be an integer. Then we can find a sufficiently small real number $r > 0$ such that $[c - r] = c - 1$ whereas $[c + r] = c$.

This, in terms of limits mean that

$$\lim_{x \to c^-} f(x) = c - 1, \quad \lim_{x \to c^+} f(x) = c$$

Since these limits cannot be equal to each other for any c, the function is discontinuous at every integral point.

5.2.1 *Algebra of continuous functions*

In the previous class, after having understood the concept of limits, we learnt some algebra of limits. Analogously, now we will study some algebra of continuous functions. Since continuity of a function at a point is entirely dictated by the limit of the function at that point, it is reasonable to expect results analogous to the case of limits.

Theorem 1 Suppose f and g be two real functions continuous at a real number c. Then

(1) $f + g$ is continuous at $x = c$.

(2) $f - g$ is continuous at $x = c$.

(3) $f \,.\, g$ is continuous at $x = c$.

(4) $\left(\dfrac{f}{g}\right)$ is continuous at $x = c$, (provided $g(c) \neq 0$).

Proof We are investigating continuity of $(f + g)$ at $x = c$. Clearly it is defined at $x = c$. We have

$$\begin{aligned}
\lim_{x \to c} (f + g)(x) &= \lim_{x \to c} [f(x) + g(x)] && \text{(by definition of } f + g\text{)} \\
&= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) && \text{(by the theorem on limits)} \\
&= f(c) + g(c) && \text{(as } f \text{ and } g \text{ are continuous)} \\
&= (f + g)(c) && \text{(by definition of } f + g\text{)}
\end{aligned}$$

Hence, $f + g$ is continuous at $x = c$.

Proofs for the remaining parts are similar and left as an exercise to the reader.

Remarks

(i) As a special case of (3) above, if f is a constant function, i.e., $f(x) = \lambda$ for some real number λ, then the function $(\lambda \,.\, g)$ defined by $(\lambda \,.\, g)(x) = \lambda \,.\, g(x)$ is also continuous. In particular if $\lambda = -1$, the continuity of f implies continuity of $-f$.

(ii) As a special case of (4) above, if f is the constant function $f(x) = \lambda$, then the function $\frac{\lambda}{g}$ defined by $\frac{\lambda}{g}(x) = \frac{\lambda}{g(x)}$ is also continuous wherever $g(x) \neq 0$. In particular, the continuity of g implies continuity of $\frac{1}{g}$.

The above theorem can be exploited to generate many continuous functions. They also aid in deciding if certain functions are continuous or not. The following examples illustrate this:

Example 16 Prove that every rational function is continuous.

Solution Recall that every rational function f is given by

$$f(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$$

where p and q are polynomial functions. The domain of f is all real numbers except points at which q is zero. Since polynomial functions are continuous (Example 14), f is continuous by (4) of Theorem 1.

Example 17 Discuss the continuity of sine function.

Solution To see this we use the following facts

$$\lim_{x \to 0} \sin x = 0$$

We have not proved it, but is intuitively clear from the graph of $\sin x$ near 0.

Now, observe that $f(x) = \sin x$ is defined for every real number. Let c be a real number. Put $x = c + h$. If $x \to c$ we know that $h \to 0$. Therefore

$$\begin{aligned} \lim_{x \to c} f(x) &= \lim_{x \to c} \sin x \\ &= \lim_{h \to 0} \sin(c + h) \\ &= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] \\ &= \lim_{h \to 0} [\sin c \cos h] + \lim_{h \to 0} [\cos c \sin h] \\ &= \sin c + 0 = \sin c = f(c) \end{aligned}$$

Thus $\lim_{x \to c} f(x) = f(c)$ and hence f is a continuous function.

Remark A similar proof may be given for the continuity of cosine function.

Example 18 Prove that the function defined by $f(x) = \tan x$ is a continuous function.

Solution The function $f(x) = \tan x = \frac{\sin x}{\cos x}$. This is defined for all real numbers such that $\cos x \neq 0$, i.e., $x \neq (2n+1)\frac{\pi}{2}$. We have just proved that both sine and cosine functions are continuous. Thus tan x being a quotient of two continuous functions is continuous wherever it is defined.

An interesting fact is the behaviour of continuous functions with respect to composition of functions. Recall that if f and g are two real functions, then

$$(f \circ g)(x) = f(g(x))$$

is defined whenever the range of g is a subset of domain of f. The following theorem (stated without proof) captures the continuity of composite functions.

Theorem 2 Suppose f and g are real valued functions such that $(f \circ g)$ is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c.

The following examples illustrate this theorem.

Example 19 Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.

Solution Observe that the function is defined for every real number. The function f may be thought of as a composition $g \circ h$ of the two functions g and h, where $g(x) = \sin x$ and $h(x) = x^2$. Since both g and h are continuous functions, by Theorem 2, it can be deduced that f is a continuous function.

Example 20 Show that the function f defined by

$$f(x) = |1 - x + |x||,$$

where x is any real number, is a continuous function.

Solution Define g by $g(x) = 1 - x + |x|$ and h by $h(x) = |x|$ for all real x. Then

$$\begin{aligned}(h \circ g)(x) &= h(g(x)) \\ &= h(1 - x + |x|) \\ &= |1 - x + |x|| = f(x)\end{aligned}$$

In Example 7, we have seen that h is a continuous function. Hence g being a sum of a polynomial function and the modulus function is continuous. But then f being a composite of two continuous functions is continuous.

EXERCISE 5.1

1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

2. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

3. Examine the following functions for continuity.

(a) $f(x) = x - 5$ (b) $f(x) = \dfrac{1}{x-5}, x \neq 5$

(c) $f(x) = \dfrac{x^2 - 25}{x+5}, x \neq -5$ (d) $f(x) = |x - 5|$

4. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

5. Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Find all points of discontinuity of f, where f is defined by

6. $f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

7. $f(x) = \begin{cases} |x|+3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \geq 3 \end{cases}$

8. $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

10. $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

11. $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$

12. $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$

13. Is the function defined by

$$f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Discuss the continuity of the function f, where f is defined by

14. $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

15. $f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 1\\ 4x, & \text{if } x>1\end{cases}$

16. $f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$

17. Find the relationship between a and b so that the function f defined by

$$f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+3, & \text{if } x>3\end{cases}$$

is continuous at $x = 3$.

18. For what value of λ is the function defined by

$$f(x)=\begin{cases}\lambda(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

21. Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \cdot \cos x$

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

23. Find all points of discontinuity of f, where

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\ge 0\end{cases}$$

24. Determine if f defined by

$$f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$$

is a continuous function?

25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$$

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

26. $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ at $x = \dfrac{\pi}{2}$

27. $f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$ at $x = 2$

28. $f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ at $x = \pi$

29. $f(x) = \begin{cases} kx + 1, & \text{if } x \leq 5 \\ 3x - 5, & \text{if } x > 5 \end{cases}$ at $x = 5$

30. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

is a continuous function.

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

33. Examine that $\sin |x|$ is a continuous function.

34. Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

5.3. Differentiability

Recall the following facts from previous class. We had defined the derivative of a real function as follows:

Suppose f is a real function and c is a point in its domain. The derivative of f at c is defined by

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided this limit exists. Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))|_c$. The function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f. The derivative of f is denoted by $f'(x)$ or $\frac{d}{dx}(f(x))$ or if $y = f(x)$ by $\frac{dy}{dx}$ or y'. The process of finding derivative of a function is called differentiation. We also use the phrase *differentiate $f(x)$ with respect to x* to mean *find $f'(x)$*.

The following rules were established as a part of algebra of derivatives:

(1) $(u \pm v)' = u' \pm v'$

(2) $(uv)' = u'v + uv'$ (Leibnitz or product rule)

(3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, wherever $v \neq 0$ (Quotient rule).

The following table gives a list of derivatives of certain standard functions:

Table 5.3

$f(x)$	x^n	$\sin x$	$\cos x$	$\tan x$
$f'(x)$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$

Whenever we defined derivative, we had put a caution *provided the limit exists*. Now the natural question is; what if it doesn't? The question is quite pertinent and so is its answer. If $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ does not exist, we say that f is not differentiable at c. In other words, we say that a function f is differentiable at a point c in its domain if both $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$ and $\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$ are finite and equal. A function is said to be differentiable in an interval $[a, b]$ if it is differentiable at every point of $[a, b]$. As in case of continuity, at the end points a and b, we take the right hand limit and left hand limit, which are nothing but left hand derivative and right hand derivative of the function at a and b respectively. Similarly, a function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b).

Theorem 3 If a function f is differentiable at a point c, then it is also continuous at that point.

Proof Since f is differentiable at c, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

But for $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$

Therefore $$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$$

or $$\lim_{x \to c} [f(x)] - \lim_{x \to c} [f(c)] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right] \cdot \lim_{x \to c} [(x - c)]$$

$$= f'(c) \cdot 0 = 0$$

or $$\lim_{x \to c} f(x) = f(c)$$

Hence f is continuous at $x = c$.

Corollary 1 Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by $f(x) = |x|$ is a continuous function. Consider the left hand limit

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \frac{-h}{h} = -1$$

The right hand limit

$$\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \frac{h}{h} = 1$$

Since the above left and right hand limits at 0 are not equal, $\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$ does not exist and hence f is not differentiable at 0. Thus f is not a differentiable function.

5.3.1 *Derivatives of composite functions*

To study derivative of composite functions, we start with an illustrative example. Say, we want to find the derivative of f, where

$$f(x) = (2x + 1)^3$$

One way is to expand $(2x + 1)^3$ using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[(2x+1)^3\right]$$

$$= \frac{d}{dx}(8x^3+12x^2+6x+1)$$

$$= 24x^2 + 24x + 6$$

$$= 6\ (2x + 1)^2$$

Now, observe that $f(x) = (h \text{ o } g)\ (x)$

where $g(x) = 2x + 1$ and $h(x) = x^3$. Put $t = g(x) = 2x + 1$. Then $f(x) = h(t) = t^3$. Thus

$$\frac{df}{dx} = 6\ (2x + 1)^2 = 3(2x + 1)^2 \ .\ 2 = 3t^2 \ .\ 2 = \frac{dh}{dt}\cdot\frac{dt}{dx}$$

The advantage with such observation is that it simplifies the calculation in finding the derivative of, say, $(2x + 1)^{100}$. We may formalise this observation in the following theorem called the chain rule.

Theorem 4 (Chain Rule) Let f be a real valued function which is a composite of two functions u and v; i.e., $f = v \text{ o } u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have

$$\frac{df}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}$$

We skip the proof of this theorem. Chain rule may be extended as follows. Suppose f is a real valued function which is a composite of three functions u, v and w; i.e.,

$f = (w \text{ o } u) \text{ o } v$. If $t = v(x)$ and $s = u(t)$, then

$$\frac{df}{dx} = \frac{d(w \text{o} u)}{dt}\cdot\frac{dt}{dx} = \frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is invited to formulate chain rule for composite of more functions.

Example 21 Find the derivative of the function given by $f(x) = \sin(x^2)$.

Solution Observe that the given function is a composite of two functions. Indeed, if $t = u(x) = x^2$ and $v(t) = \sin t$, then

$$f(x) = (v \text{ o } u)\ (x) = v(u(x)) = v(x^2) = \sin x^2$$

Put $t = u(x) = x^2$. Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of x. Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

Alternatively, We can also directly proceed as follows:

$$y = \sin (x^2) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin x^2)$$

$$= \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2$$

Example 22 Find the derivative of tan $(2x + 3)$.

Solution Let $f(x) = \tan (2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \text{ o } u)(x) = v(u(x)) = v(2x + 3) = \tan (2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2 (2x+3)$$

Example 23 Differentiate sin (cos (x^2)) with respect to x.

Solution The function $f(x) = \sin (\cos (x^2))$ is a composition $f(x) = (w \text{ o } v \text{ o } u)(x)$ of the three functions u, v and w, where $u(x) = x^2$, $v(t) = \cos t$ and $w(s) = \sin s$. Put $t = u(x) = x^2$ and $s = v(t) = \cos t$. Observe that $\frac{dw}{ds} = \cos s$, $\frac{ds}{dt} = -\sin t$ and $\frac{dt}{dx} = 2x$ exist for all real x. Hence by a generalisation of chain rule, we have

$$\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} = (\cos s) \cdot (-\sin t) \cdot (2x) = -2x \sin x^2 \cdot \cos (\cos x^2)$$

Alternatively, we can proceed as follows:

$$y = \sin(\cos x^2)$$

Therefore $$\frac{dy}{dx} = \frac{d}{dx}\sin(\cos x^2) = \cos(\cos x^2)\ \frac{d}{dx}(\cos x^2)$$

$$= \cos(\cos x^2)\ (-\sin x^2)\ \frac{d}{dx}(x^2)$$

$$= -\sin x^2 \cos(\cos x^2)\ (2x)$$

$$= -2x \sin x^2 \cos(\cos x^2)$$

EXERCISE 5.2

Differentiate the functions with respect to x in Exercises 1 to 8.

1. $\sin(x^2 + 5)$ **2.** $\cos(\sin x)$ **3.** $\sin(ax + b)$

4. $\sec(\tan(\sqrt{x}))$ **5.** $\dfrac{\sin(ax+b)}{\cos(cx+d)}$ **6.** $\cos x^3 \cdot \sin^2(x^5)$

7. $2\sqrt{\cot(x^2)}$ **8.** $\cos(\sqrt{x})$

9. Prove that the function f given by

$$f(x) = |x - 1|,\ x \in \mathbf{R}$$

is not differentiable at $x = 1$.

10. Prove that the greatest integer function defined by

$$f(x) = [x],\ 0 < x < 3$$

is not differentiable at $x = 1$ and $x = 2$.

5.3.2 *Derivatives of implicit functions*

Until now we have been differentiating various functions given in the form $y = f(x)$. But it is not necessary that functions are always expressed in this form. For example, consider one of the following relationships between x and y:

$$x - y - \pi = 0$$

$$x + \sin xy - y = 0$$

In the first case, we can *solve for* y and rewrite the relationship as $y = x - \pi$. In the second case, it does not seem that there is an easy way to *solve for* y. Nevertheless, there is no doubt about the dependence of y on x in either of the cases. When a relationship between x and y is expressed in a way that it is easy to *solve for* y and write $y = f(x)$, we say that y is given as an *explicit function* of x. In the latter case it is implicit that y is a function of x and we say that the relationship of the second type, above, gives function *implicitly*. In this subsection, we learn to differentiate implicit functions.

Example 24 Find $\frac{dy}{dx}$ if $x - y = \pi$.

Solution One way is to solve for y and rewrite the above as

$$y = x - \pi$$

But then $$\frac{dy}{dx} = 1$$

Alternatively, *directly* differentiating the relationship w.r.t., x, we have

$$\frac{d}{dx}(x - y) = \frac{d\pi}{dx}$$

Recall that $\frac{d\pi}{dx}$ means to differentiate the constant function taking value π everywhere w.r.t., x. Thus

$$\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$$

which implies that

$$\frac{dy}{dx} = \frac{dx}{dx} = 1$$

Example 25 Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$.

Solution We differentiate the relationship directly with respect to x, i.e.,

$$\frac{dy}{dx} + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$$

which implies using chain rule

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = -\sin x$$

This gives $$\frac{dy}{dx} = -\frac{\sin x}{1 + \cos y}$$

where $$y \neq (2n + 1)\pi$$

5.3.3 *Derivatives of inverse trigonometric functions*

We remark that inverse trigonometric functions are continuous functions, but we will not prove this. Now we use chain rule to find derivatives of these functions.

Example 26 Find the derivative of f given by $f(x) = \sin^{-1} x$ assuming it exists.

Solution Let $y = \sin^{-1} x$. Then, $x = \sin y$.

Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Observe that this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$, i.e., $x \in (-1, 1)$.

To make this result a bit more attractive, we carry out the following manipulation. Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1} x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y$ is positive and hence $\cos y = \sqrt{1-x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Example 27 Find the derivative of f given by $f(x) = \tan^{-1} x$ assuming it exists.

Solution Let $y = \tan^{-1} x$. Then, $x = \tan y$.

Differentiating both sides w.r.t. x, we get

$$1 = \sec^2 y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+(\tan(\tan^{-1} x))^2} = \frac{1}{1+x^2}$$

Finding of the derivatives of other inverse trigonometric functions is left as exercise. The following table gives the derivatives of the remaining inverse trigonometric functions (Table 5.4):

Table 5.4

$f(x)$	$\cos^{-1}x$	$\cot^{-1}x$	$\sec^{-1}x$	$\text{cosec}^{-1}x$
$f'(x)$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{-1}{1+x^2}$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
Domain of f'	$(-1, 1)$	**R**	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1) \cup (1, \infty)$

EXERCISE 5.3

Find $\frac{dy}{dx}$ in the following:

1. $2x + 3y = \sin x$ **2.** $2x + 3y = \sin y$ **3.** $ax + by^2 = \cos y$

4. $xy + y^2 = \tan x + y$ **5.** $x^2 + xy + y^2 = 100$ **6.** $x^3 + x^2y + xy^2 + y^3 = 81$

7. $\sin^2 y + \cos xy = \kappa$ **8.** $\sin^2 x + \cos^2 y = 1$ **9.** $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

10. $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

11. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

12. $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

13. $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

14. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

15. $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

5.4 Exponential and Logarithmic Functions

Till now we have learnt some aspects of different classes of functions like polynomial functions, rational functions and trigonometric functions. In this section, we shall learn about a new class of (related) functions called exponential functions and logarithmic functions. It needs to be emphasized that many statements made in this section are motivational and precise proofs of these are well beyond the scope of this text.

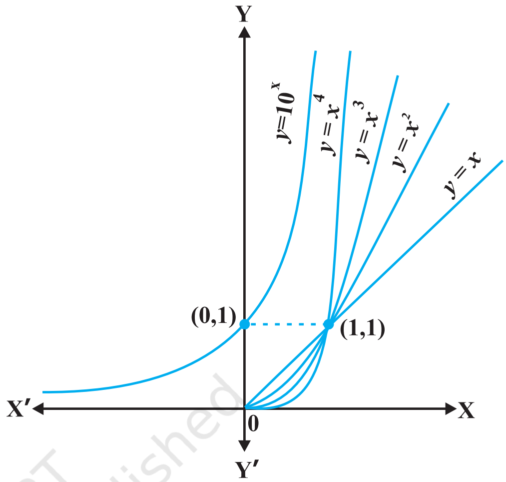

Fig 5.9

The Fig 5.9 gives a sketch of $y = f_1(x) = x$, $y = f_2(x) = x^2$, $y = f_3(x) = x^3$ and $y = f_4(x) = x^4$. Observe that the curves get steeper as the power of x increases. Steeper the curve, faster is the rate of growth. What this means is that for a fixed increment in the value of $x (> 1)$, the increment in the value of $y = f_n(x)$ increases as n increases for $n = 1, 2, 3, 4$. It is conceivable that such a statement is true for all positive values of n, where $f_n(x) = x^n$. Essentially, this means that the graph of $y = f_n(x)$ *leans* more towards the y-axis as n increases. For example, consider $f_{10}(x) = x^{10}$ and $f_{15}(x) = x^{15}$. If x increases from 1 to 2, f_{10} increases from 1 to 2^{10} whereas f_{15} increases from 1 to 2^{15}. Thus, for the same increment in x, f_{15} grow faster than f_{10}.

Upshot of the above discussion is that the growth of polynomial functions is dependent on the degree of the polynomial function – higher the degree, greater is the growth. The next natural question is: Is there a function which grows faster than any polynomial function. The answer is in affirmative and an example of such a function is

$$y = f(x) = 10^x.$$

Our claim is that this function f grows faster than $f_n(x) = x^n$ for any positive integer n. For example, we can prove that 10^x grows faster than $f_{100}(x) = x^{100}$. For large values of x like $x = 10^3$, note that $f_{100}(x) = (10^3)^{100} = 10^{300}$ whereas $f(10^3) = 10^{10^3} = 10^{1000}$. Clearly $f(x)$ is much greater than $f_{100}(x)$. It is not difficult to prove that for all $x > 10^3$, $f(x) > f_{100}(x)$. But we will not attempt to give a proof of this here. Similarly, by choosing large values of x, one can verify that $f(x)$ grows faster than $f_n(x)$ for any positive integer n.

Definition 3 The exponential function with positive base $b > 1$ is the function

$$y = f(x) = b^x$$

The graph of $y = 10^x$ is given in the Fig 5.9.

It is advised that the reader plots this graph for particular values of b like 2, 3 and 4. Following are some of the salient features of the exponential functions:

(1) Domain of the exponential function is **R**, the set of all real numbers.

(2) Range of the exponential function is the set of all positive real numbers.

(3) The point (0, 1) is always on the graph of the exponential function (this is a restatement of the fact that $b^0 = 1$ for any real $b > 1$).

(4) Exponential function is ever increasing; i.e., as we move from left to right, the graph rises above.

(5) For very large negative values of x, the exponential function is very close to 0. In other words, in the second quadrant, the graph approaches x-axis (but never meets it).

Exponential function with base 10 is called the *common exponential function*. In the Appendix A.1.4 of Class XI, it was observed that the sum of the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \ldots$$

is a number between 2 and 3 and is denoted by e. Using this e as the base we obtain an extremely important exponential function $y = e^x$.

This is called *natural exponential function*.

It would be interesting to know if the inverse of the exponential function exists and has *nice* interpretation. This search motivates the following definition.

Definition 4 Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$.

Logarithm of a to base b is denoted by $\log_b a$. Thus $\log_b a = x$ if $b^x = a$. Let us work with a few explicit examples to get a feel for this. We know $2^3 = 8$. In terms of logarithms, we may rewrite this as $\log_2 8 = 3$. Similarly, $10^4 = 10000$ is equivalent to saying $\log_{10} 10000 = 4$. Also, $625 = 5^4 = 25^2$ is equivalent to saying $\log_5 625 = 4$ or $\log_{25} 625 = 2$.

On a slightly more mature note, fixing a base $b > 1$, we may look at logarithm as a function from positive real numbers to all real numbers. This function, called the *logarithmic function*, is defined by

$$\log_b : \mathbf{R}^+ \to \mathbf{R}$$
$$x \to \log_b x = y \quad \text{if } b^y = x$$

As before if the base $b = 10$, we say it is *common logarithms* and if $b = e$, then we say it is *natural logarithms*. Often natural logarithm is denoted by *ln*. *In this chapter, log x denotes the logarithm function to base e*, i.e., $ln\ x$ will be written as simply $\log x$. The Fig 5.10 gives the plots of logarithm function to base 2, e and 10.

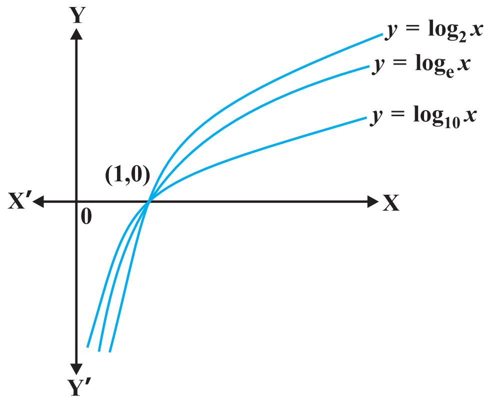

Fig 5.10

Some of the important observations about the logarithm function to any base $b > 1$ are listed below:

(1) We cannot make a meaningful definition of logarithm of non-positive numbers and hence the domain of log function is $\mathbf{R}^+$.

(2) The range of log function is the set of all real numbers.

(3) The point (1, 0) is always on the graph of the log function.

(4) The log function is ever increasing, i.e., as we move from left to right the graph rises above.

(5) For x very near to zero, the value of $\log x$ can be made lesser than any given real number. In other words in the fourth quadrant the graph approaches y-axis (but never meets it).

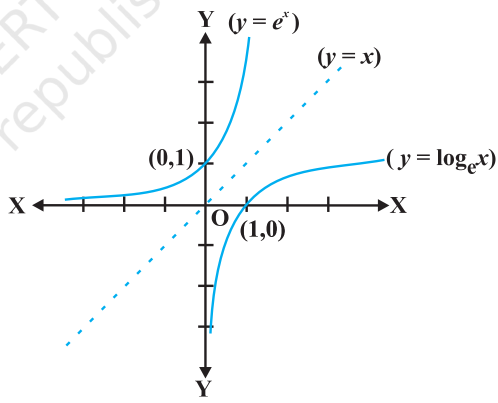

Fig 5.11

(6) Fig 5.11 gives the plot of $y = e^x$ and $y = ln\ x$. It is of interest to observe that the two curves are the mirror images of each other reflected in the line $y = x$.

Two properties of 'log' functions are proved below:

(1) There is a standard change of base rule to obtain $\log_a p$ in terms of $\log_b p$. Let $\log_a p = \alpha$, $\log_b p = \beta$ and $\log_b a = \gamma$. This means $a^\alpha = p$, $b^\beta = p$ and $b^\gamma = a$.

Substituting the third equation in the first one, we have

$$(b^\gamma)^\alpha = b^{\gamma\alpha} = p$$

Using this in the second equation, we get

$$b^\beta = p = b^{\gamma\alpha}$$

which implies $\beta = \alpha\gamma$ or $\alpha = \dfrac{\beta}{\gamma}$. But then

$$\log_a p = \frac{\log_b p}{\log_b a}$$

(2) Another interesting property of the log function is its effect on products. Let $\log_b pq = \alpha$. Then $b^\alpha = pq$. If $\log_b p = \beta$ and $\log_b q = \gamma$, then $b^\beta = p$ and $b^\gamma = q$. But then $b^\alpha = pq = b^\beta b^\gamma = b^{\beta + \gamma}$

which implies $\alpha = \beta + \gamma$, i.e.,

$$\log_b pq = \log_b p + \log_b q$$

A particularly interesting and important consequence of this is when $p = q$. In this case the above may be rewritten as

$$\log_b p^2 = \log_b p + \log_b p = 2 \log p$$

An easy generalisation of this (left as an exercise!) is

$$\log_b p^n = n \log p$$

for any positive integer n. In fact this is true for any real number n, but we will not attempt to prove this. On the similar lines the reader is invited to verify

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

Example 28 Is it true that $x = e^{\log x}$ for all real x?

Solution First, observe that the domain of log function is set of all positive real numbers. So the above equation is not true for non-positive real numbers. Now, let $y = e^{\log x}$. If $y > 0$, we may take logarithm which gives us $\log y = \log (e^{\log x}) = \log x \,.\, \log e = \log x$. Thus $y = x$. Hence $x = e^{\log x}$ is true only for positive values of x.

One of the striking properties of the natural exponential function in differential calculus is that it doesn't change during the process of differentiation. This is captured in the following theorem whose proof we skip.

Theorem 5*

(1) The derivative of e^x w.r.t., x is e^x; i.e., $\dfrac{d}{dx}(e^x) = e^x$.

(2) The derivative of $\log x$ w.r.t., x is $\dfrac{1}{x}$; i.e., $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$.

* *Please see supplementary material on Page 286.*

Example 29 Differentiate the following w.r.t. x:

(i) e^{-x} (ii) $\sin(\log x),\ x > 0$ (iii) $\cos^{-1}(e^x)$ (iv) $e^{\cos x}$

Solution

(i) Let $y = e^{-x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{-x} \cdot \frac{d}{dx}(-x) = -e^{-x}$$

(ii) Let $y = \sin(\log x)$. Using chain rule, we have

$$\frac{dy}{dx} = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

(iii) Let $y = \cos^{-1}(e^x)$. Using chain rule, we have

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x) = \frac{-e^x}{\sqrt{1-e^{2x}}}$$

(iv) Let $y = e^{\cos x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{\cos x} \cdot (-\sin x) = -(\sin x)\, e^{\cos x}$$

EXERCISE 5.4

Differentiate the following w.r.t. x:

1. $\frac{e^x}{\sin x}$
2. $e^{\sin^{-1} x}$
3. e^{x^3}
4. $\sin(\tan^{-1} e^{-x})$
5. $\log(\cos e^x)$
6. $e^x + e^{x^2} + \ldots + e^{x^5}$
7. $\sqrt{e^{\sqrt{x}}},\ x > 0$
8. $\log(\log x),\ x > 1$
9. $\frac{\cos x}{\log x},\ x > 0$
10. $\cos(\log x + e^x),\ x > 0$

5.5. Logarithmic Differentiation

In this section, we will learn to differentiate certain special class of functions given in the form

$$y = f(x) = [u(x)]^{v(x)}$$

By taking logarithm (to base e) the above may be rewritten as

$$\log y = v(x)\log[u(x)]$$

Using chain rule we may differentiate this to get

$$\frac{1}{y}\cdot\frac{dy}{dx} = v(x)\cdot\frac{1}{u(x)}\cdot u'(x) + v'(x)\cdot \log\,[u(x)]$$

which implies that

$$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}\cdot u'(x) + v'(x)\cdot\log\left[u(x)\right]\right]$$

The main point to be noted in this method is that $f(x)$ and $u(x)$ must always be positive as otherwise their logarithms are not defined. This process of differentiation is known as *logarithms differentiation* and is illustrated by the following examples:

Example 30 Differentiate $\sqrt{\dfrac{(x-3)\,(x^2+4)}{3x^2+4x+5}}$ w.r.t. x.

Solution Let $y = \sqrt{\dfrac{(x-3)\,(x^2+4)}{(3x^2+4x+5)}}$

Taking logarithm on both sides, we have

$$\log y = \frac{1}{2}\,[\log\,(x-3) + \log\,(x^2+4) - \log\,(3x^2+4x+5)]$$

Now, differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

or

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$= \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

Example 31 Differentiate a^x w.r.t. x, where a is a positive constant.

Solution Let $y = a^x$. Then

$$\log y = x \log a$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \log a$$

or $$\frac{dy}{dx} = y \log a$$

Thus $$\frac{d}{dx}(a^x) = a^x \log a$$

Alternatively $$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\log a}) = e^{x\log a}\frac{d}{dx}(x\log a)$$

$$= e^{x \log a} . \log a = a^x \log a.$$

Example 32 Differentiate $x^{\sin x}$, $x > 0$ w.r.t. x.

Solution Let $y = x^{\sin x}$. Taking logarithm on both sides, we have

$$\log y = \sin x \log x$$

Therefore $$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sin x)$$

or $$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{1}{x} + \log x \cos x$$

or $$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x - 1}\cdot\sin x + x^{\sin x}\cdot\cos x \log x$$

Example 33 Find $\frac{dy}{dx}$, if $y^x + x^y + x^x = a^b$.

Solution Given that $y^x + x^y + x^x = a^b$.

Putting $u = y^x$, $v = x^y$ and $w = x^x$, we get $u + v + w = a^b$

Therefore $$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \quad \text{... (1)}$$

Now, $u = y^x$. Taking logarithm on both sides, we have

$$\log u = x \log y$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)$$

$$= x\frac{1}{y}\cdot\frac{dy}{dx} + \log y\cdot 1$$

So
$$\frac{du}{dx} = u\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \quad \text{... (2)}$$

Also $v = x^y$

Taking logarithm on both sides, we have

$$\log v = y \log x$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\cdot\frac{dv}{dx} = y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx}$$

$$= y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

So
$$\frac{dv}{dx} = v\left[\frac{y}{x} + \log x\frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right] \quad \text{... (3)}$$

Again $w = x^x$

Taking logarithm on both sides, we have

$$\log w = x \log x.$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{w}\cdot\frac{dw}{dx} = x\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{x} + \log x\cdot 1$$

i.e.
$$\frac{dw}{dx} = w\,(1 + \log x)$$

$$= x^x\,(1 + \log x) \quad \text{... (4)}$$

From (1), (2), (3), (4), we have

$$y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)+x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+x^x(1+\log x)=0$$

or $$(x\,.\,y^{x-1}+x^y\,.\,\log x)\frac{dy}{dx}=-x^x(1+\log x)-y\,.\,x^{y-1}-y^x\log y$$

Therefore $$\frac{dy}{dx}=\frac{-[y^x\log y+y\,.\,x^{y-1}+x^x(1+\log x)]}{x\,.\,y^{x-1}+x^y\log x}$$

EXERCISE 5.5

Differentiate the functions given in Exercises 1 to 11 w.r.t. x.

1. $\cos x\,.\,\cos 2x\,.\,\cos 3x$

2. $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

3. $(\log x)^{\cos x}$

4. $x^x-2^{\sin x}$

5. $(x+3)^2\,.\,(x+4)^3\,.\,(x+5)^4$

6. $\left(x+\dfrac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$

7. $(\log x)^x+x^{\log x}$

8. $(\sin x)^x+\sin^{-1}\sqrt{x}$

9. $x^{\sin x}+(\sin x)^{\cos x}$

10. $x^{x\cos x}+\dfrac{x^2+1}{x^2-1}$

11. $(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Find $\dfrac{dy}{dx}$ of the functions given in Exercises 12 to 15.

12. $x^y+y^x=1$

13. $y^x=x^y$

14. $(\cos x)^y=(\cos y)^x$

15. $xy=e^{(x-y)}$

16. Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

17. Differentiate $(x^2-5x+8)(x^3+7x+9)$ in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial.

(iii) by logarithmic differentiation.

Do they all give the same answer?

18. If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.\ v.\ w) = \frac{du}{dx}\ v.\ w + u\ .\ \frac{dv}{dx}\ .\ w + u\ .\ v\ \frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

5.6 Derivatives of Functions in Parametric Forms

Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between them is expressed via a third variable. The third variable is called the parameter. More precisely, a relation expressed between two variables x and y in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with t as a parameter.

In order to find derivative of function in such form, we have by chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left(\text{whenever } \frac{dx}{dt} \neq 0\right)$$

Thus $$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \left(\text{as } \frac{dy}{dt} = g'(t) \text{ and } \frac{dx}{dt} = f'(t)\right) \text{ [provided } f'(t) \neq 0]$$

Example 34 Find $\frac{dy}{dx}$, if $x = a \cos \theta$, $y = a \sin \theta$.

Solution Given that

$$x = a \cos \theta,\ y = a \sin \theta$$

Therefore $$\frac{dx}{d\theta} = -a \sin \theta,\ \frac{dy}{d\theta} = a \cos \theta$$

Hence $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \cos\theta}{-a \sin\theta} = -\cot\theta$$

Example 35 Find $\frac{dy}{dx}$, if $x = at^2$, $y = 2at$.

Solution Given that $x = at^2$, $y = 2at$

So $$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

Example 36 Find $\frac{dy}{dx}$, if $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$.

Solution We have $\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = a\,(\sin\theta)$

Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2}$$

Note It may be noted here that $\frac{dy}{dx}$ is expressed in terms of parameter only without directly involving the main variables x and y.

Example 37 Find $\frac{dy}{dx}$, if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution Let $x = a\cos^3\theta$, $y = a\sin^3\theta$. Then

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a\cos^3\theta)^{\frac{2}{3}} + (a\sin^3\theta)^{\frac{2}{3}}$$
$$= a^{\frac{2}{3}}(\cos^2\theta + (\sin^2\theta) = a^{\frac{2}{3}}$$

Hence, $x = a\cos^3\theta$, $y = a\sin^3\theta$ is parametric equation of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Now $$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta = -\sqrt[3]{\frac{y}{x}}$$

EXERCISE 5.6

If x and y are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find $\frac{dy}{dx}$.

1. $x = 2at^2,\ y = at^4$
2. $x = a\cos\theta,\ y = b\cos\theta$
3. $x = \sin t,\ y = \cos 2t$
4. $x = 4t,\ y = \frac{4}{t}$
5. $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$
6. $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$
7. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$
8. $x = a\left(\cos t + \log\tan\frac{t}{2}\right)\ y = a\sin t$
9. $x = a\sec\theta,\ y = b\tan\theta$
10. $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$
11. If $x = \sqrt{a^{\sin^{-1}t}},\ y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

5.7 Second Order Derivative

Let $y = f(x)$. Then

$$\frac{dy}{dx} = f'(x) \qquad \ldots (1)$$

If $f'(x)$ is differentiable, we may differentiate (1) again w.r.t. x. Then, the left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is called the *second order derivative* of y w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$. It is also denoted by $D^2 y$ or y'' or y_2 if $y = f(x)$. We remark that higher order derivatives may be defined similarly.

Example 38 Find $\frac{d^2 y}{dx^2}$, if $y = x^3 + \tan x$.

Solution Given that $y = x^3 + \tan x$. Then

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

Therefore $$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(3x^2 + \sec^2 x\right)$$

$$= 6x + 2 \sec x \,.\, \sec x \tan x = 6x + 2 \sec^2 x \tan x$$

Example 39 If $y = A \sin x + B \cos x$, then prove that $\frac{d^2 y}{dx^2} + y = 0$.

Solution We have

$$\frac{dy}{dx} = A \cos x - B \sin x$$

and $$\frac{d^2 y}{dx^2} = \frac{d}{dx}(A \cos x - B \sin x)$$

$$= - A \sin x - B \cos x = - y$$

Hence $$\frac{d^2 y}{dx^2} + y = 0$$

Example 40 If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Solution Given that $y = 3e^{2x} + 2e^{3x}$. Then

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x} = 6\,(e^{2x} + e^{3x})$$

Therefore $$\frac{d^2 y}{dx^2} = 12e^{2x} + 18e^{3x} = 6\,(2e^{2x} + 3e^{3x})$$

Hence $$\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 6\,(2e^{2x} + 3e^{3x})$$

$$- 30\,(e^{2x} + e^{3x}) + 6\,(3e^{2x} + 2e^{3x}) = 0$$

Example 41 If $y = \sin^{-1} x$, show that $(1 - x^2) \dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} = 0$.

Solution We have $y = \sin^{-1}x$. Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

or
$$\sqrt{(1-x^2)}\,\frac{dy}{dx} = 1$$

So
$$\frac{d}{dx}\left(\sqrt{(1-x^2)}\,.\,\frac{dy}{dx}\right) = 0$$

or
$$\sqrt{(1-x^2)}\cdot\frac{d^2 y}{dx^2} + \frac{dy}{dx}\cdot\frac{d}{dx}\left(\sqrt{(1-x^2)}\right) = 0$$

or
$$\sqrt{(1-x^2)}\cdot\frac{d^2 y}{dx^2} - \frac{dy}{dx}\cdot\frac{2x}{2\sqrt{1-x^2}} = 0$$

Hence
$$(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} = 0$$

Alternatively, Given that $y = \sin^{-1} x$, we have

$$y_1 = \frac{1}{\sqrt{1-x^2}}, \text{ i.e., } \left(1-x^2\right) y_1^2 = 1$$

So
$$(1-x^2)\,.\,2y_1 y_2 + y_1^2(0-2x) = 0$$

Hence
$$(1 - x^2)\, y_2 - x y_1 = 0$$

EXERCISE 5.7

Find the second order derivatives of the functions given in Exercises 1 to 10.

1. $x^2 + 3x + 2$ **2.** x^{20} **3.** $x\,.\cos x$

4. $\log x$ **5.** $x^3 \log x$ **6.** $e^x \sin 5x$

7. $e^{6x} \cos 3x$ **8.** $\tan^{-1} x$ **9.** $\log(\log x)$

10. $\sin(\log x)$

11. If $y = 5\cos x - 3\sin x$, prove that $\dfrac{d^2 y}{dx^2} + y = 0$

12. If $y = \cos^{-1} x$, Find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

13. If $y = 3 \cos (\log x) + 4 \sin (\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

14. If $y = Ae^{mx} + Be^{nx}$, show that $\dfrac{d^2y}{dx^2} - (m+n)\dfrac{dy}{dx} + mny = 0$

15. If $y = 500e^{7x} + 600e^{-7x}$, show that $\dfrac{d^2y}{dx^2} = 49y$

16. If $e^y (x + 1) = 1$, show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

17. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

5.8 Mean Value Theorem

In this section, we will state two fundamental results in Calculus without proof. We shall also learn the geometric interpretation of these theorems.

Theorem 6 (Rolle's Theorem) Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b), such that $f(a) = f(b)$, where a and b are some real numbers. Then there exists some c in (a, b) such that $f'(c) = 0$.

In Fig 5.12 and 5.13, graphs of a few typical differentiable functions satisfying the hypothesis of Rolle's theorem are given.

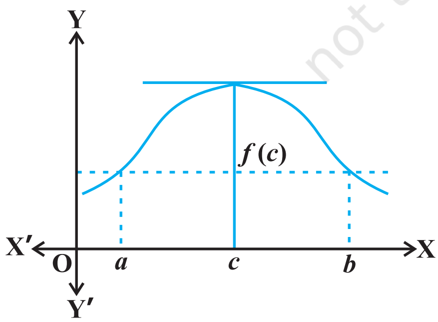

Fig 5.12

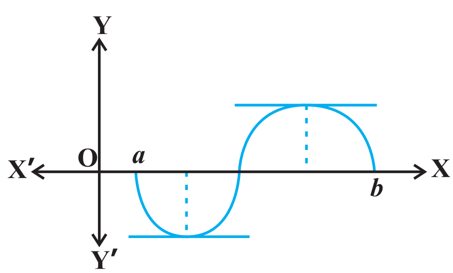

Fig 5.13

Observe what happens to the slope of the tangent to the curve at various points between a and b. In each of the graphs, the slope becomes zero at least at one point. That is precisely the claim of the Rolle's theorem as the slope of the tangent at any point on the graph of $y = f(x)$ is nothing but the derivative of $f(x)$ at that point.

Theorem 7 (Mean Value Theorem) Let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Observe that the Mean Value Theorem (MVT) is an extension of Rolle's theorem. Let us now understand a geometric interpretation of the MVT. The graph of a function $y = f(x)$ is given in the Fig 5.14. We have already interpreted $f'(c)$ as the slope of the tangent to the curve $y = f(x)$ at $(c, f(c))$. From the Fig 5.14 it is clear that $\frac{f(b) - f(a)}{b - a}$ is the slope of the secant drawn between $(a, f(a))$ and $(b, f(b))$. The MVT states that there is a point c in (a, b) such that the slope of the tangent at $(c, f(c))$ is same as the slope of the secant between $(a, f(a))$ and $(b, f(b))$. In other words, there is a point c in (a, b) such that the tangent at $(c, f(c))$ is parallel to the secant between $(a, f(a))$ and $(b, f(b))$.

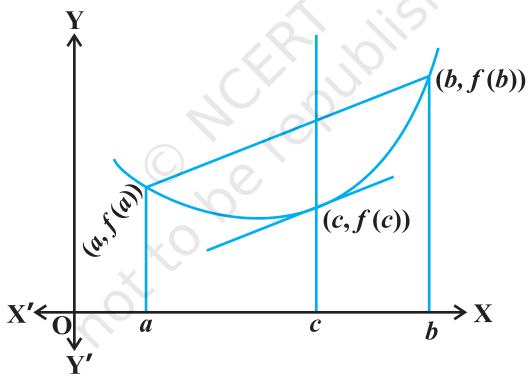

Fig 5.14

Example 42 Verify Rolle's theorem for the function $y = x^2 + 2$, $a = -2$ and $b = 2$.

Solution The function $y = x^2 + 2$ is continuous in $[-2, 2]$ and differentiable in $(-2, 2)$. Also $f(-2) = f(2) = 6$ and hence the value of $f(x)$ at -2 and 2 coincide. Rolle's theorem states that there is a point $c \in (-2, 2)$, where $f'(c) = 0$. Since $f'(x) = 2x$, we get $c = 0$. Thus at $c = 0$, we have $f'(c) = 0$ and $c = 0 \in (-2, 2)$.

Example 43 Verify Mean Value Theorem for the function $f(x) = x^2$ in the interval $[2, 4]$.

Solution The function $f(x) = x^2$ is continuous in $[2, 4]$ and differentiable in $(2, 4)$ as its derivative $f'(x) = 2x$ is defined in $(2, 4)$.

Now, $f(2) = 4$ and $f(4) = 16$. Hence

$$\frac{f(b)-f(a)}{b-a}=\frac{16-4}{4-2}=6$$

MVT states that there is a point $c \in (2, 4)$ such that $f'(c) = 6$. But $f'(x) = 2x$ which implies $c = 3$. Thus at $c = 3 \in (2, 4)$, we have $f'(c) = 6$.

EXERCISE 5.8

1. Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.
2. Examine if Rolle's theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's theorem from these example?
   (i) $f(x) = [x]$ for $x \in [5, 9]$ (ii) $f(x) = [x]$ for $x \in [-2, 2]$
   (iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$
3. If $f : [-5, 5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.
4. Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.
5. Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.
6. Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Miscellaneous Examples

Example 44 Differentiate w.r.t. x, the following function:

(i) $\sqrt{3x+2}+\dfrac{1}{\sqrt{2x^2+4}}$ (ii) $e^{\sec^2 x}+3\cos^{-1}x$ (iii) $\log_7(\log x)$

Solution

(i) Let $y = \sqrt{3x+2}+\dfrac{1}{\sqrt{2x^2+4}} = (3x+2)^{\frac{1}{2}}+(2x^2+4)^{-\frac{1}{2}}$

Note that this function is defined at all real numbers $x>-\dfrac{2}{3}$. Therefore

$$\frac{dy}{dx}=\frac{1}{2}(3x+2)^{\frac{1}{2}-1}\cdot\frac{d}{dx}(3x+2)+\left(-\frac{1}{2}\right)(2x^2+4)^{-\frac{1}{2}-1}\cdot\frac{d}{dx}(2x^2+4)$$

$$= \frac{1}{2}(3x+2)^{-\frac{1}{2}} \cdot (3) - \left(\frac{1}{2}\right)(2x^2+4)^{-\frac{3}{2}} \cdot 4x$$

$$= \frac{3}{2\sqrt{3x+2}} - \frac{2x}{(2x^2+4)^{\frac{3}{2}}}$$

This is defined for all real numbers $x > -\frac{2}{3}$.

(ii) Let $y = e^{\sec^2 x} + 3\cos^{-1} x$

This is defined at every real number in $[-1, 1]$. Therefore

$$\frac{dy}{dx} = e^{\sec^2 x} \cdot \frac{d}{dx}(\sec^2 x) + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= e^{\sec^2 x} \cdot 2\sec x \frac{d}{dx}(\sec x) + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= 2\sec x\,(\sec x \tan x)\, e^{\sec^2 x} + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= 2\sec^2 x \tan x\, e^{\sec^2 x} + 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

Observe that the derivative of the given function is valid only in (– 1, 1) as the derivative of $\cos^{-1} x$ exists only in (– 1, 1).

(iii) Let $y = \log_7 (\log x) = \frac{\log(\log x)}{\log 7}$ (by change of base formula).

The function is defined for all real numbers $x > 1$. Therefore

$$\frac{dy}{dx} = \frac{1}{\log 7}\frac{d}{dx}(\log(\log x))$$

$$= \frac{1}{\log 7}\frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{x \log 7 \log x}$$

Example 45 Differentiate the following w.r.t. x.

(i) $\cos^{-1}(\sin x)$ (ii) $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$ (iii) $\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$

Solution

(i) Let $f(x) = \cos^{-1}(\sin x)$. Observe that this function is defined for all real numbers. We may rewrite this function as

$$f(x) = \cos^{-1}(\sin x)$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= \frac{\pi}{2} - x$$

Thus $f'(x) = -1$.

(ii) Let $f(x) = \tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$. Observe that this function is defined for all real numbers, where $\cos x \neq -1$; i.e., at all odd multiples of π. We may rewrite this function as

$$f(x) = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{x}{2}\right)\right] = \frac{x}{2}$$

Observe that we could cancel $\cos\left(\dfrac{x}{2}\right)$ in both numerator and denominator as it is not equal to zero. Thus $f'(x) = \dfrac{1}{2}$.

(iii) Let $f(x) = \sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$. To find the domain of this function we need to find all x such that $-1 \leq \dfrac{2^{x+1}}{1+4^x} \leq 1$. Since the quantity in the middle is always positive, we need to find all x such that $\frac{2^{x+1}}{1+4^x} \le 1$, i.e., all x such that $2^{x+1} \le 1 + 4^x$. We may rewrite this as $2 \le \frac{1}{2^x} + 2^x$ which is true for all x. Hence the function is defined at every real number. By putting $2^x = \tan\theta$, this function may be rewritten as

$$f(x) = \sin^{-1}\left[\frac{2^{x+1}}{1+4^x}\right]$$

$$= \sin^{-1}\left[\frac{2^x \cdot 2}{1+(2^x)^2}\right]$$

$$= \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= \sin^{-1}[\sin 2\theta]$$

$$= 2\theta = 2\tan^{-1}(2^x)$$

Thus $$f'(x) = 2\cdot\frac{1}{1+(2^x)^2}\cdot\frac{d}{dx}(2^x)$$

$$= \frac{2}{1+4^x}\cdot(2^x)\log 2$$

$$= \frac{2^{x+1}\log 2}{1+4^x}$$

Example 46 Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $0 < x < \pi$.

Solution The function $y = (\sin x)^{\sin x}$ is defined for all positive real numbers. Taking logarithms, we have

$$\log y = \log(\sin x)^{\sin x} = \sin x \log(\sin x)$$

Then $$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\sin x \log(\sin x))$$

$$= \cos x \log(\sin x) + \sin x \,.\, \frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)$$

$$= \cos x \log(\sin x) + \cos x$$

$$= (1 + \log(\sin x))\cos x$$

Thus $\dfrac{dy}{dx} = y((1 + \log(\sin x))\cos x) = (1 + \log(\sin x))(\sin x)^{\sin x}\cos x$

Example 47 For a positive constant a find $\dfrac{dy}{dx}$, where

$$y = a^{t+\frac{1}{t}}, \text{ and } x = \left(t + \frac{1}{t}\right)^a$$

Solution Observe that both y and x are defined for all real $t \neq 0$. Clearly

$$\frac{dy}{dt} = \frac{d}{dt}\left(a^{t+\frac{1}{t}}\right) = a^{t+\frac{1}{t}}\frac{d}{dt}\left(t+\frac{1}{t}\right)\cdot\log a$$

$$= a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a$$

Similarly

$$\frac{dx}{dt} = a\left[t+\frac{1}{t}\right]^{a-1}\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right)$$

$$= a\left[t+\frac{1}{t}\right]^{a-1}\cdot\left(1-\frac{1}{t^2}\right)$$

$\dfrac{dx}{dt} \neq 0$ only if $t \neq \pm 1$. Thus for $t \neq \pm 1$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a}{a\left[t+\frac{1}{t}\right]^{a-1}\cdot\left(1-\frac{1}{t^2}\right)}$$

$$= \frac{a^{t+\frac{1}{t}}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$$

Example 48 Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.

Solution Let $u(x) = \sin^2 x$ and $v(x) = e^{\cos x}$. We want to find $\dfrac{du}{dv} = \dfrac{du/dx}{dv/dx}$. Clearly

$$\frac{du}{dx} = 2\sin x\cos x \text{ and } \frac{dv}{dx} = e^{\cos x}(-\sin x) = -(\sin x)\,e^{\cos x}$$

Thus $$\frac{du}{dv} = \frac{2\sin x \cos x}{-\sin x\, e^{\cos x}} = -\frac{2\cos x}{e^{\cos x}}$$

Miscellaneous Exercise on Chapter 5

Differentiate w.r.t. x the function in Exercises 1 to 11.

1. $(3x^2 - 9x + 5)^9$

2. $\sin^3 x + \cos^6 x$

3. $(5x)^{3\cos 2x}$

4. $\sin^{-1}(x\sqrt{x}),\ 0 \le x \le 1$

5. $\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},\ -2 < x < 2$

6. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right],\ 0 < x < \dfrac{\pi}{2}$

7. $(\log x)^{\log x},\ x > 1$

8. $\cos(a\cos x + b\sin x)$, for some constant a and b.

9. $(\sin x - \cos x)^{(\sin x - \cos x)},\ \dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$

10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

11. $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$

12. Find $\dfrac{dy}{dx}$, if $y = 12(1 - \cos t),\ x = 10(t - \sin t),\ -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

13. Find $\dfrac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2},\ 0 < x < 1$

14. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for , $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

15. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

16. If cos $y = x$ cos $(a + y)$, with cos $a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$.

17. If $x = a$ (cos $t + t$ sin t) and $y = a$ (sin $t - t$ cos t), find $\dfrac{d^2y}{dx^2}$.

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

19. Using mathematical induction prove that $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

20. Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

22. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

23. If $y = e^{a\cos^{-1}x}$, $-1 \leq x \leq 1$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2y = 0$.

Summary

- A real valued function is ***continuous*** at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.
- Sum, difference, product and quotient of continuous functions are continuous. i.e., if f and g are continuous functions, then

  $(f \pm g)(x) = f(x) \pm g(x)$ is continuous.

  $(f \,.\, g)(x) = f(x) \,.\, g(x)$ is continuous.

  $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ (wherever $g(x) \neq 0$) is continuous.
- Every differentiable function is continuous, but the converse is not true.

- Chain rule is rule to differentiate composites of functions. If $f = v \circ u$, $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist then

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

- Following are some of the standard derivatives (in appropriate domains):

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(e^x\right) = e^x \qquad \frac{d}{dx}\left(\log x\right) = \frac{1}{x}$$

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$. Here both $f(x)$ and $u(x)$ need to be positive for this technique to make sense.
- ***Rolle's Theorem***: If $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there exists some c in (a, b) such that $f'(c) = 0$.
- ***Mean Value Theorem***: If $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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Chapter 6

APPLICATION OF DERIVATIVES

❖ *With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature." — WHITEHEAD* ❖

6.1 Introduction

In Chapter 5, we have learnt how to find derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithmic functions. In this chapter, we will study applications of the derivative in various disciplines, e.g., in engineering, science, social science, and many other fields. For instance, we will learn how the derivative can be used (i) to determine rate of change of quantities, (ii) to find the equations of tangent and normal to a curve at a point, (iii) to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value (locally) of a function occurs. We will also use derivative to find intervals on which a function is increasing or decreasing. Finally, we use the derivative to find approximate value of certain quantities.

6.2 Rate of Change of Quantities

Recall that by the derivative $\frac{ds}{dt}$, we mean the rate of change of distance s with respect to the time t. In a similar fashion, whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

Further, if two variables x and y are varying with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to t.

Let us consider some examples.

Example 1 Find the rate of change of the area of a circle per second with respect to its radius r when $r = 5$ cm.

Solution The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of change of the area A with respect to its radius r is given by $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$.

When $r = 5$ cm, $\frac{dA}{dr} = 10\pi$. Thus, the area of the circle is changing at the rate of 10π cm^2/s.

Example 2 The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres ?

Solution Let x be the length of a side, V be the volume and S be the surface area of the cube. Then, $V = x^3$ and $S = 6x^2$, where x is a function of time t.

Now $\frac{dV}{dt} = 9\text{cm}^3/\text{s}$ (Given)

Therefore $9 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3) \cdot \frac{dx}{dt}$ (By Chain Rule)

$$= 3x^2 \cdot \frac{dx}{dt}$$

or $\frac{dx}{dt} = \frac{3}{x^2}$ ... (1)

Now $\frac{dS}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2) \cdot \frac{dx}{dt}$ (By Chain Rule)

$$= 12x \cdot \left(\frac{3}{x^2}\right) = \frac{36}{x} \quad \text{(Using (1))}$$

Hence, when $x = 10$ cm, $\frac{dS}{dt} = 3.6 \text{ cm}^2/\text{s}$

Example 3 A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Solution The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of change of area A with respect to time t is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{(By Chain Rule)}$$

It is given that $\qquad \frac{dr}{dt} = 4\text{cm/s}$

Therefore, when $r = 10$ cm, $\qquad \frac{dA}{dt} = 2\pi(10)(4) = 80\pi$

Thus, the enclosed area is increasing at the rate of 80π cm²/s, when $r = 10$ cm.

> **Note** $\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

Example 4 The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

Solution Since the length x is decreasing and the width y is increasing with respect to time, we have

$$\frac{dx}{dt} = -3 \text{ cm/min} \qquad \text{and} \qquad \frac{dy}{dt} = 2 \text{ cm/min}$$

(a) The perimeter P of a rectangle is given by

$$P = 2(x + y)$$

Therefore $\qquad \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-3+2) = -2 \text{ cm/min}$

(b) The area A of the rectangle is given by

$$A = x \cdot y$$

Therefore $\qquad \frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$

$$= -3(6) + 10(2) \qquad (\text{as } x = 10 \text{ cm and } y = 6 \text{ cm})$$

$$= 2 \text{ cm}^2/\text{min}$$

Example 5 The total cost C(x) in Rupees, associated with the production of x units of an item is given by

$$C(x) = 0.005\, x^3 - 0.02\, x^2 + 30x + 5000$$

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Solution Since marginal cost is the rate of change of total cost with respect to the output, we have

$$\text{Marginal cost (MC)} = \frac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$$

$$\text{When } x = 3, \text{ MC} = 0.015(3^2) - 0.04(3) + 30$$

$$= 0.135 - 0.12 + 30 = 30.015$$

Hence, the required marginal cost is ₹ 30.02 (nearly).

Example 6 The total revenue in Rupees received from the sale of x units of a product is given by R(x) = $3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

Solution Since marginal revenue is the rate of change of total revenue with respect to the number of units sold, we have

$$\text{Marginal Revenue (MR)} = \frac{dR}{dx} = 6x + 36$$

$$\text{When } x = 5, \text{ MR} = 6(5) + 36 = 66$$

Hence, the required marginal revenue is ₹ 66.

EXERCISE 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when
   (a) $r = 3$ cm (b) $r = 4$ cm
2. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?
3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.
4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?
7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$cm and $y = 6$cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.
8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.
9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.
10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?
11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.
12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?
13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.
14. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?
15. The total cost C (x) in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$

Find the marginal cost when 17 units are produced.
16. The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15.$$

Find the marginal revenue when $x = 7$.

Choose the correct answer for questions 17 and 18.

17. The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is
(A) 10π (B) 12π (C) 8π (D) 11π

18. The total revenue in Rupees received from the sale of x units of a product is given by
$R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is
(A) 116 (B) 96 (C) 90 (D) 126

6.3 Increasing and Decreasing Functions

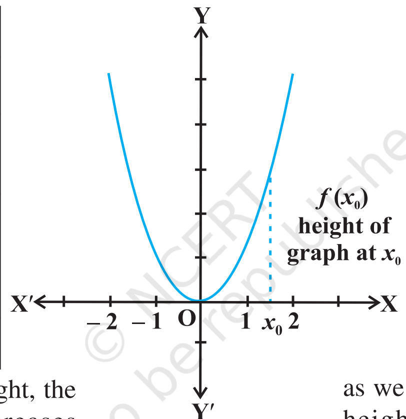
In this section, we will use differentiation to find out whether a function is increasing or decreasing or none.

Consider the function f given by $f(x) = x^2$, $x \in \mathbf{R}$. The graph of this function is a parabola as given in Fig 6.1.

Values left to origin

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0

as we move from left to right, the height of the graph decreases



Values right to origin

x	$f(x) = x^2$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

as we move from left to right, the height of the graph increases

Fig 6.1

First consider the graph (Fig 6.1) to the right of the origin. Observe that as we move from left to right along the graph, the height of the graph continuously increases. For this reason, the function is said to be increasing for the real numbers $x > 0$.

Now consider the graph to the left of the origin and observe here that as we move from left to right along the graph, the height of the graph continuously decreases. Consequently, the function is said to be decreasing for the real numbers $x < 0$.

We shall now give the following analytical definitions for a function which is increasing or decreasing on an interval.

Definition 1 Let I be an interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

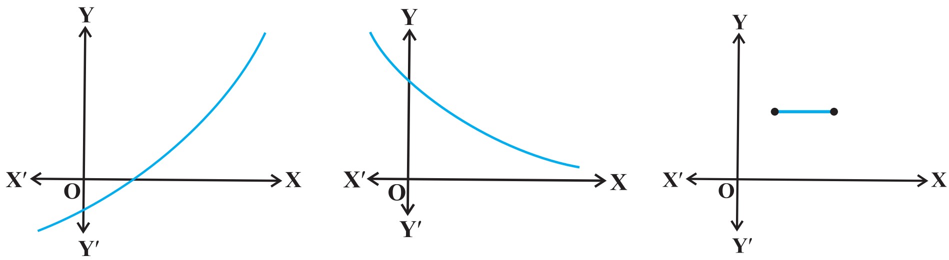
(ii) decreasing on I, if x_1, x_2 in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

(iii) constant on I, if $f(x) = c$ for all $x \in$ I, where c is a constant.

(iv) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

(v) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

For graphical representation of such functions see Fig 6.2.



Strictly Increasing function
(i)

Strictly Decreasing function
(ii)

Neither Increasing nor Decreasing function
(iii)

Fig 6.2

We shall now define when a function is increasing or decreasing at a point.

Definition 2 Let x_0 be a point in the domain of definition of a real valued function f. Then f is said to be increasing, decreasing at x_0 if there exists an open interval I containing x_0 such that f is increasing, decreasing, respectively, in I.

Let us clarify this definition for the case of increasing function.

Example 7 Show that the function given by $f(x) = 7x - 3$ is increasing on **R**.

Solution Let x_1 and x_2 be any two numbers in **R**. Then

$$x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$$

Thus, by Definition 1, it follows that f is strictly increasing on **R**.

We shall now give the first derivative test for increasing and decreasing functions. The proof of this test requires the Mean Value Theorem studied in Chapter 5.

Theorem 1 Let f be continuous on $[a, b]$ and differentiable on the open interval (a,b). Then

(a) f is increasing in $[a,b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in $[a,b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in $[a,b]$ if $f'(x) = 0$ for each $x \in (a, b)$

Proof (a) Let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$.

Then, by Mean Value Theorem (Theorem 8 in Chapter 5), there exists a point c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)\ (x_2 - x_1)$$

i.e. $\quad f(x_2) - f(x_1) > 0 \quad$ (as $f'(c) > 0$ (given))

i.e. $\quad f(x_2) > f(x_1)$

Thus, we have

$$x_1 < x_2 \quad f(x_1) \quad f(x_2), \text{ for all } x_1, x_2 \quad [a,b]$$

Hence, f is an increasing function in $[a,b]$.

The proofs of part (b) and (c) are similar. It is left as an exercise to the reader.

Remarks

There is a more generalised theorem, which states that if $f¢(x) > 0$ for x in an interval excluding the end points and f is continuous in the interval, then f is increasing. Similarly, if $f¢(x) < 0$ for x in an interval excluding the end points and f is continuous in the interval, then f is decreasing.

Example 8 Show that the function f given by

$$f(x) = x^3 - 3x^2 + 4x,\ x \in \mathbf{R}$$

is increasing on **R**.

Solution Note that

$$f'(x) = 3x^2 - 6x + 4$$
$$= 3(x^2 - 2x + 1) + 1$$
$$= 3(x - 1)^2 + 1 > 0, \text{ in every interval of } \mathbf{R}$$

Therefore, the function f is increasing on **R**.

Example 9 Prove that the function given by $f(x) = \cos x$ is

(a) decreasing in $(0, \pi)$

(b) increasing in $(\pi, 2\pi)$, and

(c) neither increasing nor decreasing in $(0, 2\pi)$.

Solution Note that $f'(x) = -\sin x$

(a) Since for each $x \in (0, \pi)$, $\sin x > 0$, we have $f'(x) < 0$ and so f is decreasing in $(0, \pi)$.

(b) Since for each $x \in (\pi, 2\pi)$, $\sin x < 0$, we have $f'(x) > 0$ and so f is increasing in $(\pi, 2\pi)$.

(c) Clearly by (a) and (b) above, f is neither increasing nor decreasing in $(0, 2\pi)$.

Example 10 Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is

(a) increasing (b) decreasing

Solution We have

$$f(x) = x^2 - 4x + 6$$

or $$f'(x) = 2x - 4$$

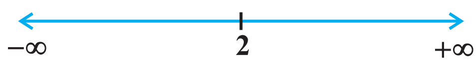


Fig 6.3

Therefore, $f'(x) = 0$ gives $x = 2$. Now the point $x = 2$ divides the real line into two disjoint intervals namely, $(-\infty, 2)$ and $(2, \infty)$ (Fig 6.3). In the interval $(-\infty, 2)$, $f'(x) = 2x - 4 < 0$.

Therefore, f is decreasing in this interval. Also, in the interval $(2, \infty)$, $f'(x) > 0$ and so the function f is increasing in this interval.

Example 11 Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = 4x^3 - 6x^2 - 72x + 30$$

or
$$\begin{aligned} f'(x) &= 12x^2 - 12x - 72 \\ &= 12(x^2 - x - 6) \\ &= 12(x - 3)(x + 2) \end{aligned}$$

Therefore, $f'(x) = 0$ gives $x = -2, 3$. The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Fig 6.4

In the intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in the interval $(-2, 3)$, $f'(x)$ is negative. Consequently, the function f is increasing in the intervals $(-\infty, -2)$ and $(3, \infty)$ while the function is decreasing in the interval $(-2, 3)$. However, f is neither increasing nor decreasing in **R**.

Interval	Sign of $f'(x)$	Nature of function f
$(-\infty, -2)$	$(-)\ (-) > 0$	f is increasing
$(-2, 3)$	$(-)\ (+) < 0$	f is decreasing
$(3, \infty)$	$(+)\ (+) > 0$	f is increasing

Example 12 Find intervals in which the function given by $f(x) = \sin 3x$, $x \in \left[0, \frac{\pi}{2}\right]$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = \sin 3x$$

or

$$f'(x) = 3\cos 3x$$

Therefore, $f'(x) = 0$ gives $\cos 3x = 0$ which in turn gives $3x = \frac{\pi}{2}, \frac{3\pi}{2}$ (as $x \in \left[0, \frac{\pi}{2}\right]$ implies $3x \in \left[0, \frac{3\pi}{2}\right]$). So $x = \frac{\pi}{6}$ and $\frac{\pi}{2}$. The point $x = \frac{\pi}{6}$ divides the interval $\left[0, \frac{\pi}{2}\right]$ into two disjoint intervals $\left[0, \frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{\pi}{2}\right]$.

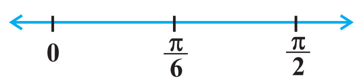


Fig 6.5

Now, $f'(x) > 0$ for all $x \in \left[0, \frac{\pi}{6}\right)$ as $0 \le x < \frac{\pi}{6} \Rightarrow 0 \le 3x < \frac{\pi}{2}$ and $f'(x) < 0$ for all $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ as $\frac{\pi}{6} < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3x < \frac{3\pi}{2}$.

Therefore, f is increasing in $\left[0, \frac{\pi}{6}\right)$ and decreasing in $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$.

Also, the given function is continuous at $x = 0$ and $x = \frac{\pi}{6}$. Therefore, by Theorem 1, f is increasing on $\left[0, \frac{\pi}{6}\right]$ and decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Example 13 Find the intervals in which the function f given by

$$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$$

is increasing or decreasing.

Solution We have

$$f(x) = \sin x + \cos x,$$

or $$f'(x) = \cos x - \sin x$$

Now $f'(x) = 0$ gives $\sin x = \cos x$ which gives that $x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \le x \le 2\pi$

The points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ divide the interval $[0, 2\pi]$ into three disjoint intervals, namely, $\left[0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$.

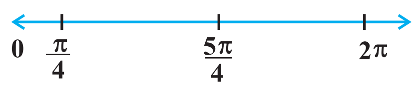


Fig 6.6

Note that $f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

or f is increasing in the intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$

Also $f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

or f is decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Interval	Sign of $f'(x)$	Nature of function
$\left[0, \frac{\pi}{4}\right)$	> 0	f is increasing
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	< 0	f is decreasing
$\left(\frac{5\pi}{4}, 2\pi\right]$	> 0	f is increasing

EXERCISE 6.2

1. Show that the function given by $f(x) = 3x + 17$ is increasing on **R**.

2. Show that the function given by $f(x) = e^{2x}$ is increasing on **R**.

3. Show that the function given by $f(x) = \sin x$ is

(a) increasing in $\left(0, \frac{\pi}{2}\right)$ (b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(c) neither increasing nor decreasing in $(0, \pi)$

4. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is

(a) increasing (b) decreasing

5. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) increasing (b) decreasing

6. Find the intervals in which the following functions are strictly increasing or decreasing:

(a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$

(c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$

(e) $(x + 1)^3 (x - 3)^3$

7. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.

8. Find the values of x for which $y = [x(x-2)]^2$ is an increasing function.

9. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

10. Prove that the logarithmic function is increasing on $(0, \infty)$.

11. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on $(-1, 1)$.

12. Which of the following functions are decreasing on $\left(0, \frac{\pi}{2}\right)$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

13. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ decreasing ?

(A) $(0,1)$ (B) $\left(\frac{\pi}{2}, \pi\right)$ (C) $\left(0, \frac{\pi}{2}\right)$ (D) None of these

14. For what values of a the function f given by $f(x) = x^2 + ax + 1$ is increasing on $[1, 2]$?

15. Let I be any interval disjoint from $[-1, 1]$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I.

16. Prove that the function f given by $f(x) = \log \sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

17. Prove that the function f given by $f(x) = \log |\cos x|$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$.

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**.

19. The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

6.4 Tangents and Normals

In this section, we shall use differentiation to find the equation of the tangent line and the normal line to a curve at a given point.

Recall that the equation of a straight line passing through a given point (x_0, y_0) having finite slope m is given by

$$y - y_0 = m(x - x_0)$$

Note that the slope of the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by $\left.\frac{dy}{dx}\right]_{(x_0, y_0)} (= f'(x_0))$. So the equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by

$$y - y_0 = f'(x_0)(x - x_0)$$

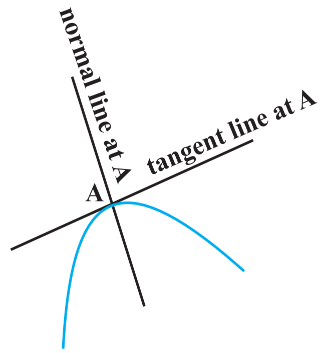


Fig 6.7

Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_0, y_0) is $\frac{-1}{f'(x_0)}$, if $f'(x_0) \neq 0$. Therefore, the equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is given by

$$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$$

i.e. $\quad (y - y_0) f'(x_0) + (x - x_0) = 0$

Note If a tangent line to the curve $y = f(x)$ makes an angle θ with x-axis in the positive direction, then $\frac{dy}{dx}$ = slope of the tangent = $\tan\theta$.

Particular cases

(i) If slope of the tangent line is zero, then $\tan\theta = 0$ and so $\theta = 0$ which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point (x_0, y_0) is given by $y = y_0$.

(ii) If $\theta \to \frac{\pi}{2}$, then $\tan\theta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $x = x_0$ (Why?).

Example 14 Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.

Solution The slope of the tangent at $x = 2$ is given by

$$\left.\frac{dy}{dx}\right]_{x=2} = \left.3x^2 - 1\right]_{x=2} = 11.$$

Example 15 Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

Solution Slope of tangent to the given curve at (x, y) is

$$\frac{dy}{dx} = \frac{1}{2}(4x-3)^{\frac{-1}{2}} 4 = \frac{2}{\sqrt{4x-3}}$$

The slope is given to be $\frac{2}{3}$.

So $$\frac{2}{\sqrt{4x-3}} = \frac{2}{3}$$

or $$4x - 3 = 9$$

or $$x = 3$$

Now $y = \sqrt{4x-3} - 1$. So when $x = 3$, $y = \sqrt{4(3)-3} - 1 = 2$.

Therefore, the required point is (3, 2).

Example 16 Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$.

Solution Slope of the tangent to the given curve at any point (x,y) is given by

$$\frac{dy}{dx} = \frac{2}{(x-3)^2}$$

But the slope is given to be 2. Therefore

$$\frac{2}{(x-3)^2} = 2$$

or $$(x-3)^2 = 1$$

or $$x - 3 = \pm 1$$

or $$x = 2, 4$$

Now $x = 2$ gives $y = 2$ and $x = 4$ gives $y = -2$. Thus, there are two tangents to the given curve with slope 2 and passing through the points (2, 2) and (4, – 2). The equation of tangent through (2, 2) is given by

$$y - 2 = 2(x - 2)$$

or $$y - 2x + 2 = 0$$

and the equation of the tangent through (4, – 2) is given by

$$y - (-2) = 2(x - 4)$$

or $$y - 2x + 10 = 0$$

Example 17 Find points on the curve $\frac{x^2}{4}+\frac{y^2}{25}=1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

Solution Differentiating $\frac{x^2}{4}+\frac{y^2}{25}=1$ with respect to x, we get

$$\frac{x}{2}+\frac{2y}{25}\frac{dy}{dx}=0$$

or

$$\frac{dy}{dx}=\frac{-25}{4}\frac{x}{y}$$

(i) Now, the tangent is parallel to the x-axis if the slope of the tangent is zero which gives $\frac{-25}{4}\frac{x}{y}=0$. This is possible if $x = 0$. Then $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $x = 0$ gives $y^2 = 25$, i.e., $y = \pm 5$.

Thus, the points at which the tangents are parallel to the x-axis are (0, 5) and (0, – 5).

(ii) The tangent line is parallel to y-axis if the slope of the normal is 0 which gives $\frac{4y}{25x}=0$, i.e., $y = 0$. Therefore, $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $y = 0$ gives $x = \pm 2$. Hence, the points at which the tangents are parallel to the y-axis are (2, 0) and (–2, 0).

Example 18 Find the equation of the tangent to the curve $y=\frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Solution Note that on x-axis, $y = 0$. So the equation of the curve, when $y = 0$, gives $x = 7$. Thus, the curve cuts the x-axis at (7, 0). Now differentiating the equation of the curve with respect to x, we obtain

$$\frac{dy}{dx}=\frac{1-y(2x-5)}{(x-2)(x-3)} \qquad \text{(Why?)}$$

or

$$\left.\frac{dy}{dx}\right]_{(7,0)}=\frac{1-0}{(5)(4)}=\frac{1}{20}$$

Therefore, the slope of the tangent at (7, 0) is $\frac{1}{20}$. Hence, the equation of the tangent at (7, 0) is

$$y - 0 = \frac{1}{20}(x - 7) \qquad \text{or} \qquad 20y - x + 7 = 0$$

Example 19 Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1).

Solution Differentiating $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ with respect to x, we get

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Therefore, the slope of the tangent at (1, 1) is $\left.\frac{dy}{dx}\right]_{(1,\,1)} = -1$.

So the equation of the tangent at (1, 1) is

$$y - 1 = -1\,(x - 1) \qquad \text{or} \qquad y + x - 2 = 0$$

Also, the slope of the normal at (1, 1) is given by

$$\frac{-1}{\text{slope of the tangent at }(1,1)} = 1$$

Therefore, the equation of the normal at (1, 1) is

$$y - 1 = 1\,(x - 1) \qquad \text{or} \qquad y - x = 0$$

Example 20 Find the equation of tangent to the curve given by

$$x = a\sin^3 t\,, \qquad y = b\cos^3 t \qquad \dots (1)$$

at a point where $t = \frac{\pi}{2}$.

Solution Differentiating (1) with respect to t, we get

$$\frac{dx}{dt} = 3a\sin^2 t\cos t \qquad \text{and} \qquad \frac{dy}{dt} = -3b\cos^2 t\sin t$$

or $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3b\cos^2 t \sin t}{3a\sin^2 t \cos t} = \frac{-b}{a}\frac{\cos t}{\sin t}$$

Therefore, slope of the tangent at $t = \frac{\pi}{2}$ is

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{2}} = \frac{-b\cos\frac{\pi}{2}}{a\sin\frac{\pi}{2}} = 0$$

Also, when $t = \frac{\pi}{2}$, $x = a$ and $y = 0$. Hence, the equation of tangent to the given curve at $t = \frac{\pi}{2}$, i.e., at $(a, 0)$ is

$$y - 0 = 0\,(x - a), \text{ i.e., } y = 0.$$

EXERCISE 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.

5. Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

6. Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

7. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

10. Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}, x \neq 1$.

11. Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}, x \neq 3$.

12. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.

13. Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

(i) parallel to x-axis (ii) parallel to y-axis.

14. Find the equations of the tangent and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(iii) $y = x^3$ at (1, 1)

(iv) $y = x^2$ at (0, 0)

(v) $x = \cos t, y = \sin t$ at $t = \frac{\pi}{4}$

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(a) parallel to the line $2x - y + 9 = 0$

(b) perpendicular to the line $5y - 15x = 13$.

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles* if $8k^2 = 1$.

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

25. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Choose the correct answer in Exercises 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is

(A) 3 (B) $\frac{1}{3}$ (C) –3 (D) $-\frac{1}{3}$

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1) (C) (1, – 2) (D) (– 1, 2)

6.5 Approximations

In this section, we will use differentials to approximate values of certain quantities.

Let $f: D \to \mathbf{R}$, $D \subset \mathbf{R}$, be a given function and let $y = f(x)$. Let Δx denote a small increment in x. Recall that the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x) - f(x)$. We define the following

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$.

(ii) The differential of y, denoted by dy, is defined by $dy = f'(x)\, dx$ or

$$dy = \left(\frac{dy}{dx}\right)\Delta x.$$

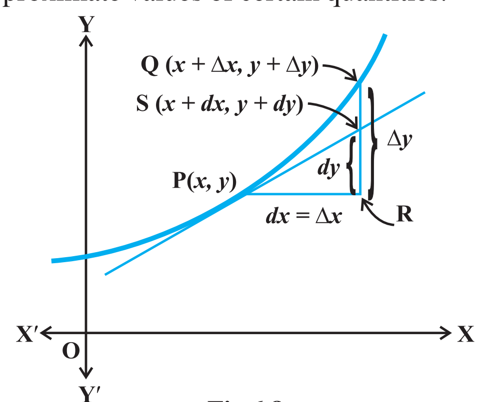


Fig 6.8

* Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.

In case $dx = \Delta x$ is relatively small when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

For geometrical meaning of Δx, Δy, dx and dy, one may refer to Fig 6.8.

Note In view of the above discussion and Fig 6.8, we may note that the differential of the dependent variable is not equal to the increment of the variable where as the differential of independent variable is equal to the increment of the variable.

Example 21 Use differential to approximate $\sqrt{36.6}$.

Solution Take $y = \sqrt{x}$. Let $x = 36$ and let $\Delta x = 0.6$. Then

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36} = \sqrt{36.6} - 6$$

or $\sqrt{36.6} = 6 + \Delta y$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05 \qquad (\text{as } y = \sqrt{x})$$

Thus, the approximate value of $\sqrt{36.6}$ is $6 + 0.05 = 6.05$.

Example 22 Use differential to approximate $(25)^{\frac{1}{3}}$.

Solution Let $y = x^{\frac{1}{3}}$. Let $x = 27$ and let $\Delta x = -2$. Then

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (25)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (25)^{\frac{1}{3}} - 3$$

or $(25)^{\frac{1}{3}} = 3 + \Delta y$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{\frac{2}{3}}}(-2) \qquad (\text{as } y = x^{\frac{1}{3}})$$

$$= \frac{1}{3((27)^{\frac{1}{3}})^2}(-2) = \frac{-2}{27} = -0.074$$

Thus, the approximate value of $(25)^{\frac{1}{3}}$ is given by

$$3 + (-0.074) = 2.926$$

Example 23 Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.

Solution Let $x = 3$ and $\Delta x = 0.02$. Then

$$f(3.02) = f(x + \Delta x) = 3(x + \Delta x)^2 + 5(x + \Delta x) + 3$$

Note that $\Delta y = f(x + \Delta x) - f(x)$. Therefore

$$f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x)\,\Delta x \qquad \text{(as } dx = \Delta x\text{)}$$

or

$$f(3.02) \approx (3x^2 + 5x + 3) + (6x + 5)\,\Delta x$$

$$= (3(3)^2 + 5(3) + 3) + (6(3) + 5)(0.02) \qquad \text{(as } x = 3, \Delta x = 0.02\text{)}$$

$$= (27 + 15 + 3) + (18 + 5)(0.02)$$

$$= 45 + 0.46 = 45.46$$

Hence, approximate value of $f(3.02)$ is 45.46.

Example 24 Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%.

Solution Note that

$$V = x^3$$

or

$$dV = \left(\frac{dV}{dx}\right)\Delta x = (3x^2)\,\Delta x$$

$$= (3x^2)(0.02x) = 0.06x^3 \text{ m}^3 \qquad \text{(as 2\% of } x \text{ is } 0.02x\text{)}$$

Thus, the approximate change in volume is $0.06\,x^3\,\text{m}^3$.

Example 25 If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Solution Let r be the radius of the sphere and Δr be the error in measuring the radius. Then $r = 9$ cm and $\Delta r = 0.03$ cm. Now, the volume V of the sphere is given by

$$V = \frac{4}{3}\pi r^3$$

or

$$\frac{dV}{dr} = 4\pi r^2$$

Therefore

$$dV = \left(\frac{dV}{dr}\right)\Delta r = (4\pi r^2)\Delta r$$

$$= 4\pi(9)^2(0.03) = 9.72\pi \text{ cm}^3$$

Thus, the approximate error in calculating the volume is 9.72π cm^3.

EXERCISE 6.4

1. Using differentials, find the approximate value of each of the following up to 3 places of decimal.

(i) $\sqrt{25.3}$ (ii) $\sqrt{49.5}$ (iii) $\sqrt{0.6}$

(iv) $(0.009)^{\frac{1}{3}}$ (v) $(0.999)^{\frac{1}{10}}$ (vi) $(15)^{\frac{1}{4}}$

(vii) $(26)^{\frac{1}{3}}$ (viii) $(255)^{\frac{1}{4}}$ (ix) $(82)^{\frac{1}{4}}$

(x) $(401)^{\frac{1}{2}}$ (xi) $(0.0037)^{\frac{1}{2}}$ (xii) $(26.57)^{\frac{1}{3}}$

(xiii) $(81.5)^{\frac{1}{4}}$ (xiv) $(3.968)^{\frac{3}{2}}$ (xv) $(32.15)^{\frac{1}{5}}$

2. Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.

3. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.

4. Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 1%.

5. Find the approximate change in the surface area of a cube of side x metres caused by decreasing the side by 1%.

6. If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.

7. If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating its surface area.

8. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is

(A) 47.66 (B) 57.66 (C) 67.66 (D) 77.66

9. The approximate change in the volume of a cube of side x metres caused by increasing the side by 3% is

(A) $0.06\, x^3$ m^3 (B) $0.6\, x^3$ m^3 (C) $0.09\, x^3$ m^3 (D) $0.9\, x^3$ m^3

6.6 Maxima and Minima

In this section, we will use the concept of derivatives to calculate the maximum or minimum values of various functions. In fact, we will find the 'turning points' of the graph of a function and thus find points at which the graph reaches its highest (or lowest) *locally*. The knowledge of such points is very useful in sketching the graph of a given function. Further, we will also find the absolute maximum and absolute minimum of a function that are necessary for the solution of many applied problems.

Let us consider the following problems that arise in day to day life.

(i) The profit from a grove of orange trees is given by $P(x) = ax + bx^2$, where a,b are constants and x is the number of orange trees per acre. How many trees per acre will maximise the profit?

(ii) A ball, thrown into the air from a building 60 metres high, travels along a path given by $h(x) = 60 + x - \frac{x^2}{60}$, where x is the horizontal distance from the building and $h(x)$ is the height of the ball . What is the maximum height the ball will reach?

(iii) An Apache helicopter of enemy is flying along the path given by the curve $f(x) = x^2 + 7$. A soldier, placed at the point (1, 2), wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there is something common, i.e., we wish to find out the maximum or minimum values of the given functions. In order to tackle such problems, we first formally define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

Definition 3 Let f be a function defined on an interval I. Then

(a) f is said to have a *maximum value* in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in$ I.

The number $f(c)$ is called the maximum value of f in I and the point c is called a *point of maximum value* of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in$ I.

The number $f(c)$, in this case, is called the minimum value of f in I and the point c, in this case, is called a *point of minimum value* of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

The number $f(c)$, in this case, is called an *extreme value* of f in I and the point c is called an *extreme point*.

Remark In Fig 6.9(a), (b) and (c), we have exhibited that graphs of certain particular functions help us to find maximum value and minimum value at a point. Infact, through graphs, we can even find maximum/minimum value of a function at a point at which it is not even differentiable (Example 27).

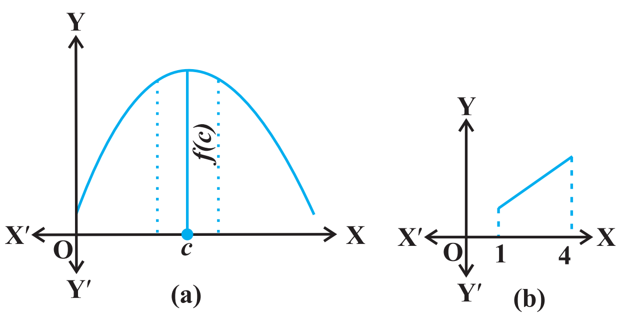


Fig 6.9

Example 26 Find the maximum and the minimum values, if any, of the function f given by

$$f(x) = x^2, x \in \mathbf{R}.$$

Solution From the graph of the given function (Fig 6.10), we have $f(x) = 0$ if $x = 0$. Also

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R}.$$

Therefore, the minimum value of f is 0 and the point of minimum value of f is $x = 0$. Further, it may be observed from the graph of the function that f has no maximum value and hence no point of maximum value of f in **R**.

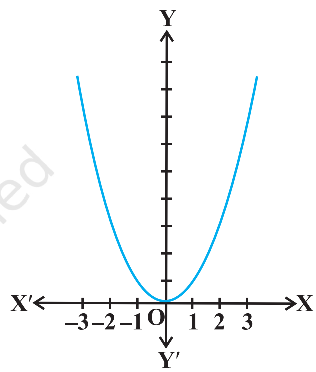


Fig 6.10

Note If we restrict the domain of f to [– 2, 1] only, then f will have maximum value$(-2)^2 = 4$ at $x = -2$.

Example 27 Find the maximum and minimum values of f, if any, of the function given by $f(x) = |x|, x \in \mathbf{R}$.

Solution From the graph of the given function (Fig 6.11) , note that

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R} \text{ and } f(x) = 0 \text{ if } x = 0.$$

Therefore, the function f has a minimum value 0 and the point of minimum value of f is $x = 0$. Also, the graph clearly shows that f has no maximum value in **R** and hence no point of maximum value in **R**.

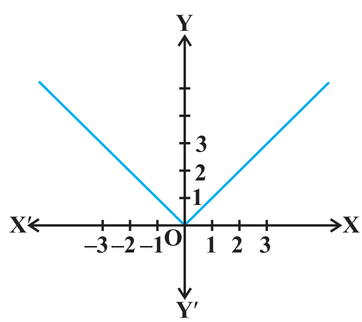


Fig 6.11

Note

(i) If we restrict the domain of f to [– 2, 1] only, then f will have maximum value $|-2| = 2$.

(ii) One may note that the function f in Example 27 is not differentiable at $x = 0$.

Example 28 Find the maximum and the minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1).$$

Solution The given function is an increasing (strictly) function in the given interval (0, 1). From the graph (Fig 6.12) of the function f, it seems that, it should have the minimum value at a point closest to 0 on its right and the maximum value at a point closest to 1 on its left. Are such points available? Of course, not. It is not possible to locate such points. Infact, if a point x_0 is closest to 0, then we find $\frac{x_0}{2} < x_0$ for all $x_0 \in (0,1)$. Also, if x_1 is closest to 1, then $\frac{x_1+1}{2} > x_1$ for all $x_1 \in (0,1)$.

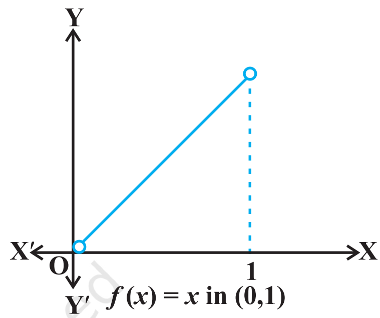


Fig 6.12

Therefore, the given function has neither the maximum value nor the minimum value in the interval (0,1).

Remark The reader may observe that in Example 28, if we include the points 0 and 1 in the domain of f, i.e., if we extend the domain of f to [0,1], then the function f has minimum value 0 at $x = 0$ and maximum value 1 at $x = 1$. Infact, we have the following results (The proof of these results are beyond the scope of the present text)

Every monotonic function assumes its maximum/minimum value at the end points of the domain of definition of the function.

A more general result is

Every continuous function on a closed interval has a maximum and a minimum value.

Note By a monotonic function f in an interval I, we mean that f is either increasing in I or decreasing in I.

Maximum and minimum values of a function defined on a closed interval will be discussed later in this section.

Let us now examine the graph of a function as shown in Fig 6.13. Observe that at points A, B, C and D on the graph, the function changes its nature from decreasing to increasing or vice-versa. These points may be called *turning points* of the given function. Further, observe that at turning points, the graph has either a little hill or a little valley. Roughly speaking, the function has minimum value in some neighbourhood (interval) of each of the points A and C which are at the bottom of their respective

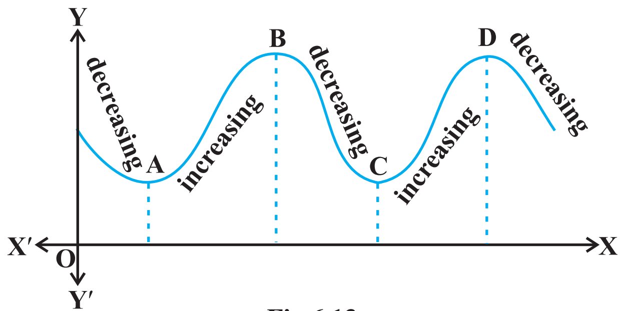


Fig 6.13

valleys. Similarly, the function has maximum value in some neighbourhood of points B and D which are at the top of their respective hills. For this reason, the points A and C may be regarded as points of *local minimum value* (or *relative minimum value*) and points B and D may be regarded as points of *local maximum value* (or *relative maximum value*) for the function. The *local maximum value* and *local minimum value* of the function are referred to as *local maxima* and *local minima*, respectively, of the function.

We now formally give the following definition

Definition 4 Let f be a real valued function and let c be an interior point in the domain of f. Then

(a) c is called a point of *local maxima* if there is an $h > 0$ such that

$$f(c) \geq f(x), \text{ for all } x \text{ in } (c - h, c + h), x \neq c$$

The value $f(c)$ is called the *local maximum value* of f.

(b) c is called a point of *local minima* if there is an $h > 0$ such that

$$f(c) \leq f(x), \text{ for all } x \text{ in } (c - h, c + h)$$

The value $f(c)$ is called the *local minimum value* of f.

Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of f around c will be as shown in Fig 6.14(a). Note that the function f is increasing (i.e., $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c + h)$.

This suggests that $f'(c)$ must be zero.

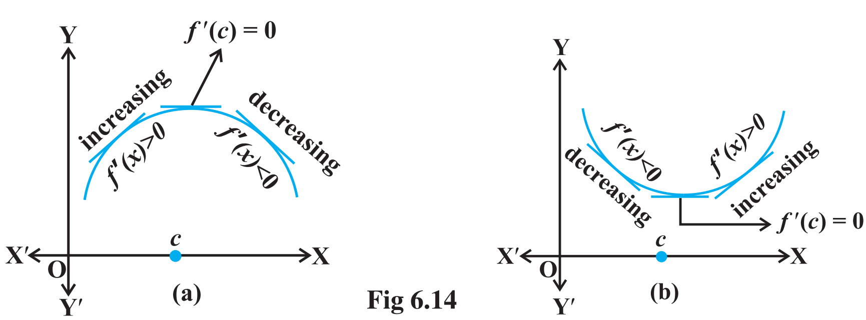


Fig 6.14

Similarly, if c is a point of local minima of f, then the graph of f around c will be as shown in Fig 6.14(b). Here f is decreasing (i.e., $f'(x) < 0$) in the interval $(c-h, c)$ and increasing (i.e., $f'(x) > 0$) in the interval $(c, c+h)$. This again suggest that $f'(c)$ must be zero.

The above discussion lead us to the following theorem (without proof).

Theorem 2 Let f be a function defined on an open interval I. Suppose $c \in$ I be any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

Remark The converse of above theorem need not be true, that is, a point at which the derivative vanishes need not be a point of local maxima or local minima. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and so $f'(0) = 0$. But 0 is neither a point of local maxima nor a point of local minima (Fig 6.15).

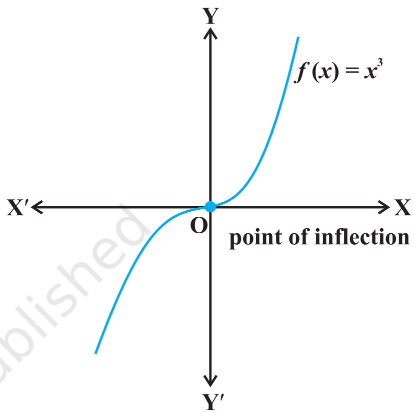


Fig 6.15

> **Note** A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a *critical point* of f. Note that if f is continuous at c and $f'(c) = 0$, then there exists an $h > 0$ such that f is differentiable in the interval $(c-h, c+h)$.

We shall now give a working rule for finding points of local maxima or points of local minima using only the first order derivatives.

Theorem 3 (First Derivative Test) Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflection* (Fig 6.15).

Note If c is a point of local maxima of f, then $f(c)$ is a local maximum value of f. Similarly, if c is a point of local minima of f, then $f(c)$ is a local minimum value of f.

Figures 6.15 and 6.16, geometrically explain Theorem 3.

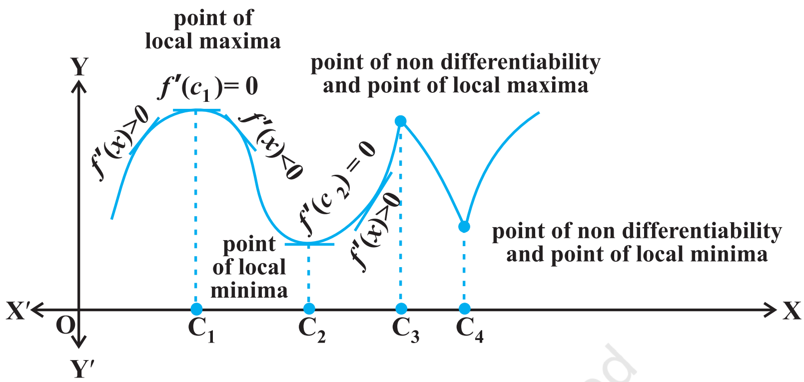


Fig 6.16

Example 29 Find all points of local maxima and local minima of the function f given by

$$f(x) = x^3 - 3x + 3.$$

Solution We have

$$f(x) = x^3 - 3x + 3$$

or $$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$$

or $$f'(x) = 0 \text{ at } x = 1 \text{ and } x = -1$$

Thus, $x = \pm 1$ are the only critical points which could possibly be the points of local maxima and/or local minima of f. Let us first examine the point $x = 1$.

Note that for values close to 1 and to the right of 1, $f'(x) > 0$ and for values close to 1 and to the left of 1, $f'(x) < 0$. Therefore, by first derivative test, $x = 1$ is a point of local minima and local minimum value is $f(1) = 1$. In the case of $x = -1$, note that $f'(x) > 0$, for values close to and to the left of -1 and $f'(x) < 0$, for values close to and to the right of -1. Therefore, by first derivative test, $x = -1$ is a point of local maxima and local maximum value is $f(-1) = 5$.

Values of x		Sign of $f'(x) = 3(x - 1)(x + 1)$
Close to 1	to the right (say 1.1 etc.)	>0
	to the left (say 0.9 etc.)	<0
Close to -1	to the right (say -0.9 etc.)	<0
	to the left (say -1.1 etc.)	>0

Example 30 Find all the points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or $$f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2$$

or $$f'(x) = 0 \quad \text{at} \quad x = 1$$

Thus, $x = 1$ is the only critical point of f. We shall now examine this point for local maxima and/or local minima of f. Observe that $f'(x) \geq 0$, for all $x \in \mathbf{R}$ and in particular $f'(x) > 0$, for values close to 1 and to the left and to the right of 1. Therefore, by first derivative test, the point $x = 1$ is neither a point of local maxima nor a point of local minima. Hence $x = 1$ is a point of inflexion.

Remark One may note that since $f'(x)$, in Example 30, never changes its sign on $\mathbf{R}$, graph of f has no turning points and hence no point of local maxima or local minima.

We shall now give another test to examine local maxima and local minima of a given function. This test is often easier to apply than the first derivative test.

Theorem 4 (Second Derivative Test) Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Note As f is twice differentiable at c, we mean second order derivative of f exists at c.

Example 31 Find local minimum value of the function f given by $f(x) = 3 + |x|, x \in \mathbf{R}$.

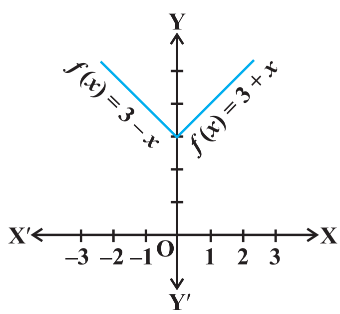


Fig 6.17

Solution Note that the given function is not differentiable at $x = 0$. So, second derivative test fails. Let us try first derivative test. Note that 0 is a critical point of f. Now to the left of 0, $f(x) = 3 - x$ and so $f'(x) = -1 < 0$. Also to the right of 0, $f(x) = 3 + x$ and so $f'(x) = 1 > 0$. Therefore, by first derivative test, $x = 0$ is a point of local minima of f and local minimum value of f is $f(0) = 3$.

Example 32 Find local maximum and local minimum values of the function f given by

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

Solution We have

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

or $$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x-1)(x+2)$$

or $$f'(x) = 0 \text{ at } x = 0,\ x = 1 \text{ and } x = -2.$$

Now $$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

or $$\begin{cases} f''(0) = -24 < 0 \\ f''(1) = 36 > 0 \\ f''(-2) = 72 > 0 \end{cases}$$

Therefore, by second derivative test, $x = 0$ is a point of local maxima and local maximum value of f at $x = 0$ is $f(0) = 12$ while $x = 1$ and $x = -2$ are the points of local minima and local minimum values of f at $x = -1$ and -2 are $f(1) = 7$ and $f(-2) = -20$, respectively.

Example 33 Find all the points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or $$\begin{cases} f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2 \\ f''(x) = 12(x-1) \end{cases}$$

Now $f'(x) = 0$ gives $x = 1$. Also $f''(1) = 0$. Therefore, the second derivative test fails in this case. So, we shall go back to the first derivative test.

We have already seen (Example 30) that, using first derivative test, $x = 1$ is neither a point of local maxima nor a point of local minima and so it is a point of inflexion.

Example 34 Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Solution Let one of the numbers be x. Then the other number is $(15 - x)$. Let $S(x)$ denote the sum of the squares of these numbers. Then

$$S(x) = x^2 + (15 - x)^2 = 2x^2 - 30x + 225$$

or
$$\begin{cases} S'(x) = 4x - 30 \\ S''(x) = 4 \end{cases}$$

Now $S'(x) = 0$ gives $x = \frac{15}{2}$. Also $S''\left(\frac{15}{2}\right) = 4 > 0$. Therefore, by second derivative test, $x = \frac{15}{2}$ is the point of local minima of S. Hence the sum of squares of numbers is minimum when the numbers are $\frac{15}{2}$ and $15 - \frac{15}{2} = \frac{15}{2}$.

Remark Proceeding as in Example 34 one may prove that the two positive numbers, whose sum is k and the sum of whose squares is minimum, are $\frac{k}{2}$ and $\frac{k}{2}$.

Example 35 Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.

Solution Let (h, k) be any point on the parabola $y = x^2$. Let D be the required distance between (h, k) and $(0, c)$. Then

$$D = \sqrt{(h-0)^2 + (k-c)^2} = \sqrt{h^2 + (k-c)^2} \qquad \text{... (1)}$$

Since (h, k) lies on the parabola $y = x^2$, we have $k = h^2$. So (1) gives

$$D \equiv D(k) = \sqrt{k + (k-c)^2}$$

or
$$D'(k) = \frac{1 + 2(k-c)}{2\sqrt{k + (k-c)^2}}$$

Now
$$D'(k) = 0 \text{ gives } k = \frac{2c-1}{2}$$

Observe that when $k < \frac{2c-1}{2}$, then $2(k-c)+1 < 0$, i.e., $D'(k) < 0$. Also when $k > \frac{2c-1}{2}$, then $D'(k) > 0$. So, by first derivative test, D (k) is minimum at $k = \frac{2c-1}{2}$.

Hence, the required shortest distance is given by

$$D\left(\frac{2c-1}{2}\right)=\sqrt{\frac{2c-1}{2}+\left(\frac{2c-1}{2}-c\right)^2}=\frac{\sqrt{4c-1}}{2}$$

Note The reader may note that in Example 35, we have used first derivative test instead of the second derivative test as the former is easy and short.

Example 36 Let AP and BQ be two vertical poles at points A and B, respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

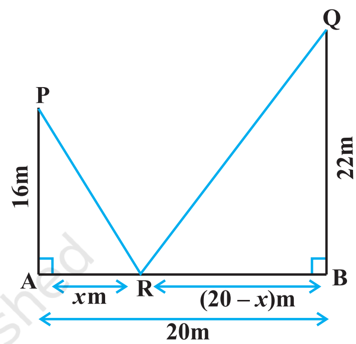


Fig 6.18

Solution Let R be a point on AB such that AR = x m. Then RB = $(20 - x)$ m (as AB = 20 m). From Fig 6.18, we have

$$RP^2 = AR^2 + AP^2$$

and $$RQ^2 = RB^2 + BQ^2$$

Therefore $$RP^2 + RQ^2 = AR^2 + AP^2 + RB^2 + BQ^2$$
$$= x^2 + (16)^2 + (20 - x)^2 + (22)^2$$
$$= 2x^2 - 40x + 1140$$

Let $$S \equiv S(x) = RP^2 + RQ^2 = 2x^2 - 40x + 1140.$$

Therefore $$S'(x) = 4x - 40.$$

Now $S'(x) = 0$ gives $x = 10$. Also $S''(x) = 4 > 0$, for all x and so $S''(10) > 0$. Therefore, by second derivative test, $x = 10$ is the point of local minima of S. Thus, the distance of R from A on AB is AR = x =10 m.

Example 37 If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.

Solution The required trapezium is as given in Fig 6.19. Draw perpendiculars DP and

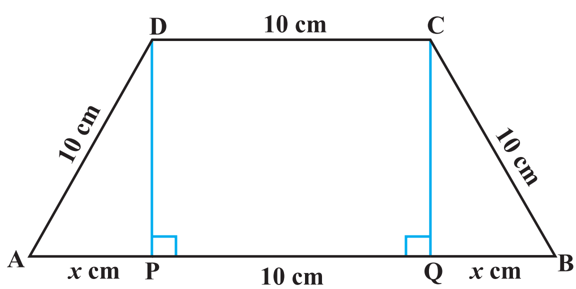


Fig 6.19

CQ on AB. Let AP = x cm. Note that ΔAPD ~ ΔBQC. Therefore, QB = x cm. Also, by Pythagoras theorem, DP = QC = $\sqrt{100-x^2}$. Let A be the area of the trapezium. Then

$$A \equiv A(x) = \frac{1}{2}\text{(sum of parallel sides) (height)}$$

$$= \frac{1}{2}(2x+10+10)\left(\sqrt{100-x^2}\right)$$

$$= (x+10)\left(\sqrt{100-x^2}\right)$$

or $$A'(x) = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}} + \left(\sqrt{100-x^2}\right)$$

$$= \frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

Now $A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$, i.e., $x = 5$ and $x = -10$.

Since x represents distance, it can not be negative.

So, $x = 5$. Now

$$A''(x) = \frac{\sqrt{100-x^2}(-4x-10) - (-2x^2-10x+100)\dfrac{(-2x)}{2\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}} \text{ (on simplification)}$$

or $$A''(5) = \frac{2(5)^3-300(5)-1000}{(100-(5)^2)^{\frac{3}{2}}} = \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and the area is given by

$$A(5) = (5+10)\sqrt{100-(5)^2} = 15\sqrt{75} = 75\sqrt{3}\text{ cm}^2$$

Example 38 Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Solution Let OC = r be the radius of the cone and OA = h be its height. Let a cylinder with radius OE = x inscribed in the given cone (Fig 6.20). The height QE of the cylinder is given by

$$\frac{QE}{OA} = \frac{EC}{OC} \quad \text{(since } \Delta QEC \sim \Delta AOC\text{)}$$

or $$\frac{QE}{h} = \frac{r-x}{r}$$

or $$QE = \frac{h(r-x)}{r}$$

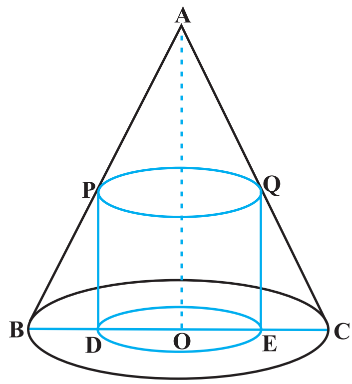


Fig 6.20

Let S be the curved surface area of the given cylinder. Then

$$S \equiv S(x) = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

or $$\begin{cases} S'(x) = \dfrac{2\pi h}{r}(r-2x) \\ S''(x) = \dfrac{-4\pi h}{r} \end{cases}$$

Now $S'(x) = 0$ gives $x = \frac{r}{2}$. Since $S''(x) < 0$ for all x, $S''\left(\frac{r}{2}\right) < 0$. So $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

6.6.1 Maximum and Minimum Values of a Function in a Closed Interval

Let us consider a function f given by

$$f(x) = x + 2, x \in (0, 1)$$

Observe that the function is continuous on (0, 1) and neither has a maximum value nor has a minimum value. Further, we may note that the function even has neither a local maximum value nor a local minimum value.

However, if we extend the domain of f to the closed interval [0, 1], then f still may not have a local maximum (minimum) values but it certainly does have maximum value $3 = f(1)$ and minimum value $2 = f(0)$. The maximum value 3 of f at $x = 1$ is called *absolute maximum* value (*global maximum* or *greatest value*) of f on the interval [0, 1]. Similarly, the minimum value 2 of f at $x = 0$ is called the *absolute minimum* value (*global minimum* or *least value*) of f on [0, 1].

Consider the graph given in Fig 6.21 of a continuous function defined on a closed interval $[a, d]$. Observe that the function f has a local minima at $x = b$ and local

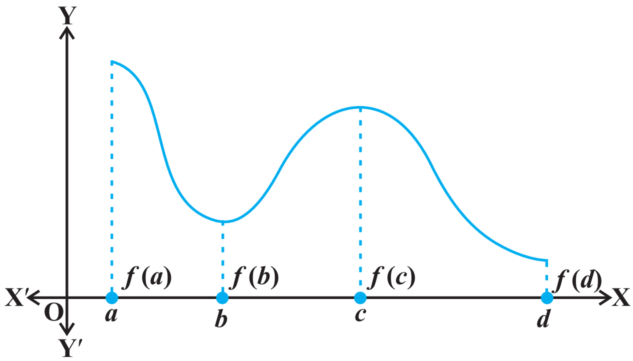


Fig 6.21

minimum value is $f(b)$. The function also has a local maxima at $x = c$ and local maximum value is $f(c)$.

Also from the graph, it is evident that f has absolute maximum value $f(a)$ and absolute minimum value $f(d)$. Further note that the absolute maximum (minimum) value of f is different from local maximum (minimum) value of f.

We will now state two results (without proof) regarding absolute maximum and absolute minimum values of a function on a closed interval I.

Theorem 5 Let f be a continuous function on an interval I = $[a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Theorem 6 Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(i) $f'(c) = 0$ if f attains its absolute maximum value at c.

(ii) $f'(c) = 0$ if f attains its absolute minimum value at c.

In view of the above results, we have the following working rule for finding absolute maximum and/or absolute minimum values of a function in a given closed interval $[a, b]$.

Working Rule

Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f.

Example 39 Find the absolute maximum and minimum values of a function f given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \text{ on the interval } [1, 5].$$

Solution We have

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

or
$$f'(x) = 6x^2 - 30x + 36 = 6(x - 3)(x - 2)$$

Note that $f'(x) = 0$ gives $x = 2$ and $x = 3$.

We shall now evaluate the value of f at these points and at the end points of the interval [1, 5], i.e., at $x = 1$, $x = 2$, $x = 3$ and at $x = 5$. So

$$f(1) = 2(1^3) - 15(1^2) + 36(1) + 1 = 24$$

$$f(2) = 2(2^3) - 15(2^2) + 36(2) + 1 = 29$$

$$f(3) = 2(3^3) - 15(3^2) + 36(3) + 1 = 28$$

$$f(5) = 2(5^3) - 15(5^2) + 36(5) + 1 = 56$$

Thus, we conclude that absolute maximum value of f on [1, 5] is 56, occurring at $x = 5$, and absolute minimum value of f on [1, 5] is 24 which occurs at $x = 1$.

Example 40 Find absolute maximum and minimum values of a function f given by

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}},\ x \in [-1, 1]$$

Solution We have

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$$

or
$$f'(x) = 16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}} = \frac{2(8x-1)}{x^{\frac{2}{3}}}$$

Thus, $f'(x) = 0$ gives $x = \frac{1}{8}$. Further note that $f'(x)$ is not defined at $x = 0$. So the critical points are $x = 0$ and $x = \frac{1}{8}$. Now evaluating the value of f at critical points $x = 0, \frac{1}{8}$ and at end points of the interval $x = -1$ and $x = 1$, we have

$$f(-1) = 12(-1)^{\frac{4}{3}} - 6(-1)^{\frac{1}{3}} = 18$$

$$f(0) = 12(0) - 6(0) = 0$$

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{\frac{4}{3}} - 6\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{-9}{4}$$

$$f(1) = 12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}} = 6$$

Hence, we conclude that absolute maximum value of f is 18 that occurs at $x = -1$ and absolute minimum value of f is $\frac{-9}{4}$ that occurs at $x = \frac{1}{8}$.

Example 41 An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution For each value of x, the helicopter's position is at point $(x, x^2 + 7)$. Therefore, the distance between the helicopter and the soldier placed at (3,7) is

$$\sqrt{(x-3)^2 + (x^2 + 7 - 7)^2}, \text{ i.e., } \sqrt{(x-3)^2 + x^4}.$$

Let $\qquad f(x) = (x - 3)^2 + x^4$

or $\qquad f'(x) = 2(x - 3) + 4x^3 = 2(x - 1)(2x^2 + 2x + 3)$

Thus, $f'(x) = 0$ gives $x = 1$ or $2x^2 + 2x + 3 = 0$ for which there are no real roots. Also, there are no end points of the interval to be added to the set for which f' is zero, i.e., there is only one point, namely, $x = 1$. The value of f at this point is given by $f(1) = (1 - 3)^2 + (1)^4 = 5$. Thus, the distance between the solider and the helicopter is $\sqrt{f(1)} = \sqrt{5}$.

Note that $\sqrt{5}$ is either a maximum value or a minimum value. Since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5},$$

it follows that $\sqrt{5}$ is the minimum value of $\sqrt{f(x)}$. Hence, $\sqrt{5}$ is the minimum distance between the soldier and the helicopter.

EXERCISE 6.5

1. Find the maximum and minimum values, if any, of the following functions given by

   (i) $f(x) = (2x - 1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$

   (iii) $f(x) = -(x - 1)^2 + 10$ (iv) $g(x) = x^3 + 1$

2. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x + 2| - 1$ (ii) $g(x) = -|x + 1| + 3$

(iii) $h(x) = \sin(2x) + 5$ (iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$ (ii) $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$ (vi) $g(x) = \frac{x}{2} + \frac{2}{x},\ x > 0$

(vii) $g(x) = \frac{1}{x^2 + 2}$ (viii) $f(x) = x\sqrt{1-x},\ 0 < x < 1$

4. Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ (ii) $g(x) = \log x$

(iii) $h(x) = x^3 + x^2 + x + 1$

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3, x \in [-2, 2]$ (ii) $f(x) = \sin x + \cos x,\ x \in [0, \pi]$

(iii) $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right]$ (iv) $f(x) = (x-1)^2 + 3,\ x \in [-3, 1]$

6. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2$$

7. Find both the maximum value and the minimum value of

$3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

8. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

9. What is the maximum value of the function $\sin x + \cos x$?

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

12. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

13. Find two numbers whose sum is 24 and whose product is as large as possible.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

15. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Choose the correct answer in Questions 27 and 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2},4)$ (B) $(2\sqrt{2},0)$ (C) (0, 0) (D) (2, 2)

28. For all real values of x, the minimum value of $\dfrac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) 1 (C) 3 (D) $\dfrac{1}{3}$

29. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}$, $0 \le x \le 1$ is

(A) $\left(\dfrac{1}{3}\right)^{\frac{1}{3}}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 0

Miscellaneous Examples

Example 42 A car starts from a point P at time $t = 0$ seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by

$$x = t^2\left(2-\frac{t}{3}\right)$$

Find the time taken by it to reach Q and also find distance between P and Q.

Solution Let v be the velocity of the car at t seconds.

Now $$x = t^2\left(2-\frac{t}{3}\right)$$

Therefore $$v = \frac{dx}{dt} = 4t - t^2 = t(4-t)$$

Thus, $v = 0$ gives $t = 0$ and/or $t = 4$.

Now $v = 0$ at P as well as at Q and at P, $t = 0$. So, at Q, $t = 4$. Thus, the car will reach the point Q after 4 seconds. Also the distance travelled in 4 seconds is given by

$$x]_{t=4} = 4^2\left(2-\frac{4}{3}\right) = 16\left(\frac{2}{3}\right) = \frac{32}{3}\text{ m}$$

Example 43 A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

Solution Let r, h and α be as in Fig 6.22. Then $\tan\alpha = \frac{r}{h}$.

So $$\alpha = \tan^{-1}\left(\frac{r}{h}\right).$$

But $$\alpha = \tan^{-1}(0.5) \quad \text{(given)}$$

or $$\frac{r}{h} = 0.5$$

or $$r = \frac{h}{2}$$

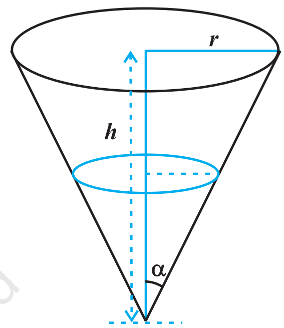


Fig 6.22

Let V be the volume of the cone. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

Therefore $$\frac{dV}{dt} = \frac{d}{dh}\left(\frac{\pi h^3}{12}\right)\cdot\frac{dh}{dt} \quad \text{(by Chain Rule)}$$

$$= \frac{\pi}{4}h^2\frac{dh}{dt}$$

Now rate of change of volume, i.e., $\frac{dV}{dt} = 5\ \text{m}^3/\text{h}$ and $h = 4$ m.

Therefore $$5 = \frac{\pi}{4}(4)^2\cdot\frac{dh}{dt}$$

or $$\frac{dh}{dt} = \frac{5}{4\pi} = \frac{35}{88}\ \text{m/h} \quad \left(\pi = \frac{22}{7}\right)$$

Thus, the rate of change of water level is $\frac{35}{88}$ m/h.

Example 44 A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.

Solution In Fig 6.23, Let AB be the lamp-post, the lamp being at the position B and let MN be the man at a particular time t and let AM = l metres. Then, MS is the shadow of the man. Let MS = s metres.

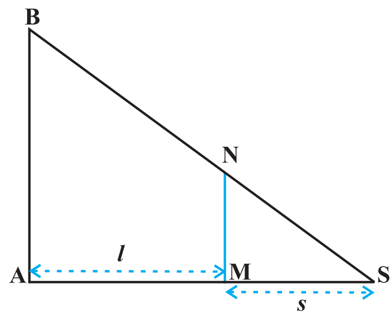


Fig 6.23

Note that $\Delta MSN \sim \Delta ASB$

or $$\frac{MS}{AS} = \frac{MN}{AB}$$

or $AS = 3s$ (as MN = 2 and AB = 6 (given))

Thus $AM = 3s - s = 2s$. But $AM = l$

So $l = 2s$

Therefore $$\frac{dl}{dt} = 2\frac{ds}{dt}$$

Since $\frac{dl}{dt} = 5$ km/h. Hence, the length of the shadow increases at the rate $\frac{5}{2}$ km/h.

Example 45 Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).

Solution Differentiating $x^2 = 4y$ with respect to x, we get

$$\frac{dy}{dx} = \frac{x}{2}$$

Let (h, k) be the coordinates of the point of contact of the normal to the curve $x^2 = 4y$. Now, slope of the tangent at (h, k) is given by

$$\left.\frac{dy}{dx}\right]_{(h,k)} = \frac{h}{2}$$

Hence, slope of the normal at $(h, k) = \frac{-2}{h}$

Therefore, the equation of normal at (h, k) is

$$y - k = \frac{-2}{h}(x - h) \qquad \ldots (1)$$

Since it passes through the point (1, 2), we have

$$2 - k = \frac{-2}{h}(1 - h) \quad \text{or} \quad k = 2 + \frac{2}{h}(1 - h) \qquad \ldots (2)$$

Since (h, k) lies on the curve $x^2 = 4y$, we have

$$h^2 = 4k \qquad \text{... (3)}$$

From (2) and (3), we have $h = 2$ and $k = 1$. Substituting the values of h and k in (1), we get the required equation of normal as

$$y - 1 = \frac{-2}{2}(x - 2) \text{ or } x + y = 3$$

Example 46 Find the equation of tangents to the curve

$$y = \cos(x + y), -2\pi \leq x \leq 2\pi$$

that are parallel to the line $x + 2y = 0$.

Solution Differentiating $y = \cos(x + y)$ with respect to x, we have

$$\frac{dy}{dx} = \frac{-\sin(x+y)}{1+\sin(x+y)}$$

or $\qquad$ slope of tangent at $(x, y) = \dfrac{-\sin(x+y)}{1+\sin(x+y)}$

Since the tangents to the given curve are parallel to the line $x + 2y = 0$, whose slope is $\dfrac{-1}{2}$, we have

$$\frac{-\sin(x+y)}{1+\sin(x+y)} = \frac{-1}{2}$$

or $\qquad \sin(x + y) = 1$

or $\qquad x + y = n\pi + (-1)^n \dfrac{\pi}{2}, \; n \in \mathbf{Z}$

Then $\qquad y = \cos(x + y) = \cos\left(n\pi + (-1)^n \dfrac{\pi}{2}\right), \; n \in \mathbf{Z}$

$\qquad = 0$, for all $n \in \mathbf{Z}$

Also, since $-2\pi \leq x \leq 2\pi$, we get $x = \dfrac{-3\pi}{2}$ and $x = \dfrac{\pi}{2}$. Thus, tangents to the given curve are parallel to the line $x + 2y = 0$ only at points $\left(\dfrac{-3\pi}{2}, 0\right)$ and $\left(\dfrac{\pi}{2}, 0\right)$.

Therefore, the required equation of tangents are

$$y - 0 = \frac{-1}{2}\left(x + \frac{3\pi}{2}\right) \quad \text{or} \quad 2x + 4y + 3\pi = 0$$

and
$$y - 0 = \frac{-1}{2}\left(x - \frac{\pi}{2}\right) \quad \text{or} \quad 2x + 4y - \pi = 0$$

Example 47 Find intervals in which the function given by

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

is (a) increasing (b) decreasing.

Solution We have

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

Therefore
$$f'(x) = \frac{3}{10}(4x^3) - \frac{4}{5}(3x^2) - 3(2x) + \frac{36}{5}$$

$$= \frac{6}{5}(x-1)(x+2)(x-3) \quad \text{(on simplification)}$$

Now $f'(x) = 0$ gives $x = 1$, $x = -2$, or $x = 3$. The points $x = 1, -2$, and 3 divide the real line into four disjoint intervals namely, $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$ (Fig 6.24).

Fig 6.24

Consider the interval $(-\infty, -2)$, i.e., when $-\infty < x < -2$.

In this case, we have $x - 1 < 0$, $x + 2 < 0$ and $x - 3 < 0$.

(In particular, observe that for $x = -3$, $f'(x) = (x-1)(x+2)(x-3) = (-4)(-1)(-6) < 0$)

Therefore, $f'(x) < 0$ when $-\infty < x < -2$.

Thus, the function f is decreasing in $(-\infty, -2)$.

Consider the interval $(-2, 1)$, i.e., when $-2 < x < 1$.

In this case, we have $x - 1 < 0$, $x + 2 > 0$ and $x - 3 < 0$

(In particular, observe that for $x = 0$, $f'(x) = (x-1)(x+2)(x-3) = (-1)(2)(-3) = 6 > 0$)

So $f'(x) > 0$ when $-2 < x < 1$.

Thus, f is increasing in $(-2, 1)$.

Now consider the interval (1, 3), i.e., when $1 < x < 3$. In this case, we have $x - 1 > 0$, $x + 2 > 0$ and $x - 3 < 0$.

So, $f'(x) < 0$ when $1 < x < 3$.

Thus, f is decreasing in (1, 3).

Finally, consider the interval $(3, \infty)$, i.e., when $x > 3$. In this case, we have $x - 1 > 0$, $x + 2 > 0$ and $x - 3 > 0$. So $f'(x) > 0$ when $x > 3$.

Thus, f is increasing in the interval $(3, \infty)$.

Example 48 Show that the function f given by

$$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$$

is always an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Solution We have

$$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$$

Therefore $$f'(x) = \frac{1}{1+(\sin x+\cos x)^2}(\cos x-\sin x)$$

$$= \frac{\cos x-\sin x}{2+\sin 2x} \qquad \text{(on simplification)}$$

Note that $2 + \sin 2x > 0$ for all x in $0, \frac{\pi}{4}$.

Therefore $f'(x) > 0$ if $\cos x - \sin x > 0$

or $f'(x) > 0$ if $\cos x > \sin x$ or $\cot x > 1$

Now $\cot x > 1$ if $\tan x < 1$, i.e., if $0 < x < \frac{\pi}{4}$

Thus $f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$

Hence f is increasing function in $\left(0, \frac{\pi}{4}\right)$.

Example 49 A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm.

Solution Let r be the radius of the given disc and A be its area. Then

$$A = \pi r^2$$

or
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{(by Chain Rule)}$$

Now approximate rate of increase of radius $= dr = \frac{dr}{dt}\Delta t = 0.05$ cm/s.

Therefore, the approximate rate of increase in area is given by

$$dA = \frac{dA}{dt}(\Delta t) = 2\pi r\left(\frac{dr}{dt}\Delta t\right)$$

$$= 2\pi\,(3.2)\,(0.05) = 0.320\pi \text{ cm}^2\text{/s} \qquad (r = 3.2 \text{ cm})$$

Example 50 An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

Solution Let x metre be the length of a side of the removed squares. Then, the height of the box is x, length is $8 - 2x$ and breadth is $3 - 2x$ (Fig 6.25). If V(x) is the volume of the box, then

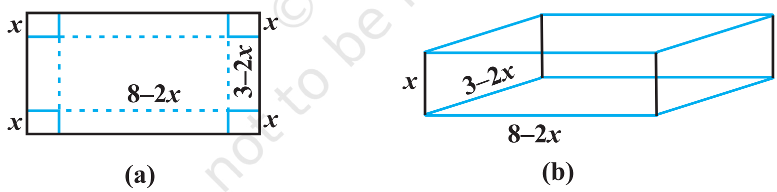


Fig 6.25

$$V(x) = x(3 - 2x)(8 - 2x)$$
$$= 4x^3 - 22x^2 + 24x$$

Therefore
$$\begin{cases} V'(x) = 12x^2 - 44x + 24 = 4(x-3)(3x-2) \\ V''(x) = 24x - 44 \end{cases}$$

Now $\quad V'(x) = 0$ gives $x = 3, \frac{2}{3}$. But $x \neq 3$ (Why?)

Thus, we have $x = \frac{2}{3}$. Now $V''\left(\frac{2}{3}\right) = 24\left(\frac{2}{3}\right) - 44 = -28 < 0$.

Therefore, $x = \frac{2}{3}$ is the point of maxima, i.e., if we remove a square of side $\frac{2}{3}$ metre from each corner of the sheet and make a box from the remaining sheet, then the volume of the box such obtained will be the largest and it is given by

$$V\left(\frac{2}{3}\right) = 4\left(\frac{2}{3}\right)^3 - 22\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right)$$

$$= \frac{200}{27} \text{ m}^3$$

Example 51 Manufacturer can sell x items at a price of rupees $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is Rs $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Solution Let $S(x)$ be the selling price of x items and let $C(x)$ be the cost price of x items. Then, we have

$$S(x) = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$$

and $$C(x) = \frac{x}{5} + 500$$

Thus, the profit function $P(x)$ is given by

$$P(x) = S(x) - C(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

i.e. $$P(x) = \frac{24}{5}x - \frac{x^2}{100} - 500$$

or $$P'(x) = \frac{24}{5} - \frac{x}{50}$$

Now $P'(x) = 0$ gives $x = 240$. Also $P''(x) = \frac{-1}{50}$. So $P''(240) = \frac{-1}{50} < 0$

Thus, $x = 240$ is a point of maxima. Hence, the manufacturer can earn maximum profit, if he sells 240 items.

Miscellaneous Exercise on Chapter 6

1. Using differentials, find the approximate value of each of the following:

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ (b) $(33)^{-\frac{1}{5}}$

2. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

3. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?

4. Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

5. Show that the normal at any point θ to the curve

$x = a \cos\theta + a\,\theta \sin\theta$, $y = a \sin\theta - a\theta \cos\theta$

is at a constant distance from the origin.

6. Find the intervals in which the function f given by

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$$

is (i) increasing (ii) decreasing.

7. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

(i) increasing (ii) decreasing.

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

10. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

11. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

12. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.

Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

13. Find the points at which the function f given by $f(x) = (x-2)^4 (x+1)^3$ has

(i) local maxima (ii) local minima

(iii) point of inflexion

14. Find the absolute maximum and minimum values of the function f given by

$$f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$$

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

16. Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

18. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

Choose the correct answer in the questions from 19 to 24.

19. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h (B) 0.1 m/h

(C) 1.1 m/h (D) 0.5 m/h

20. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is

(A) $\frac{22}{7}$ (B) $\frac{6}{7}$ (C) $\frac{7}{6}$ (D) $\frac{-6}{7}$

21. The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is

(A) 1 (B) 2 (C) 3 (D) $\frac{1}{2}$

22. The normal at the point (1,1) on the curve $2y + x^2 = 3$ is

(A) $x + y = 0$ (B) $x - y = 0$

(C) $x + y + 1 = 0$ (D) $x - y = 1$

23. The normal to the curve $x^2 = 4y$ passing (1,2) is

(A) $x + y = 3$ (B) $x - y = 3$

(C) $x + y = 1$ (D) $x - y = 1$

24. The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

(A) $\left(4, \pm\frac{8}{3}\right)$ (B) $4, \frac{-8}{3}$

(C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm 4, \frac{3}{8}\right)$

Summary

- If a quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.
- If two variables x and y are varying with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0.$$

- A function f is said to be

(a) increasing on an interval (a, b) if
$x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.

Alternatively, if $f'(x) \geq 0$ for each x in (a, b)

(b) decreasing on (a,b) if

$$x_1 < x_2 \text{ in } (a, b) \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in (a, b).$$

(c) constant in (a, b), if $f(x) = c$ for all $x \in (a, b)$, where c is a constant.

- The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by

$$y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0)$$

- If $\frac{dy}{dx}$ does not exist at the point (x_0, y_0), then the tangent at this point is parallel to the y-axis and its equation is $x = x_0$.
- If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x-axis, then $\left.\frac{dy}{dx}\right]_{x=x_0} = 0$.
- Equation of the normal to the curve $y = f(x)$ at a point (x_0, y_0) is given by

$$y - y_0 = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}} (x - x_0)$$

- If $\frac{dy}{dx}$ at the point (x_0, y_0) is zero, then equation of the normal is $x = x_0$.
- If $\frac{dy}{dx}$ at the point (x_0, y_0) does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$.
- Let $y = f(x)$, Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then dy given by

$$dy = f'(x)dx \quad \text{or} \quad dy = \left(\frac{dy}{dx}\right)\Delta x.$$

is a good approximation of Δy when $dx = \Delta x$ is relatively small and we denote it by $dy \approx \Delta y$.
- A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a *critical point* of f.

- ***First Derivative Test*** Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then
  (i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.
  (ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.
  (iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflexion*.
- ***Second Derivative Test*** Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then
  (i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$
  The values $f(c)$ is local maximum value of f.
  (ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
  In this case, $f(c)$ is local minimum value of f.
  (iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.
  In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.
- Working rule for finding absolute maxima and/or absolute minima
  Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.
  Step 2: Take the end points of the interval.
  Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.
  Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

MATHEMATICS

PART II

Textbook for Class XII

12080

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
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Foreword

The National Curriculum Framework, 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

NCERT appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisor for this book, Professor P.K. Jain for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

Director
National Council of Educational
Research and Training

New Delhi
20 November 2006

Preface

The National Council of Educational Research and Training (NCERT) had constituted 21 Focus Groups on Teaching of various subjects related to School Education, to review the National Curriculum Framework for School Education - 2000 (NCFSE - 2000) in face of new emerging challenges and transformations occurring in the fields of content and pedagogy under the contexts of National and International spectrum of school education. These Focus Groups made general and specific comments in their respective areas. Consequently, based on these reports of Focus Groups, National Curriculum Framework (NCF)-2005 was developed.

NCERT designed the new syllabi and constituted Textbook Development Teams for Classes XI and XII to prepare textbooks in Mathematics under the new guidelines and new syllabi. The textbook for Class XI is already in use, which was brought in 2005.

The first draft of the present book (Class XII) was prepared by the team consisting of NCERT faculty, experts and practicing teachers. The draft was refined by the development team in different meetings. This draft of the book was exposed to a group of practicing teachers teaching Mathematics at higher secondary stage in different parts of the country, in a review workshop organised by the NCERT at Delhi. The teachers made useful comments and suggestions which were incorporated in the draft textbook. The draft textbook was finalised by an editorial board constituted out of the development team. Finally, the Advisory Group in Science and Mathematics and the Monitoring Committee constituted by the HRD Ministry, Government of India have approved the draft of the textbook.

In the fitness of things, let us cite some of the essential features dominating the textbook. These characteristics have reflections in almost all the chapters. The existing textbook contains thirteen main chapters and two appendices. Each chapter contains the followings :

- Introduction: Highlighting the importance of the topic; connection with earlier studied topics; brief mention about the new concepts to be discussed in the chapter.
- Organisation of chapter into sections comprising one or more concepts/ subconcepts.
- Motivating and introducing the concepts/subconcepts. Illustrations have been provided wherever possible.

- Proofs/problem solving involving deductive or inductive reasoning, multiplicity of approaches wherever possible have been inducted.
- Geometric viewing / visualisation of concepts have been emphasized whenever needed.
- Applications of mathematical concepts have also been integrated with allied subjects like Science and Social Sciences.
- Adequate and variety of examples/exercises have been given in each section.
- For refocusing and strengthening the understanding and skill of problem solving and applicabilities, miscellaneous types of examples/exercises have been provided involving two or more subconcepts at a time at the end of the chapter. The scope of challenging problems to talented minority have been reflected conducive to the recommendation as reflected in NCF-2005.
- For more motivational purpose, brief historical background of topics have been provided at the end of the chapter and at the beginning of each chapter, relevant quotation and photograph of eminent mathematician who have contributed significantly in the development of the topic undertaken, are also provided.
- Lastly, for direct recapitulation of main concepts, formulas and results, brief summary of the chapter has also been provided.

I am thankful to Professor Krishan Kumar, Director, NCERT who constituted the team and invited me to join this national endeavour for the improvement of Mathematics education. He has provided us with an enlightened perspective and a very conducive environment. This made the task of preparing the book much more enjoyable and rewarding. I express my gratitude to Professor J.V. Narlikar, Chairperson of the Advisory Group in Science and Mathematics, for his specific suggestions and advice towards the improvement of the book from time to time. I, also, thank Professor G. Ravindra, Joint Director, NCERT for his help from time to time.

I express my sincere thanks to Professor Hukum Singh, Chief Coordinator and Head, DESM, Dr. V. P. Singh, Coordinator and Professor, S. K. Singh Gautam who have been helping for the success of this project academically as well as administratively. Also, I would like to place on records my appreciation and thanks to all the members of the team and the teachers who have been associated with this noble cause in one or the other form.

PAWAN K. JAIN
Chief Advisor

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THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

---

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

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Contents of
MATHEMATICS PART I
For Class XII

Contents

Part II

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Chapter 7

INTEGRALS

❖ *Just as a mountaineer climbs a mountain – because it is there, so a good mathematics student studies new material because it is there.* — *JAMES B. BRISTOL* ❖

7.1 Introduction

G .W. Leibnitz (1646 -1716)

Differential Calculus is centred on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.

If a function f is differentiable in an interval I, i.e., its derivative f' exists at each point of I, then a natural question arises that given f' at each point of I, can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function. Further, the formula that gives all these anti derivatives is called the ***indefinite integral*** of the function and such process of finding anti derivatives is called integration. Such type of problems arise in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types:

(a) the problem of finding a function whenever its derivative is given,

(b) the problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the ***Integral Calculus***.

There is a connection, known as the ***Fundamental Theorem of Calculus***, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering. The definite integral is also used to solve many interesting problems from various disciplines like economics, finance and probability.

In this Chapter, we shall confine ourselves to the study of indefinite and definite integrals and their elementary properties including some techniques of integration.

7.2 Integration as an Inverse Process of Differentiation

Integration is the inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called *integration* or *anti differentiation*.
Let us consider the following examples:

We know that $$\frac{d}{dx}(\sin x) = \cos x \qquad \text{... (1)}$$

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \qquad \text{... (2)}$$

and $$\frac{d}{dx}(e^x) = e^x \qquad \text{... (3)}$$

We observe that in (1), the function cos x is the derived function of sin x. We say that sin x is an anti derivative (or an integral) of cos x. Similarly, in (2) and (3), $\frac{x^3}{3}$ and e^x are the anti derivatives (or integrals) of x^2 and e^x, respectively. Again, we note that for any real number C, treated as constant function, its derivative is zero and hence, we can write (1), (2) and (3) as follows :

$$\frac{d}{dx}(\sin x + \mathrm{C}) = \cos x\ , \ \frac{d}{dx}\left(\frac{x^3}{3} + \mathrm{C}\right) = x^2 \text{ and } \frac{d}{dx}(e^x + \mathrm{C}) = e^x$$

Thus, anti derivatives (or integrals) of the above cited functions are not unique. Actually, there exist infinitely many anti derivatives of each of these functions which can be obtained by choosing C arbitrarily from the set of real numbers. For this reason C is customarily referred to as ***arbitrary constant***. In fact, C is the ***parameter*** by varying which one gets different anti derivatives (or integrals) of the given function.

More generally, if there is a function F such that $\frac{d}{dx}\mathrm{F}(x) = f(x)$, $\forall\, x \in \mathrm{I}$ (interval), then for any arbitrary real number C, (also called *constant of integration*)

$$\frac{d}{dx}\left[\mathrm{F}(x) + \mathrm{C}\right] = f(x),\ x \in \mathrm{I}$$

Thus, $\{F + C, C \in \mathbf{R}\}$ denotes a family of anti derivatives of f.

Remark Functions with same derivatives differ by a constant. To show this, let g and h be two functions having the same derivatives on an interval I.

Consider the function $f = g - h$ defined by $f(x) = g(x) - h(x)$, $\forall\, x \in$ I

Then $\quad \frac{df}{dx} = f' = g' - h'$ giving $f'(x) = g'(x) - h'(x)\ \forall\, x \in$ I

or $\quad f'(x) = 0$, $\forall\, x \in$ I by hypothesis,

i.e., the rate of change of f with respect to x is zero on I and hence f is constant.

In view of the above remark, it is justified to infer that the family $\{F + C, C \in \mathbf{R}\}$ provides all possible anti derivatives of f.

We introduce a new symbol, namely, $\int f(x)\, dx$ which will represent the entire class of anti derivatives read as the indefinite integral of f with respect to x.

Symbolically, we write $\int f(x)\, dx = F(x) + C$.

Notation Given that $\frac{dy}{dx} = f(x)$, we write $y = \int f(x)\, dx$.

For the sake of convenience, we mention below the following symbols/terms/phrases with their meanings as given in the Table (7.1).

Table 7.1

Symbols/Terms/Phrases	**Meaning**
$\int f(x)\, dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)\, dx$	Integrand
x in $\int f(x)\, dx$	Variable of integration
Integrate	Find the integral
An integral of f	A function F such that $F'(x) = f(x)$
Integration	The process of finding the integral
Constant of Integration	Any real number C, considered as constant function

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down immediately the corresponding formulae (referred to as standard formulae) for the integrals of these functions, as listed below which will be used to find integrals of other functions.

Derivatives	**Integrals (Anti derivatives)**
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$;	$\int x^n\, dx=\frac{x^{n+1}}{n+1}+\mathrm{C}$, $n \neq -1$
Particularly, we note that	
$\frac{d}{dx}(x)=1$;	$\int dx = x+\mathrm{C}$
(ii) $\frac{d}{dx}(\sin x)=\cos x$;	$\int \cos x\, dx=\sin x+\mathrm{C}$
(iii) $\frac{d}{dx}(-\cos x)=\sin x$;	$\int \sin x\, dx=-\cos x+\mathrm{C}$
(iv) $\frac{d}{dx}(\tan x)=\sec^2 x$;	$\int \sec^2 x\, dx=\tan x+\mathrm{C}$
(v) $\frac{d}{dx}(-\cot x)=\mathrm{cosec}^2 x$;	$\int \mathrm{cosec}^2 x\, dx=-\cot x+\mathrm{C}$
(vi) $\frac{d}{dx}(\sec x)=\sec x\tan x$;	$\int \sec x\tan x\, dx=\sec x+\mathrm{C}$
(vii) $\frac{d}{dx}(-\mathrm{cosec}\, x)=\mathrm{cosec}\, x\cot x$;	$\int \mathrm{cosec}\, x\cot x\, dx=-\mathrm{cosec}\, x+\mathrm{C}$
(viii) $\frac{d}{dx}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1} x+\mathrm{C}$
(ix) $\frac{d}{dx}\left(-\cos^{-1} x\right)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=-\cos^{-1} x+\mathrm{C}$
(x) $\frac{d}{dx}\left(\tan^{-1} x\right)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=\tan^{-1} x+\mathrm{C}$
(xi) $\frac{d}{dx}\left(-\cot^{-1} x\right)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=-\cot^{-1} x+\mathrm{C}$

(xii) $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{x\sqrt{x^2-1}}$; $\quad \displaystyle\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + \mathrm{C}$

(xiii) $\dfrac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right) = \dfrac{1}{x\sqrt{x^2-1}}$; $\quad \displaystyle\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + \mathrm{C}$

(xiv) $\dfrac{d}{dx}(e^x) = e^x$; $\quad \displaystyle\int e^x\, dx = e^x + \mathrm{C}$

(xv) $\dfrac{d}{dx}(\log|x|) = \dfrac{1}{x}$; $\quad \displaystyle\int \frac{1}{x}\, dx = \log|x| + \mathrm{C}$

(xvi) $\dfrac{d}{dx}\left(\dfrac{a^x}{\log a}\right) = a^x$; $\quad \displaystyle\int a^x\, dx = \frac{a^x}{\log a} + \mathrm{C}$

Note In practice, we normally do not mention the interval over which the various functions are defined. However, in any specific problem one has to keep it in mind.

7.2.1 *Geometrical interpretation of indefinite integral*

Let $f(x) = 2x$. Then $\int f(x)\, dx = x^2 + \mathrm{C}$. For different values of C, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + \mathrm{C}$, where C is arbitrary constant, represents a family of integrals. By assigning different values to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

Clearly, for C = 0, we obtain $y = x^2$, a parabola with its vertex on the origin. The curve $y = x^2 + 1$ for C = 1 is obtained by shifting the parabola $y = x^2$ one unit along y-axis in positive direction. For C = – 1, $y = x^2 - 1$ is obtained by shifting the parabola $y = x^2$ one unit along y-axis in the negative direction. Thus, for each positive value of C, each parabola of the family has its vertex on the positive side of the y-axis and for negative values of C, each has its vertex along the negative side of the y-axis. Some of these have been shown in the Fig 7.1.

Let us consider the intersection of all these parabolas by a line $x = a$. In the Fig 7.1, we have taken $a > 0$. The same is true when $a < 0$. If the line $x = a$ intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at P_0, P_1, P_2, P_{-1}, P_{-2} etc., respectively, then $\dfrac{dy}{dx}$ at these points equals $2a$. This indicates that the tangents to the curves at these points are parallel. Thus, $\int 2x\, dx = x^2 + \mathrm{C} = \mathrm{F_C}(x)$ (say), implies that

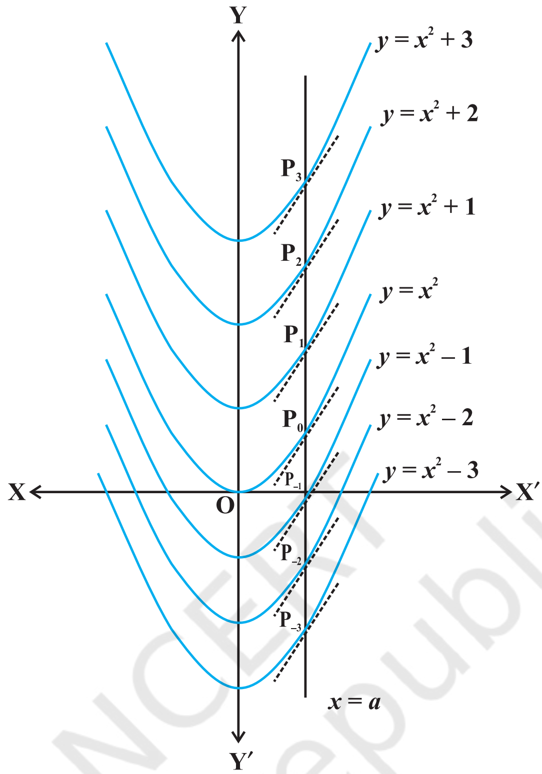

Fig 7.1

the tangents to all the curves $y = F_C(x)$, $C \in \mathbf{R}$, at the points of intersection of the curves by the line $x = a$, ($a \in \mathbf{R}$), are parallel.

Further, the following equation (statement) $\int f(x)\,dx = F(x) + C = y$ (say), represents a family of curves. The different values of C will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

7.2.2 *Some properties of indefinite integral*

In this sub section, we shall derive some properties of indefinite integrals.

(I) The process of differentiation and integration are inverses of each other in the sense of the following results :

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

and $$\int f'(x)\,dx = f(x) + C, \text{ where C is any arbitrary constant.}$$

Proof Let F be any anti derivative of f, i.e.,

$$\frac{d}{dx}\text{F}(x) = f(x)$$

Then $$\int f(x)\,dx = \text{F}(x) + \text{C}$$

Therefore $$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}(\text{F}(x)+\text{C})$$

$$= \frac{d}{dx}\text{F}(x) = f(x)$$

Similarly, we note that

$$f'(x) = \frac{d}{dx}f(x)$$

and hence $$\int f'(x)\,dx = f(x) + \text{C}$$

where C is arbitrary constant called constant of integration.

(II) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

Proof Let f and g be two functions such that

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$$

or $$\frac{d}{dx}\left[\int f(x)\,dx - \int g(x)\,dx\right] = 0$$

Hence $\int f(x)\,dx - \int g(x)\,dx =$ C, where C is any real number (Why?)

or $$\int f(x)\,dx = \int g(x)\,dx + \text{C}$$

So the families of curves $\left\{\int f(x)\,dx + \text{C}_1, \text{C}_1 \in \text{R}\right\}$

and $\left\{\int g(x)\,dx + \text{C}_2, \text{C}_2 \in \text{R}\right\}$ are identical.

Hence, in this sense, $\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent.

Note The equivalence of the families $\left\{\int f(x)\,dx + C_1, C_1 \in \mathbf{R}\right\}$ and $\left\{\int g(x)\,dx + C_2, C_2 \in \mathbf{R}\right\}$ is customarily expressed by writing $\int f(x)\,dx = \int g(x)\,dx$, without mentioning the parameter.

(III) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Proof By Property (I), we have

$$\frac{d}{dx}\left[\int [f(x) + g(x)]\,dx\right] = f(x) + g(x) \quad \text{... (1)}$$

On the otherhand, we find that

$$\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx$$

$$= f(x) + g(x) \quad \text{... (2)}$$

Thus, in view of Property (II), it follows by (1) and (2) that

$$\int \left(f(x) + g(x)\right) dx = \int f(x)\,dx + \int g(x)\,dx.$$

(IV) For any real number k, $\int k\,f(x)\,dx = k\int f(x)\,dx$

Proof By the Property (I), $\frac{d}{dx}\int k\,f(x)\,dx = k\,f(x)$.

Also $\frac{d}{dx}\left[k\int f(x)\,dx\right] = k\frac{d}{dx}\int f(x)\,dx = k\,f(x)$

Therefore, using the Property (II), we have $\int k\,f(x)\,dx = k\int f(x)\,dx$.

(V) Properties (III) and (IV) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int [k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x)]\,dx$$

$$= k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + ... + k_n\int f_n(x)\,dx.$$

To find an anti derivative of a given function, we search intuitively for a function whose derivative is the given function. The search for the requisite function for finding an anti derivative is known as integration by the method of inspection. We illustrate it through some examples.

Example 1 Write an anti derivative for each of the following functions using the method of inspection:

(i) $\cos 2x$ (ii) $3x^2 + 4x^3$ (iii) $\frac{1}{x}, x \neq 0$

Solution

(i) We look for a function whose derivative is $\cos 2x$. Recall that

$$\frac{d}{dx} \sin 2x = 2 \cos 2x$$

or $$\cos 2x = \frac{1}{2} \frac{d}{dx} (\sin 2x) = \frac{d}{dx}\left(\frac{1}{2} \sin 2x\right)$$

Therefore, an anti derivative of $\cos 2x$ is $\frac{1}{2} \sin 2x$.

(ii) We look for a function whose derivative is $3x^2 + 4x^3$. Note that

$$\frac{d}{dx}\left(x^3 + x^4\right) = 3x^2 + 4x^3.$$

Therefore, an anti derivative of $3x^2 + 4x^3$ is $x^3 + x^4$.

(iii) We know that

$$\frac{d}{dx}(\log x) = \frac{1}{x}, x > 0 \text{ and } \frac{d}{dx}[\log(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}, x < 0$$

Combining above, we get $\frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}, x \neq 0$

Therefore, $\int \frac{1}{x} dx = \log|x|$ is one of the anti derivatives of $\frac{1}{x}$.

Example 2 Find the following integrals:

(i) $\int \frac{x^3 - 1}{x^2} dx$ (ii) $\int (x^{\frac{2}{3}} + 1)\, dx$ (iii) $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})\, dx$

Solution

(i) We have

$$\int \frac{x^3 - 1}{x^2} dx = \int x\, dx - \int x^{-2}\, dx \qquad \text{(by Property V)}$$

$$= \left(\frac{x^{1+1}}{1+1} + C_1\right) - \left(\frac{x^{-2+1}}{-2+1} + C_2\right); \text{ } C_1, C_2 \text{ are constants of integration}$$

$$= \frac{x^2}{2} + C_1 - \frac{x^{-1}}{-1} - C_2 = \frac{x^2}{2} + \frac{1}{x} + C_1 - C_2$$

$$= \frac{x^2}{2} + \frac{1}{x} + C, \text{ where } C = C_1 - C_2 \text{ is another constant of integration.}$$

Note From now onwards, we shall write only one constant of integration in the final answer.

(ii) We have

$$\int (x^{\frac{2}{3}} + 1)\, dx = \int x^{\frac{2}{3}}\, dx + \int dx$$

$$= \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + x + C = \frac{3}{5} x^{\frac{5}{3}} + x + C$$

(iii) We have $\int (x^{\frac{3}{2}} + 2\,e^x - \frac{1}{x})\, dx = \int x^{\frac{3}{2}}\, dx + \int 2\,e^x\, dx - \int \frac{1}{x}\, dx$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2\,e^x - \log|x| + C$$

$$= \frac{2}{5} x^{\frac{5}{2}} + 2\,e^x - \log|x| + C$$

Example 3 Find the following integrals:

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \text{cosec } x\, (\text{cosec } x + \cot x)\, dx$

(iii) $\int \frac{1 - \sin x}{\cos^2 x}\, dx$

Solution

(i) We have

$$\int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx$$

$$= -\cos x + \sin x + C$$

(ii) We have

$$\int (\text{cosec}\, x\, (\text{cosec}\, x + \cot x)\, dx = \int \text{cosec}^2 x\, dx + \int \text{cosec}\, x \cot x\, dx$$

$$= -\cot x - \text{cosec}\, x + C$$

(iii) We have

$$\int \frac{1 - \sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx - \int \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 x\, dx - \int \tan x \sec x\, dx$$

$$= \tan x - \sec x + C$$

Example 4 Find the anti derivative F of f defined by $f(x) = 4x^3 - 6$, where F (0) = 3

Solution One anti derivative of $f(x)$ is $x^4 - 6x$ since

$$\frac{d}{dx}(x^4 - 6x) = 4x^3 - 6$$

Therefore, the anti derivative F is given by

$$F(x) = x^4 - 6x + C, \text{ where C is constant.}$$

Given that $F(0) = 3$, which gives,

$$3 = 0 - 6 \times 0 + C \quad \text{or} \quad C = 3$$

Hence, the required anti derivative is the unique function F defined by

$$F(x) = x^4 - 6x + 3.$$

Remarks

(i) We see that if F is an anti derivative of f, then so is F + C, where C is any constant. Thus, if we know one anti derivative F of a function f, we can write down an infinite number of anti derivatives of f by adding any constant to F expressed by $F(x) + C, C \in \mathbf{R}$. In applications, it is often necessary to satisfy an additional condition which then determines a specific value of C giving unique anti derivative of the given function.

(ii) Sometimes, F is not expressible in terms of elementary functions viz., polynomial, logarithmic, exponential, trigonometric functions and their inverses etc. We are therefore blocked for finding $\int f(x)\, dx$. For example, it is not possible to find $\int e^{-x^2} dx$ by inspection since we can not find a function whose derivative is e^{-x^2}

(iii) When the variable of integration is denoted by a variable other than x, the integral formulae are modified accordingly. For instance

$$\int y^4\, dy = \frac{y^{4+1}}{4+1} + C = \frac{1}{5} y^5 + C$$

7.2.3 *Comparison between differentiation and integration*

1. Both are operations on functions.
2. Both satisfy the property of linearity, i.e.,

   (i) $\frac{d}{dx}\left[k_1 f_1(x) + k_2 f_2(x)\right] = k_1 \frac{d}{dx} f_1(x) + k_2 \frac{d}{dx} f_2(x)$

   (ii) $\int \left[k_1 f_1(x) + k_2 f_2(x)\right] dx = k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx$

   Here k_1 and k_2 are constants.
3. We have already seen that all functions are not differentiable. Similarly, all functions are not integrable. We will learn more about nondifferentiable functions and nonintegrable functions in higher classes.
4. The derivative of a function, when it exists, is a unique function. The integral of a function is not so. However, they are unique upto an additive constant, i.e., any two integrals of a function differ by a constant.
5. When a polynomial function P is differentiated, the result is a polynomial whose degree is 1 less than the degree of P. When a polynomial function P is integrated, the result is a polynomial whose degree is 1 more than that of P.
6. We can speak of the derivative at a point. We never speak of the integral at a point, we speak of the integral of a function over an interval on which the integral is defined as will be seen in Section 7.7.
7. The derivative of a function has a geometrical meaning, namely, the slope of the tangent to the corresponding curve at a point. Similarly, the indefinite integral of a function represents geometrically, a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines orthogonal (perpendicular) to the axis representing the variable of integration.
8. The derivative is used for finding some physical quantities like the velocity of a moving particle, when the distance traversed at any time t is known. Similarly, the integral is used in calculating the distance traversed when the velocity at time t is known.
9. Differentiation is a process involving limits. **So is integration,** as will be seen in Section 7.7.

10. The process of differentiation and integration are inverses of each other as discussed in Section 7.2.2 (i).

EXERCISE 7.1

Find an anti derivative (or integral) of the following functions by the method of inspection.

1. $\sin 2x$ **2.** $\cos 3x$ **3.** e^{2x}

4. $(ax + b)^2$ **5.** $\sin 2x - 4\, e^{3x}$

Find the following integrals in Exercises 6 to 20:

6. $\int (4\, e^{3x} + 1)\, dx$ **7.** $\int x^2 (1 - \frac{1}{x^2})\, dx$ **8.** $\int (ax^2 + bx + c)\, dx$

9. $\int (2x^2 + e^x)\, dx$ **10.** $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ **11.** $\int \frac{x^3 + 5x^2 - 4}{x^2}\, dx$

12. $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\, dx$ **13.** $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$ **14.** $\int (1 - x)\sqrt{x}\, dx$

15. $\int \sqrt{x}(\,3x^2 + 2x + 3)\, dx$ **16.** $\int (2x - 3\cos x + e^x)\, dx$

17. $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$ **18.** $\int \sec x\, (\sec x + \tan x)\, dx$

19. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\, dx$ **20.** $\int \frac{2 - 3\sin x}{\cos^2 x}\, dx.$

Choose the correct answer in Exercises 21 and 22.

21. The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

22. If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

7.3 Methods of Integration

In previous section, we discussed integrals of those functions which were readily obtainable from derivatives of some functions. It was based on inspection, i.e., on the search of a function F whose derivative is f which led us to the integral of f. However, this method, which depends on inspection, is not very suitable for many functions. Hence, we need to develop additional techniques or methods for finding the integrals by reducing them into standard forms. Prominent among them are methods based on:

1. Integration by Substitution
2. Integration using Partial Fractions
3. Integration by Parts

7.3.1 *Integration by substitution*

In this section, we consider the method of integration by substitution.

The given integral $\int f(x)\, dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Consider $\qquad \mathrm{I} = \int f(x)\, dx$

Put $x = g(t)$ so that $\dfrac{dx}{dt} = g'(t)$.

We write $\qquad dx = g'(t)\, dt$

Thus $\qquad \mathrm{I} = \int f(x)\, dx = \int f(g(t))\, g'(t)\, dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated in the following examples.

Example 5 Integrate the following functions w.r.t. x:

(i) $\sin mx$ $\qquad$ (ii) $2x \sin (x^2 + 1)$

(iii) $\dfrac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$ $\qquad$ (iv) $\dfrac{\sin (\tan^{-1} x)}{1+x^2}$

Solution

(i) We know that derivative of mx is m. Thus, we make the substitution $mx = t$ so that $mdx = dt$.

Therefore, $\qquad \int \sin mx\, dx = \dfrac{1}{m}\int \sin t\, dt = -\dfrac{1}{m} \cos t + \mathrm{C} = -\dfrac{1}{m} \cos mx + \mathrm{C}$

(ii) Derivative of $x^2 + 1$ is $2x$. Thus, we use the substitution $x^2 + 1 = t$ so that $2x\ dx = dt$.

Therefore, $\int 2x \sin (x^2+1)\, dx = \int \sin t\, dt = -\cos t + C = -\cos (x^2 + 1) + C$

(iii) Derivative of $\sqrt{x}$ is $\frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Thus, we use the substitution

$\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}} dx = dt$ giving $dx = 2t\ dt$.

Thus, $$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \int \frac{2t \tan^4 t \sec^2 t\, dt}{t} = 2 \int \tan^4 t \sec^2 t\, dt$$

Again, we make another substitution $\tan t = u$ so that $\sec^2 t\ dt = du$

Therefore, $$2 \int \tan^4 t \sec^2 t\, dt = 2 \int u^4\, du = 2 \frac{u^5}{5} + C$$

$$= \frac{2}{5} \tan^5 t + C \text{ (since } u = \tan t)$$

$$= \frac{2}{5} \tan^5 \sqrt{x} + C \text{ (since } t = \sqrt{x})$$

Hence, $$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \frac{2}{5} \tan^5 \sqrt{x} + C$$

Alternatively, make the substitution $\tan \sqrt{x} = t$

(iv) Derivative of $\tan^{-1} x = \frac{1}{1+x^2}$. Thus, we use the substitution

$$\tan^{-1} x = t \text{ so that } \frac{dx}{1+x^2} = dt.$$

Therefore, $$\int \frac{\sin (\tan^{-1} x)}{1+x^2} dx = \int \sin t\, dt = -\cos t + C = -\cos (\tan^{-1} x) + C$$

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i) $\boldsymbol{\int \tan x\, dx = \log |\sec x| + C}$

We have

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$$

Put $\cos x = t$ so that $\sin x\, dx = -\, dt$

Then $$\int \tan x\, dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$$

or $$\int \tan x\, dx = \log|\sec x| + C$$

(ii) $\displaystyle\int \cot x\, dx = \log|\sin x| + C$

We have $\displaystyle\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$

Put $\sin x = t$ so that $\cos x\, dx = dt$

Then $$\int \cot x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$$

(iii) $\displaystyle\int \sec x\, dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x\, dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

Put $\sec x + \tan x = t$ so that $\sec x\,(\tan x + \sec x)\, dx = dt$

Therefore, $\displaystyle\int \sec x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$

(iv) $\displaystyle\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

We have

$$\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)}\, dx$$

Put $\operatorname{cosec} x + \cot x = t$ so that $-\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)\, dx = dt$

So $$\int \operatorname{cosec} x\, dx = -\int \frac{dt}{t} = -\log|t| = -\log|\operatorname{cosec} x + \cot x| + C$$

$$= -\log\left|\frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec} x - \cot x}\right| + C$$

$$= \log|\operatorname{cosec} x - \cot x| + C$$

Example 6 Find the following integrals:

(i) $\displaystyle\int \sin^3 x \cos^2 x\, dx$ (ii) $\displaystyle\int \frac{\sin x}{\sin(x+a)}\, dx$ (iii) $\displaystyle\int \frac{1}{1+\tan x}\, dx$

Solution

(i) We have

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \, (\sin x) \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \, (\sin x) \, dx$$

Put $t = \cos x$ so that $dt = -\sin x \, dx$

Therefore, $\int \sin^2 x \cos^2 x \, (\sin x) \, dx = -\int (1 - t^2) \, t^2 \, dt$

$$= -\int (t^2 - t^4) \, dt = -\left(\frac{t^3}{3} - \frac{t^5}{5}\right) + \text{C}$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + \text{C}$$

(ii) Put $x + a = t$. Then $dx = dt$. Therefore

$$\int \frac{\sin x}{\sin (x + a)} dx = \int \frac{\sin (t - a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t} dt$$

$$= \cos a \int dt - \sin a \int \cot t \, dt$$

$$= (\cos a) \, t - (\sin a) \left[\log \left|\sin t\right| + \text{C}_1\right]$$

$$= (\cos a) \, (x + a) - (\sin a) \left[\log \left|\sin (x + a)\right| + \text{C}_1\right]$$

$$= x \cos a + a \cos a - (\sin a) \log \left|\sin (x + a)\right| - \text{C}_1 \sin a$$

Hence, $\int \frac{\sin x}{\sin (x + a)} dx = x \cos a - \sin a \log |\sin (x + a)| + \text{C}$,

where, $\text{C} = -\text{C}_1 \sin a + a \cos a$, is another arbitrary constant.

(iii) $$\int \frac{dx}{1 + \tan x} = \int \frac{\cos x \, dx}{\cos x + \sin x}$$

$$= \frac{1}{2} \int \frac{(\cos x + \sin x + \cos x - \sin x) \, dx}{\cos x + \sin x}$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

$$= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx \qquad \ldots (1)$$

Now, consider $\mathrm{I} = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

Put $\cos x + \sin x = t$ so that $(\cos x - \sin x)\,dx = dt$

Therefore $\qquad \mathrm{I} = \int \frac{dt}{t} = \log|t| + C_2 = \log|\cos x + \sin x| + C_2$

Putting it in (1), we get

$$\int \frac{dx}{1+\tan x} = \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_1}{2} + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + C, \left(C = \frac{C_1}{2} + \frac{C_2}{2}\right)$$

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

1. $\frac{2x}{1+x^2}$ **2.** $\frac{(\log x)^2}{x}$ **3.** $\frac{1}{x + x\log x}$

4. $\sin x \sin(\cos x)$ **5.** $\sin(ax+b)\cos(ax+b)$

6. $\sqrt{ax+b}$ **7.** $x\sqrt{x+2}$ **8.** $x\sqrt{1+2x^2}$

9. $(4x+2)\sqrt{x^2+x+1}$ **10.** $\frac{1}{x-\sqrt{x}}$ **11.** $\frac{x}{\sqrt{x+4}}, x > 0$

12. $(x^3-1)^{\frac{1}{3}}\,x^5$ **13.** $\frac{x^2}{(2+3x^3)^3}$ **14.** $\frac{1}{x(\log x)^m}, x > 0, m \neq 1$

15. $\frac{x}{9-4x^2}$ **16.** e^{2x+3} **17.** $\frac{x}{e^{x^2}}$

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$ **19.** $\dfrac{e^{2x}-1}{e^{2x}+1}$ **20.** $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21. $\tan^2(2x-3)$ **22.** $\sec^2(7-4x)$ **23.** $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

24. $\dfrac{2\cos x-3\sin x}{6\cos x+4\sin x}$ **25.** $\dfrac{1}{\cos^2 x(1-\tan x)^2}$ **26.** $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

27. $\sqrt{\sin 2x}\cos 2x$ **28.** $\dfrac{\cos x}{\sqrt{1+\sin x}}$ **29.** $\cot x \log \sin x$

30. $\dfrac{\sin x}{1+\cos x}$ **31.** $\dfrac{\sin x}{(1+\cos x)^2}$ **32.** $\dfrac{1}{1+\cot x}$

33. $\dfrac{1}{1-\tan x}$ **34.** $\dfrac{\sqrt{\tan x}}{\sin x\cos x}$ **35.** $\dfrac{(1+\log x)^2}{x}$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$ **37.** $\dfrac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

Choose the correct answer in Exercises 38 and 39.

38. $\int \dfrac{10x^9+10^x\log_e 10\,dx}{x^{10}+10^x}$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$ (D) $\log(10^x + x^{10}) + C$

39. $\int \dfrac{dx}{\sin^2 x\cos^2 x}$ equals

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

7.3.2 *Integration using trigonometric identities*

When the integrand involves some trigonometric functions, we use some known identities to find the integral as illustrated through the following example.

Example 7 Find (i) $\int \cos^2 x\,dx$ (ii) $\int \sin 2x\cos 3x\,dx$ (iii) $\int \sin^3 x\,dx$

Solution

(i) Recall the identity $\cos 2x = 2\cos^2 x - 1$, which gives

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Therefore, $$\int \cos^2 x\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

(ii) Recall the identity $\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$ (Why?)

Then $$\int \sin 2x \cos 3x\, dx = \frac{1}{2}\left[\int \sin 5x\, dx - \int \sin x\, dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right] + C$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$$

(iii) From the identity $\sin 3x = 3\sin x - 4\sin^3 x$, we find that

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Therefore, $$\int \sin^3 x\, dx = \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

Alternatively, $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int (1-\cos^2 x)\sin x\, dx$

Put $\cos x = t$ so that $-\sin x\, dx = dt$

Therefore, $$\int \sin^3 x\, dx = -\int \left(1-t^2\right) dt = -\int dt + \int t^2\, dt = -t + \frac{t^3}{3} + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

Remark It can be shown using trigonometric identities that both answers are equivalent.

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

1. $\sin^2(2x+5)$
2. $\sin 3x \cos 4x$
3. $\cos 2x \cos 4x \cos 6x$
4. $\sin^3(2x+1)$
5. $\sin^3 x \cos^3 x$
6. $\sin x \sin 2x \sin 3x$
7. $\sin 4x \sin 8x$
8. $\dfrac{1-\cos x}{1+\cos x}$
9. $\dfrac{\cos x}{1+\cos x}$
10. $\sin^4 x$
11. $\cos^4 2x$
12. $\dfrac{\sin^2 x}{1+\cos x}$
13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$
14. $\dfrac{\cos x - \sin x}{1+\sin 2x}$
15. $\tan^3 2x \sec 2x$
16. $\tan^4 x$
17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$
18. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$
19. $\dfrac{1}{\sin x \cos^3 x}$
20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$
21. $\sin^{-1}(\cos x)$
22. $\dfrac{1}{\cos(x-a)\cos(x-b)}$

Choose the correct answer in Exercises 23 and 24.

23. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\,dx$ is equal to

    (A) $\tan x + \cot x + C$ (B) $\tan x + \operatorname{cosec} x + C$

    (C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

24. $\displaystyle\int \frac{e^x(1+x)}{\cos^2(e^x x)}\,dx$ equals

    (A) $-\cot(ex^x) + C$ (B) $\tan(xe^x) + C$

    (C) $\tan(e^x) + C$ (D) $\cot(e^x) + C$

7.4 Integrals of Some Particular Functions

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

(1) $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

(2) $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$

(3) $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

(4) $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$

(5) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

(6) $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$

We now prove the above results:

(1) We have $\displaystyle\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)}$

$$= \frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right] = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

Therefore, $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right]$

$$= \frac{1}{2a}\left[\log|(x-a)| - \log|(x+a)|\right] + C$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(2) In view of (1) above, we have

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left[\frac{(a+x)+(a-x)}{(a+x)(a-x)}\right] = \frac{1}{2a}\left[\frac{1}{a-x} + \frac{1}{a+x}\right]$$

Therefore, $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\left[\int \frac{dx}{a-x} + \int \frac{dx}{a+x}\right]$

$$= \frac{1}{2a}[-\log|a-x| + \log|a+x|] + C$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

Note The technique used in (1) will be explained in Section 7.5.

(3) Put $x = a \tan \theta$. Then $dx = a \sec^2 \theta \, d\theta$.

Therefore, $\int \frac{dx}{x^2 + a^2} = \int \frac{a \sec^2 \theta \, d\theta}{a^2 \tan^2\theta + a^2}$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4) Let $x = a \sec\theta$. Then $dx = a \sec\theta \tan\theta \, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec\theta \tan\theta \, d\theta}{\sqrt{a^2 \sec^2\theta - a^2}}$

$$= \int \sec\theta \, d\theta = \log|\sec\theta + \tan\theta| + C_1$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| - \log|a| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C, \text{ where } C = C_1 - \log|a|$$

(5) Let $x = a \sin\theta$. Then $dx = a \cos\theta \, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos\theta \, d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}}$

$$= \int d\theta = \theta + C = \sin^{-1}\frac{x}{a} + C$$

(6) Let $x = a \tan\theta$. Then $dx = a \sec^2\theta \, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \sec^2\theta \, d\theta}{\sqrt{a^2 \tan^2\theta + a^2}}$

$$= \int \sec\theta \, d\theta = \log|(\sec\theta + \tan\theta)| + C_1$$

$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right|+C_1$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|-\log|a|+C_1$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|+C \text{, where } C = C_1 - \log|a|$$

Applying these standard formulae, we now obtain some more formulae which are useful from applications point of view and can be applied directly to evaluate other integrals.

(7) To find the integral $\int\frac{dx}{ax^2+bx+c}$, we write

$$ax^2 + bx + c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

Now, put $x+\frac{b}{2a}=t$ so that $dx = dt$ and writing $\frac{c}{a}-\frac{b^2}{4a^2}=\pm k^2$. We find the integral reduced to the form $\frac{1}{a}\int\frac{dt}{t^2\pm k^2}$ depending upon the sign of $\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)$ and hence can be evaluated.

(8) To find the integral of the type $\int\frac{dx}{\sqrt{ax^2+bx+c}}$, proceeding as in (7), we obtain the integral using the standard formulae.

(9) To find the integral of the type $\int\frac{px+q}{ax^2+bx+c}dx$, where p, q, a, b, c are constants, we are to find real numbers A, B such that

$$px+q = A\frac{d}{dx}(ax^2+bx+c) + B = A\,(2ax+b) + B$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

(10) For the evaluation of the integral of the type $\int \frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}}$, we proceed as in (9) and transform the integral into known standard forms.

Let us illustrate the above methods by some examples.

Example 8 Find the following integrals:

(i) $\int \frac{dx}{x^2-16}$ (ii) $\int \frac{dx}{\sqrt{2x-x^2}}$

Solution

(i) We have $\int \frac{dx}{x^2-16} = \int \frac{dx}{x^2-4^2} = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + C$ [by 7.4 (1)]

(ii) $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

Put $x - 1 = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + C$ [by 7.4 (5)]

$$= \sin^{-1}(x-1) + C$$

Example 9 Find the following integrals :

(i) $\int \frac{dx}{x^2-6x+13}$ (ii) $\int \frac{dx}{3x^2+13x-10}$ (iii) $\int \frac{dx}{\sqrt{5x^2-2x}}$

Solution

(i) We have $x^2 - 6x + 13 = x^2 - 6x + 3^2 - 3^2 + 13 = (x-3)^2 + 4$

So, $\int \frac{dx}{x^2-6x+13} = \int \frac{1}{(x-3)^2+2^2}\,dx$

Let $x - 3 = t$. Then $dx = dt$

Therefore, $\int \frac{dx}{x^2-6x+13} = \int \frac{dt}{t^2+2^2} = \frac{1}{2}\tan^{-1}\frac{t}{2} + C$ [by 7.4 (3)]

$$= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

(ii) The given integral is of the form 7.4 (7). We write the denominator of the integrand,

$$3x^2 + 13x - 10 = 3\left(x^2 + \frac{13x}{3} - \frac{10}{3}\right)$$

$$= 3\left[\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2\right] \text{ (completing the square)}$$

Thus $\displaystyle\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3}\int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}$

Put $x + \frac{13}{6} = t$. Then $dx = dt$.

Therefore, $\displaystyle\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3}\int \frac{dt}{t^2 - \left(\frac{17}{6}\right)^2}$

$$= \frac{1}{3 \times 2 \times \frac{17}{6}} \log\left|\frac{t - \frac{17}{6}}{t + \frac{17}{6}}\right| + C_1 \qquad \text{[by 7.4 (i)]}$$

$$= \frac{1}{17} \log\left|\frac{x + \frac{13}{6} - \frac{17}{6}}{x + \frac{13}{6} + \frac{17}{6}}\right| + C_1$$

$$= \frac{1}{17} \log\left|\frac{6x - 4}{6x + 30}\right| + C_1$$

$$= \frac{1}{17} \log\left|\frac{3x - 2}{x + 5}\right| + C_1 + \frac{1}{17} \log\frac{1}{3}$$

$$= \frac{1}{17} \log\left|\frac{3x - 2}{x + 5}\right| + C, \text{ where } C = C_1 + \frac{1}{17} \log\frac{1}{3}$$

(iii) We have $\int \frac{dx}{\sqrt{5x^2 - 2x}} = \int \frac{dx}{\sqrt{5\left(x^2 - \frac{2x}{5}\right)}}$

$$= \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{\left(x - \frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}} \quad \text{(completing the square)}$$

Put $x - \frac{1}{5} = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{5x^2 - 2x}} = \frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{5}\right)^2}}$

$$= \frac{1}{\sqrt{5}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + C \qquad \text{[by 7.4 (4)]}$$

$$= \frac{1}{\sqrt{5}} \log \left| x - \frac{1}{5} + \sqrt{x^2 - \frac{2x}{5}} \right| + C$$

Example 10 Find the following integrals:

(i) $\int \frac{x+2}{2x^2 + 6x + 5} dx$ (ii) $\int \frac{x+3}{\sqrt{5 - 4x - x^2}} dx$

Solution

(i) Using the formula 7.4 (9), we express

$$x + 2 = A \frac{d}{dx}\left(2x^2 + 6x + 5\right) + B = A(4x + 6) + B$$

Equating the coefficients of x and the constant terms from both sides, we get

$4A = 1$ and $6A + B = 2$ or $A = \frac{1}{4}$ and $B = \frac{1}{2}$.

Therefore, $\int \frac{x+2}{2x^2 + 6x + 5} = \frac{1}{4} \int \frac{4x+6}{2x^2 + 6x + 5} dx + \frac{1}{2} \int \frac{dx}{2x^2 + 6x + 5}$

$$= \frac{1}{4} I_1 + \frac{1}{2} I_2 \quad \text{(say)} \qquad \text{... (1)}$$

In I_1, put $2x^2 + 6x + 5 = t$, so that $(4x + 6)\,dx = dt$

Therefore, $$I_1 = \int \frac{dt}{t} = \log|t| + C_1$$

$$= \log|2x^2 + 6x + 5| + C_1 \qquad \text{... (2)}$$

and $$I_2 = \int \frac{dx}{2x^2 + 6x + 5} = \frac{1}{2}\int \frac{dx}{x^2 + 3x + \frac{5}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

Put $x + \frac{3}{2} = t$, so that $dx = dt$, we get

$$I_2 = \frac{1}{2}\int \frac{dt}{t^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2 \times \frac{1}{2}} \tan^{-1} 2t + C_2 \qquad \text{[by 7.4 (3)]}$$

$$= \tan^{-1} 2\left(x + \frac{3}{2}\right) + C_2 = \tan^{-1}(2x + 3) + C_2 \qquad \text{... (3)}$$

Using (2) and (3) in (1), we get

$$\int \frac{x + 2}{2x^2 + 6x + 5}\,dx = \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{2}\tan^{-1}(2x + 3) + C$$

where, $$C = \frac{C_1}{4} + \frac{C_2}{2}$$

(ii) This integral is of the form given in 7.4 (10). Let us express

$$x + 3 = A\frac{d}{dx}(5 - 4x - x^2) + B = A(-4 - 2x) + B$$

Equating the coefficients of x and the constant terms from both sides, we get

$-2A = 1$ and $-4A + B = 3$, i.e., $A = -\frac{1}{2}$ and $B = 1$

Therefore, $\displaystyle\int \frac{x+3}{\sqrt{5-4x-x^2}}\,dx = -\frac{1}{2}\int \frac{(-4-2x)\,dx}{\sqrt{5-4x-x^2}} + \int \frac{dx}{\sqrt{5-4x-x^2}}$

$$= -\frac{1}{2}\,\mathrm{I}_1 + \mathrm{I}_2 \qquad \text{... (1)}$$

In I_1, put $5-4x-x^2 = t$, so that $(-4-2x)\,dx = dt$.

Therefore, $\displaystyle \mathrm{I}_1 = \int \frac{(-4-2x)\,dx}{\sqrt{5-4x-x^2}} = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + \mathrm{C}_1$

$$= 2\sqrt{5-4x-x^2} + \mathrm{C}_1 \qquad \text{... (2)}$$

Now consider $\displaystyle \mathrm{I}_2 = \int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}}$

Put $x + 2 = t$, so that $dx = dt$.

Therefore, $\displaystyle \mathrm{I}_2 = \int \frac{dt}{\sqrt{3^2-t^2}} = \sin^{-1}\frac{t}{3} + \mathrm{C}_2$ [by 7.4 (5)]

$$= \sin^{-1}\frac{x+2}{3} + \mathrm{C}_2 \qquad \text{... (3)}$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{\sqrt{5-4x-x^2}} = -\sqrt{5-4x-x^2} + \sin^{-1}\frac{x+2}{3} + \mathrm{C}, \text{ where } \mathrm{C} = \mathrm{C}_2 - \frac{\mathrm{C}_1}{2}$$

EXERCISE 7.4

Integrate the functions in Exercises 1 to 23.

1. $\dfrac{3x^2}{x^6+1}$ **2.** $\dfrac{1}{\sqrt{1+4x^2}}$ **3.** $\dfrac{1}{\sqrt{(2-x)^2+1}}$

4. $\dfrac{1}{\sqrt{9-25x^2}}$ **5.** $\dfrac{3x}{1+2x^4}$ **6.** $\dfrac{x^2}{1-x^6}$

7. $\dfrac{x-1}{\sqrt{x^2-1}}$ **8.** $\dfrac{x^2}{\sqrt{x^6+a^6}}$ **9.** $\dfrac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

10. $\dfrac{1}{\sqrt{x^2+2x+2}}$ **11.** $\dfrac{1}{9x^2+6x+5}$ **12.** $\dfrac{1}{\sqrt{7-6x-x^2}}$

13. $\dfrac{1}{\sqrt{(x-1)(x-2)}}$ **14.** $\dfrac{1}{\sqrt{8+3x-x^2}}$ **15.** $\dfrac{1}{\sqrt{(x-a)(x-b)}}$

16. $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$ **17.** $\dfrac{x+2}{\sqrt{x^2-1}}$ **18.** $\dfrac{5x-2}{1+2x+3x^2}$

19. $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$ **20.** $\dfrac{x+2}{\sqrt{4x-x^2}}$ **21.** $\dfrac{x+2}{\sqrt{x^2+2x+3}}$

22. $\dfrac{x+3}{x^2-2x-5}$ **23.** $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$.

Choose the correct answer in Exercises 24 and 25.

24. $\displaystyle\int \frac{dx}{x^2+2x+2}$ equals

(A) $x \tan^{-1}(x+1) + C$ (B) $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1}x + C$ (D) $\tan^{-1}x + C$

25. $\displaystyle\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\dfrac{1}{9}\sin^{-1}\left(\dfrac{9x-8}{8}\right)+C$ (B) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{8x-9}{9}\right)+C$

(C) $\dfrac{1}{3}\sin^{-1}\left(\dfrac{9x-8}{8}\right)+C$ (D) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{9x-8}{9}\right)+C$

7.5 Integration by Partial Fractions

Recall that a rational function is defined as the ratio of two polynomials in the form $\dfrac{P(x)}{Q(x)}$, where P (x) and Q(x) are polynomials in x and Q$(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational functions can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T($x$) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function. As we know how to integrate polynomials, the integration of any rational function is reduced to the integration of a proper rational function. The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors. Assume that we want to evaluate $\int \frac{P(x)}{Q(x)} dx$, where $\frac{P(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods. The following Table 7.2 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.2

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$,
	where $x^2 + bx + c$ cannot be factorised further	

In the above table, A, B and C are real numbers to be determined suitably.

Example 11 Find $\int \frac{dx}{(x+1)(x+2)}$

Solution The integrand is a proper rational function. Therefore, by using the form of partial fraction [Table 7.2 (i)], we write

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad \ldots (1)$$

where, real numbers A and B are to be determined suitably. This gives

$$1 = A(x+2) + B(x+1).$$

Equating the coefficients of x and the constant term, we get

$$A + B = 0$$

and

$$2A + B = 1$$

Solving these equations, we get A =1 and B = – 1.

Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$$

Therefore,

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log|x+1| - \log|x+2| + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C$$

Remark The equation (1) above is an identity, i.e. a statement true for all (permissible) values of x. Some authors use the symbol '≡' to indicate that the statement is an identity and use the symbol '=' to indicate that the statement is an equation, i.e., to indicate that the statement is true only for certain values of x.

Example 12 Find $\int \frac{x^2+1}{x^2-5x+6}\,dx$

Solution Here the integrand $\frac{x^2+1}{x^2-5x+6}$ is not proper rational function, so we divide $x^2 + 1$ by $x^2 - 5x + 6$ and find that

$$\frac{x^2+1}{x^2-5x+6} = 1+\frac{5x-5}{x^2-5x+6} = 1+\frac{5x-5}{(x-2)(x-3)}$$

Let $$\frac{5x-5}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3}$$

So that $$5x-5 = A(x-3)+B(x-2)$$

Equating the coefficients of x and constant terms on both sides, we get $A + B = 5$ and $3A + 2B = 5$. Solving these equations, we get $A = -5$ and $B = 10$

Thus, $$\frac{x^2+1}{x^2-5x+6} = 1-\frac{5}{x-2}+\frac{10}{x-3}$$

Therefore, $$\int\frac{x^2+1}{x^2-5x+6}dx = \int dx-5\int\frac{1}{x-2}dx+10\int\frac{dx}{x-3}$$

$$= x-5\log|x-2|+10\log|x-3|+C.$$

Example 13 Find $\int\frac{3x-2}{(x+1)^2(x+3)}dx$

Solution The integrand is of the type as given in Table 7.2 (4). We write

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$$

So that $$3x-2 = A(x+1)(x+3)+B(x+3)+C(x+1)^2$$

$$= A(x^2+4x+3)+B(x+3)+C(x^2+2x+1)$$

Comparing coefficient of x^2, x and constant term on both sides, we get $A + C = 0$, $4A + B + 2C = 3$ and $3A + 3B + C = -2$. Solving these equations, we get $A=\frac{11}{4}, B=\frac{-5}{2}$ and $C=\frac{-11}{4}$. Thus the integrand is given by

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4(x+1)}-\frac{5}{2(x+1)^2}-\frac{11}{4(x+3)}$$

Therefore, $$\int\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4}\int\frac{dx}{x+1}-\frac{5}{2}\int\frac{dx}{(x+1)^2}-\frac{11}{4}\int\frac{dx}{x+3}$$

$$= \frac{11}{4}\log|x+1|+\frac{5}{2(x+1)}-\frac{11}{4}\log|x+3|+C$$

$$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2(x+1)}+C$$

Example 14 Find $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

Solution Consider $\frac{x^2}{(x^2+1)(x^2+4)}$ and put $x^2 = y$.

Then $$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$$

Write $$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$$

So that $$y = A(y+4) + B(y+1)$$

Comparing coefficients of y and constant terms on both sides, we get A + B = 1 and 4A + B = 0, which give

$$A = -\frac{1}{3} \quad \text{and} \quad B = \frac{4}{3}$$

Thus, $$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

Therefore, $$\int \frac{x^2 dx}{(x^2+1)(x^2+4)} = -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{4}{3} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C$$

In the above example, the substitution was made only for the partial fraction part and not for the integration part. Now, we consider an example, where the integration involves a combination of the substitution method and the partial fraction method.

Example 15 Find $\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi} d\phi$

Solution Let $y = \sin\phi$

Then $$dy = \cos\phi \, d\phi$$

Therefore, $\displaystyle\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi}\, d\phi = \int \frac{(3y-2)\, dy}{5-(1-y^2)-4y}$

$$= \int \frac{3y-2}{y^2-4y+4}\, dy$$

$$= \int \frac{3y-2}{(y-2)^2} = \text{I (say)}$$

Now, we write $\displaystyle\frac{3y-2}{(y-2)^2} = \frac{\text{A}}{y-2} + \frac{\text{B}}{(y-2)^2}$ [by Table 7.2 (2)]

Therefore, $3y - 2 = \text{A}\,(y-2) + \text{B}$

Comparing the coefficients of y and constant term, we get A = 3 and B – 2A = – 2, which gives A = 3 and B = 4.

Therefore, the required integral is given by

$$\text{I} = \int \left[\frac{3}{y-2} + \frac{4}{(y-2)^2}\right] dy = 3\int \frac{dy}{y-2} + 4\int \frac{dy}{(y-2)^2}$$

$$= 3\log\left| y-2 \right| + 4\left(-\frac{1}{y-2}\right) + \text{C}$$

$$= 3\log\left| \sin\phi - 2 \right| + \frac{4}{2-\sin\phi} + \text{C}$$

$$= 3\log(2-\sin\phi) + \frac{4}{2-\sin\phi} + \text{C} \quad \text{(since, } 2-\sin\phi \text{ is always positive)}$$

Example 16 Find $\displaystyle\int \frac{x^2+x+1\, dx}{(x+2)\,(x^2+1)}$

Solution The integrand is a proper rational function. Decompose the rational function into partial fraction [Table 2.2(5)]. Write

$$\frac{x^2+x+1}{(x^2+1)\,(x+2)} = \frac{\text{A}}{x+2} + \frac{\text{B}x+\text{C}}{(x^2+1)}$$

Therefore, $x^2 + x + 1 = \text{A}\,(x^2+1) + (\text{B}x + \text{C})\,(x+2)$

Equating the coefficients of x^2, x and of constant term of both sides, we get A + B = 1, 2B + C = 1 and A + 2C = 1. Solving these equations, we get $A = \frac{3}{5}, B = \frac{2}{5}$ and $C = \frac{1}{5}$

Thus, the integrand is given by

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{3}{5(x+2)} + \frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1} = \frac{3}{5(x+2)} + \frac{1}{5}\left(\frac{2x+1}{x^2+1}\right)$$

Therefore, $\displaystyle\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx = \frac{3}{5}\int\frac{dx}{x+2} + \frac{1}{5}\int\frac{2x}{x^2+1}dx + \frac{1}{5}\int\frac{1}{x^2+1}dx$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + C$$

EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

1. $\dfrac{x}{(x+1)(x+2)}$ **2.** $\dfrac{1}{x^2-9}$ **3.** $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$

4. $\dfrac{x}{(x-1)(x-2)(x-3)}$ **5.** $\dfrac{2x}{x^2+3x+2}$ **6.** $\dfrac{1-x^2}{x(1-2x)}$

7. $\dfrac{x}{(x^2+1)(x-1)}$ **8.** $\dfrac{x}{(x-1)^2(x+2)}$ **9.** $\dfrac{3x+5}{x^3-x^2-x+1}$

10. $\dfrac{2x-3}{(x^2-1)(2x+3)}$ **11.** $\dfrac{5x}{(x+1)(x^2-4)}$ **12.** $\dfrac{x^3+x+1}{x^2-1}$

13. $\dfrac{2}{(1-x)(1+x^2)}$ **14.** $\dfrac{3x-1}{(x+2)^2}$ **15.** $\dfrac{1}{x^4-1}$

16. $\dfrac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

17. $\dfrac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint : Put $\sin x = t$]

18. $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ **19.** $\dfrac{2x}{(x^2+1)(x^2+3)}$ **20.** $\dfrac{1}{x(x^4-1)}$

21. $\dfrac{1}{(e^x-1)}$ [Hint : Put $e^x = t$]

Choose the correct answer in each of the Exercises 22 and 23.

22. $\displaystyle\int \frac{x\,dx}{(x-1)(x-2)}$ equals

(A) $\log\left|\dfrac{(x-1)^2}{x-2}\right|+C$ (B) $\log\left|\dfrac{(x-2)^2}{x-1}\right|+C$

(C) $\log\left|\left(\dfrac{x-1}{x-2}\right)^2\right|+C$ (D) $\log|(x-1)(x-2)|+C$

23. $\displaystyle\int \frac{dx}{x(x^2+1)}$ equals

(A) $\log|x|-\dfrac{1}{2}\log(x^2+1)+C$ (B) $\log|x|+\dfrac{1}{2}\log(x^2+1)+C$

(C) $-\log|x|+\dfrac{1}{2}\log(x^2+1)+C$ (D) $\dfrac{1}{2}\log|x|+\log(x^2+1)+C$

7.6 Integration by Parts

In this section, we describe one more method of integration, that is found quite useful in integrating products of functions.

If u and v are any two differentiable functions of a single variable x (say). Then, by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx}+v\frac{du}{dx}$$

Integrating both sides, we get

$$uv = \int u\frac{dv}{dx}\,dx+\int v\frac{du}{dx}\,dx$$

or
$$\int u\frac{dv}{dx}\,dx = uv-\int v\frac{du}{dx}\,dx \qquad \ldots (1)$$

Let
$$u = f(x) \text{ and } \frac{dv}{dx} = g(x). \text{ Then}$$

$$\frac{du}{dx} = f'(x) \text{ and } v = \int g(x)\,dx$$

Therefore, expression (1) can be rewritten as

$$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int [\int g(x)\, dx]\, f'(x)\, dx$$

i.e.,
$$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int [f'(x) \int g(x)\, dx]\, dx$$

If we take f as the first function and g as the second function, then this formula may be stated as follows:

"The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Example 17 Find $\int x \cos x\, dx$

Solution Put $f(x) = x$ (first function) and $g(x) = \cos x$ (second function). Then, integration by parts gives

$$\int x \cos x\, dx = x \int \cos x\, dx - \int [\frac{d}{dx}(x) \int \cos x\, dx]\, dx$$

$$= x \sin x - \int \sin x\, dx = x \sin x + \cos x + \mathrm{C}$$

Suppose, we take $f(x) = \cos x$ and $g(x) = x$. Then

$$\int x \cos x\, dx = \cos x \int x\, dx - \int [\frac{d}{dx}(\cos x) \int x\, dx]\, dx$$

$$= (\cos x)\frac{x^2}{2} + \int \sin x \frac{x^2}{2}\, dx$$

Thus, it shows that the integral $\int x \cos x\, dx$ is reduced to the comparatively more complicated integral having more power of x. Therefore, the proper choice of the first function and the second function is significant.

Remarks

(i) It is worth mentioning that integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for $\int \sqrt{x} \sin x\, dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x}\, \sin x$.

(ii) Observe that while finding the integral of the second function, we did not add any constant of integration. If we write the integral of the second function $\cos x$

as $\sin x + k$, where k is any constant, then

$$\int x \cos x \, dx = x(\sin x + k) - \int (\sin x + k)\, dx$$

$$= x(\sin x + k) - \int (\sin x \, dx - \int k \, dx$$

$$= x(\sin x + k) - \cos x - kx + C = x \sin x + \cos x + C$$

This shows that adding a constant to the integral of the second function is superfluous so far as the final result is concerned while applying the method of integration by parts.

(iii) Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, in cases where other function is inverse trigonometric function or logarithmic function, then we take them as first function.

Example 18 Find $\int \log x \, dx$

Solution To start with, we are unable to guess a function whose derivative is $\log x$. We take $\log x$ as the first function and the constant function 1 as the second function. Then, the integral of the second function is x.

Hence, $$\int (\log x.1)\, dx = \log x \int 1\, dx - \int [\frac{d}{dx}(\log x) \int 1\, dx]\, dx$$

$$= (\log x) \cdot x - \int \frac{1}{x} x \, dx = x \log x - x + C .$$

Example 19 Find $\int x e^x dx$

Solution Take first function as x and second function as e^x. The integral of the second function is e^x.

Therefore, $$\int x e^x dx = x e^x - \int 1 \cdot e^x dx = xe^x - e^x + C.$$

Example 20 Find $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Solution Let first function be $\sin^{-1} x$ and second function be $\frac{x}{\sqrt{1-x^2}}$.

First we find the integral of the second function, i.e., $\int \frac{x\, dx}{\sqrt{1-x^2}}$.

Put $t = 1 - x^2$. Then $dt = -2x\, dx$

Therefore, $$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

Hence, $$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx = (\sin^{-1} x)\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right) dx$$
$$= -\sqrt{1-x^2}\,\sin^{-1}x + x + C = x - \sqrt{1-x^2}\,\sin^{-1}x + C$$

Alternatively, this integral can also be worked out by making substitution $\sin^{-1} x = \theta$ and then integrating by parts.

Example 21 Find $\int e^x \sin x\, dx$

Solution Take e^x as the first function and $\sin x$ as second function. Then, integrating by parts, we have

$$\mathrm{I} = \int e^x \sin x\,dx = e^x(-\cos x) + \int e^x \cos x\,dx$$
$$= -e^x \cos x + \mathrm{I}_1 \text{ (say)} \qquad \ldots (1)$$

Taking e^x and $\cos x$ as the first and second functions, respectively, in I_1, we get

$$\mathrm{I}_1 = e^x \sin x - \int e^x \sin x\,dx$$

Substituting the value of I_1 in (1), we get

$$\mathrm{I} = -e^x \cos x + e^x \sin x - \mathrm{I} \text{ or } 2\mathrm{I} = e^x(\sin x - \cos x)$$

Hence, $$\mathrm{I} = \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Alternatively, above integral can also be determined by taking $\sin x$ as the first function and e^x the second function.

7.6.1 *Integral of the type* $\int e^x [f(x) + f'(x)]\,dx$

We have $$\mathrm{I} = \int e^x [f(x) + f'(x)]\,dx = \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$
$$= \mathrm{I}_1 + \int e^x f'(x)\,dx, \text{ where } \mathrm{I}_1 = \int e^x f(x)\,dx \qquad \ldots (1)$$

Taking $f(x)$ and e^x as the first function and second function, respectively, in I_1 and integrating it by parts, we have $\mathrm{I}_1 = f(x)\,e^x - \int f'(x)\,e^x dx + C$

Substituting I_1 in (1), we get

$$\mathrm{I} = e^x f(x) - \int f'(x)\,e^x dx + \int e^x f'(x)\,dx + C = e^x f(x) + C$$

Thus, $$\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C$$

Example 22 Find (i) $\int e^x (\tan^{-1}x + \frac{1}{1+x^2})\,dx$ (ii) $\int \frac{(x^2+1)\,e^x}{(x+1)^2}\,dx$

Solution

(i) We have $I = \int e^x (\tan^{-1}x + \frac{1}{1+x^2})\,dx$

Consider $f(x) = \tan^{-1}x$, then $f'(x) = \frac{1}{1+x^2}$

Thus, the given integrand is of the form $e^x\,[f(x) + f'(x)]$.

Therefore, $I = \int e^x (\tan^{-1}x + \frac{1}{1+x^2})\,dx = e^x \tan^{-1}x + C$

(ii) We have $I = \int \frac{(x^2+1)\,e^x}{(x+1)^2}\,dx = \int e^x [\frac{x^2-1+1+1}{(x+1)^2}]\,dx$

$$= \int e^x [\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}]\,dx = \int e^x [\frac{x-1}{x+1} + \frac{2}{(x+1)^2}]\,dx$$

Consider $f(x) = \frac{x-1}{x+1}$, then $f'(x) = \frac{2}{(x+1)^2}$

Thus, the given integrand is of the form $e^x\,[f(x) + f'(x)]$.

Therefore, $$\int \frac{x^2+1}{(x+1)^2}\,e^x\,dx = \frac{x-1}{x+1}\,e^x + C$$

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

1. $x \sin x$ **2.** $x \sin 3x$ **3.** $x^2 e^x$ **4.** $x \log x$

5. $x \log 2x$ **6.** $x^2 \log x$ **7.** $x \sin^{-1}x$ **8.** $x \tan^{-1} x$

9. $x \cos^{-1} x$ **10.** $(\sin^{-1}x)^2$ **11.** $\frac{x \cos^{-1}x}{\sqrt{1-x^2}}$ **12.** $x \sec^2 x$

13. $\tan^{-1}x$ **14.** $x (\log x)^2$ **15.** $(x^2 + 1) \log x$

16. $e^x (\sin x + \cos x)$ **17.** $\dfrac{x e^x}{(1+x)^2}$ **18.** $e^x \left(\dfrac{1+\sin x}{1+\cos x}\right)$

19. $e^x \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$ **20.** $\dfrac{(x-3)\, e^x}{(x-1)^3}$ **21.** $e^{2x} \sin x$

22. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3} dx$ equals

(A) $\dfrac{1}{3} e^{x^3} + C$ (B) $\dfrac{1}{3} e^{x^2} + C$

(C) $\dfrac{1}{2} e^{x^3} + C$ (D) $\dfrac{1}{2} e^{x^2} + C$

24. $\int e^x \sec x\,(1 + \tan x)\, dx$ equals

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

7.6.2 *Integrals of some more types*

Here, we discuss some special types of standard integrals based on the technique of integration by parts :

(i) $\int \sqrt{x^2 - a^2}\, dx$ (ii) $\int \sqrt{x^2 + a^2}\, dx$ (iii) $\int \sqrt{a^2 - x^2}\, dx$

(i) Let $I = \int \sqrt{x^2 - a^2}\, dx$

Taking constant function 1 as the second function and integrating by parts, we have

$$I = x\sqrt{x^2 - a^2} - \int \frac{1}{2} \frac{2x}{\sqrt{x^2 - a^2}}\, x\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}}\, dx = x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - \mathrm{I} - a^2 \int \frac{dx}{\sqrt{x^2-a^2}}$$

or $$2\mathrm{I} = x\sqrt{x^2-a^2} - a^2 \int \frac{dx}{\sqrt{x^2-a^2}}$$

or $$\mathbf{I} = \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left| x+\sqrt{x^2-a^2} \right| + C$$

Similarly, integrating other two integrals by parts, taking constant function 1 as the second function, we get

(ii) $$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left| x+\sqrt{x^2+a^2} \right| + C$$

(iii) $$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

Alternatively, integrals (i), (ii) and (iii) can also be found by making trigonometric substitution $x = a\sec\theta$ in (i), $x = a\tan\theta$ in (ii) and $x = a\sin\theta$ in (iii) respectively.

Example 23 Find $\int \sqrt{x^2+2x+5}\, dx$

Solution Note that

$$\int \sqrt{x^2+2x+5}\, dx = \int \sqrt{(x+1)^2+4}\, dx$$

Put $x + 1 = y$, so that $dx = dy$. Then

$$\int \sqrt{x^2+2x+5}\, dx = \int \sqrt{y^2+2^2}\, dy$$

$$= \frac{1}{2}y\sqrt{y^2+4} + \frac{4}{2}\log\left| y+\sqrt{y^2+4} \right| + C \qquad \text{[using 7.6.2 (ii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log\left| x+1+\sqrt{x^2+2x+5} \right| + C$$

Example 24 Find $\int \sqrt{3-2x-x^2}\, dx$

Solution Note that $\int \sqrt{3-2x-x^2}\, dx = \int \sqrt{4-(x+1)^2}\, dx$

Put $x + 1 = y$ so that $dx = dy$.

Thus $$\int \sqrt{3-2x-x^2}\, dx = \int \sqrt{4-y^2}\, dy$$

$$= \frac{1}{2} y \sqrt{4-y^2} + \frac{4}{2} \sin^{-1} \frac{y}{2} + C \qquad \text{[using 7.6.2 (iii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C$$

EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

1. $\sqrt{4-x^2}$ **2.** $\sqrt{1-4x^2}$ **3.** $\sqrt{x^2+4x+6}$

4. $\sqrt{x^2+4x+1}$ **5.** $\sqrt{1-4x-x^2}$ **6.** $\sqrt{x^2+4x-5}$

7. $\sqrt{1+3x-x^2}$ **8.** $\sqrt{x^2+3x}$ **9.** $\sqrt{1+\frac{x^2}{9}}$

Choose the correct answer in Exercises 10 to 11.

10. $\int \sqrt{1+x^2}\, dx$ is equal to

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right| + C$

(B) $\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$ (C) $\frac{2}{3}x(1+x^2)^{\frac{3}{2}} + C$

(D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right| + C$

11. $\int \sqrt{x^2-8x+7}\, dx$ is equal to

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|x+4+\sqrt{x^2-8x+7}\right| + C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

7.7 Definite Integral

In the previous sections, we have studied about the indefinite integrals and discussed few methods of finding them including integrals of some special functions. In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)\, dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$. Here, we shall consider these two cases separately as discussed below:

7.7.1 *Definite integral as the limit of a sum*

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\, dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x = a$ and $x = b$ (Fig 7.2).

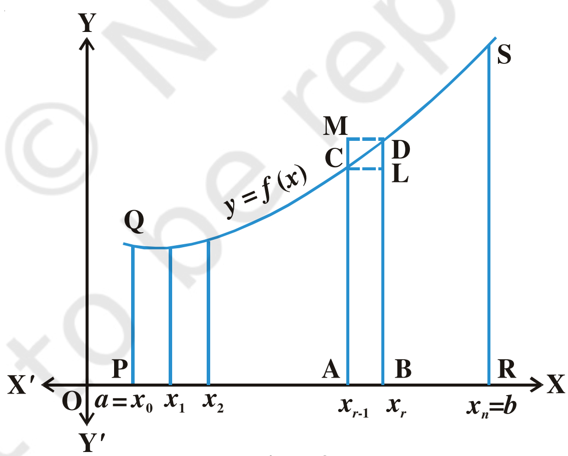

Fig 7.2

Divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1]$, $[x_1, x_2]$,..., $[x_{r-1}, x_r]$, ..., $[x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... , $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \dfrac{b-a}{h}$. We note that as $n \to \infty$, $h \to 0$.

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots, n$.

From Fig 7.2, we have

area of the rectangle (ABLC) < area of the region (ABDCA) < area of the rectangle (ABDM) ... (1)

Evidently as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h\,[f(x_0) + \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \qquad \ldots (2)$$

and
$$S_n = h\,[f(x_1) + f(x_2) + \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \qquad \ldots (3)$$

Here, s_n and S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$, respectively.

In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region PRSQP} < S_n \qquad \ldots (4)$$

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region PRSQP} = \int_a^b f(x)dx \qquad \ldots (5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h]$$

or
$$\int_a^b f(x)dx = (b-a) \lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h] \qquad \ldots (6)$$

where
$$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression (6) is known as the definition of definite integral as the *limit of sum*.

Remark The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)\,dt$ or $\int_a^b f(u)\,du$ instead of $\int_a^b f(x)\,dx$. Hence, the variable of integration is called a *dummy variable*.

Example 25 Find $\int_0^2 (x^2+1)\,dx$ as the limit of a sum.

Solution By definition

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+...+f(a+(n-1)\,h],$$

where, $h = \dfrac{b-a}{n}$

In this example, $a = 0$, $b = 2$, $f(x) = x^2 + 1$, $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

Therefore,

$$\int_0^2 (x^2+1)\,dx = 2\lim_{n\to\infty}\frac{1}{n}[f(0)+f(\frac{2}{n})+f(\frac{4}{n})+...+f(\frac{2\,(n-1)}{n})]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[1+(\frac{2^2}{n^2}+1)+(\frac{4^2}{n^2}+1)+...+\left(\frac{(2n-2)^2}{n^2}+1\right)]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[\underbrace{(1+1+...+1)}_{n\text{-}terms}+\frac{1}{n^2}(2^2+4^2+...+(2n-2)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2^2}{n^2}(1^2+2^2+...+(n-1)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{4}{n^2}\,\frac{(n-1)\,n\,(2n-1)}{6}]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2}{3}\,\frac{(n-1)\;(2n-1)}{n}]$$

$$= 2\lim_{n\to\infty}\;[1+\frac{2}{3}(1-\frac{1}{n})\;(2-\frac{1}{n})] = 2\;[1+\frac{4}{3}] = \frac{14}{3}$$

Example 26 Evaluate $\int_0^2 e^x \, dx$ as the limit of a sum.

Solution By definition

$$\int_0^2 e^x \, dx = (2-0) \lim_{n\to\infty} \frac{1}{n}\left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}}\right]$$

Using the sum to n terms of a G.P., where $a = 1$, $r = e^{\frac{2}{n}}$, we have

$$\int_0^2 e^x \, dx = 2 \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right] = 2 \lim_{n\to\infty} \frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{2\,(e^2-1)}{\lim\limits_{n\to\infty}\left[\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right]\cdot 2} = e^2 - 1 \qquad \left[\text{using } \lim_{h\to 0}\frac{(e^h-1)}{h} = 1\right]$$

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x \, dx$ **2.** $\int_0^5 (x+1) \, dx$ **3.** $\int_2^3 x^2 \, dx$

4. $\int_1^4 (x^2 - x) \, dx$ **5.** $\int_{-1}^1 e^x \, dx$ **6.** $\int_0^4 (x + e^{2x}) \, dx$

7.8 Fundamental Theorem of Calculus

7.8.1 *Area function*

We have defined $\int_a^b f(x) \, dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and x-axis. Let x be a given point in $[a, b]$. Then $\int_a^x f(x) \, dx$ represents the area of the light shaded region

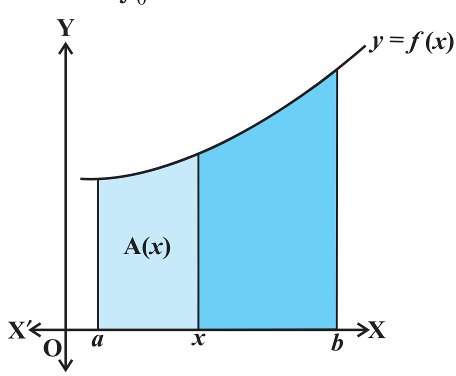

Fig 7.3

in Fig 7.3 [Here it is assumed that $f(x) > 0$ for $x \in [a, b]$, the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of x.

In other words, the area of this shaded region is a function of x. We denote this function of x by A(x). We call the function A(x) as *Area function* and is given by

$$\mathbf{A}(\boldsymbol{x}) = \int_a^x f(x)\,dx \qquad \text{... (1)}$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

7.8.2 *First fundamental theorem of integral calculus*

Theorem 1 Let f be a continuous function on the closed interval $[a, b]$ and let A (x) be the area function. Then $\mathbf{A}'(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$, for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$.

7.8.3 *Second fundamental theorem of integral calculus*

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 2 Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [\mathbf{F}(x)]_a^b = \mathbf{F}(b) - \mathbf{F}(a)$.

Remarks

(i) In words, the Theorem 2 tells us that $\int_a^b f(x)\,dx$ = (value of the anti derivative F of f at the upper limit b – value of the same anti derivative at the lower limit a).

(ii) This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

(iii) The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.

(iv) In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in $[a, b]$.

For instance, the consideration of definite integral $\int_{-2}^{3} x(x^2 - 1)^{\frac{1}{2}}\,dx$ is erroneous since the function f expressed by $f(x) = x(x^2 - 1)^{\frac{1}{2}}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

Steps for calculating $\int_a^b f(x)\, dx$.

(i) Find the indefinite integral $\int f(x)\, dx$. Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get

$$\int_a^b f(x)\, dx = [\mathrm{F}(x) + \mathrm{C}]_a^b = [\mathrm{F}(b) + \mathrm{C}] - [\mathrm{F}(a) + \mathrm{C}] = \mathrm{F}(b) - \mathrm{F}(a).$$

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate F(b) – F(a) = $[\mathrm{F}(x)]_a^b$, which is the value of $\int_a^b f(x)\, dx$.

We now consider some examples

Example 27 Evaluate the following integrals:

(i) $\int_2^3 x^2\, dx$ (ii) $\int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\, dx$

(iii) $\int_1^2 \frac{x\, dx}{(x+1)\,(x+2)}$ (iv) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\, dt$

Solution

(i) Let $\mathrm{I} = \int_2^3 x^2\, dx$. Since $\int x^2\, dx = \frac{x^3}{3} = \mathrm{F}(x)$,

Therefore, by the second fundamental theorem, we get

$$\mathrm{I} = \mathrm{F}(3) - \mathrm{F}(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

(ii) Let $\mathrm{I} = \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\, dx$. We first find the anti derivative of the integrand.

Put $30 - x^{\frac{3}{2}} = t$. Then $-\frac{3}{2}\sqrt{x}\, dx = dt$ or $\sqrt{x}\, dx = -\frac{2}{3} dt$

Thus, $$\int \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\, dx = -\frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right] = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right] = \mathrm{F}(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$\text{I} = \text{F}(9) - \text{F}(4) = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30-27)} - \frac{1}{30-8}\right] = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right] = \frac{19}{99}$$

(iii) Let $\text{I} = \int_1^2 \frac{x\,dx}{(x+1)\,(x+2)}$

Using partial fraction, we get $\frac{x}{(x+1)\,(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

So $$\int \frac{x\,dx}{(x+1)\,(x+2)} = -\log|x+1| + 2\log|x+2| = \text{F}(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$\text{I} = \text{F}(2) - \text{F}(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3]$$

$$= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right)$$

(iv) Let $\text{I} = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2\,t\,dt$. Consider $\int \sin^3 2t \cos 2\,t\,dt$

Put $\sin 2t = u$ so that $2\cos 2t\,dt = du$ or $\cos 2t\,dt = \frac{1}{2}\,du$

So $$\int \sin^3 2t \cos 2\,t\,dt = \frac{1}{2}\int u^3 du$$

$$= \frac{1}{8}[u^4] = \frac{1}{8}\sin^4 2t = \text{F}(t) \text{ say}$$

Therefore, by the second fundamental theorem of integral calculus

$$\text{I} = \text{F}\left(\frac{\pi}{4}\right) - \text{F}(0) = \frac{1}{8}\left[\sin^4\frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

1. $\int_{-1}^{1}(x+1)\,dx$ **2.** $\int_{2}^{3}\frac{1}{x}\,dx$ **3.** $\int_{1}^{2}(4x^3-5x^2+6x+9)\,dx$

4. $\int_{0}^{\frac{\pi}{4}}\sin 2x\,dx$ **5.** $\int_{0}^{\frac{\pi}{2}}\cos 2x\,dx$ **6.** $\int_{4}^{5}e^x dx$ **7.** $\int_{0}^{\frac{\pi}{4}}\tan x\,dx$

8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\text{cosec}\,x\,dx$ **9.** $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$ **10.** $\int_{0}^{1}\frac{dx}{1+x^2}$ **11.** $\int_{2}^{3}\frac{dx}{x^2-1}$

12. $\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx$ **13.** $\int_{2}^{3}\frac{x\,dx}{x^2+1}$ **14.** $\int_{0}^{1}\frac{2x+3}{5x^2+1}\,dx$ **15.** $\int_{0}^{1}x\,e^{x^2}\,dx$

16. $\int_{1}^{2}\frac{5x^2}{x^2+4x+3}$ **17.** $\int_{0}^{\frac{\pi}{4}}(2\sec^2 x+x^3+2)\,dx$ **18.** $\int_{0}^{\pi}(\sin^2\frac{x}{2}-\cos^2\frac{x}{2})\,dx$

19. $\int_{0}^{2}\frac{6x+3}{x^2+4}\,dx$ **20.** $\int_{0}^{1}(x\,e^x+\sin\frac{\pi x}{4})\,dx$

Choose the correct answer in Exercises 21 and 22.

21. $\int_{1}^{\sqrt{3}}\frac{dx}{1+x^2}$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

22. $\int_{0}^{\frac{2}{3}}\frac{dx}{4+9x^2}$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

7.9 Evaluation of Definite Integrals by Substitution

In the previous sections, we have discussed several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution.

To evaluate $\int_a^b f(x)\,dx$, by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute, $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form.
2. Integrate the new integrand with respect to the new variable without mentioning the constant of integration.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

Note In order to quicken this method, we can proceed as follows: After performing steps 1, and 2, there is no need of step 3. Here, the integral will be kept in the new variable itself, and the limits of the integral will accordingly be changed, so that we can perform the last step.

Let us illustrate this by examples.

Example 28 Evaluate $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$.

Solution Put $t = x^5 + 1$, then $dt = 5x^4\,dx$.

Therefore,
$$\int 5x^4\sqrt{x^5+1}\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$$

Hence,
$$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$$
$$= \frac{2}{3}\left[(1^5+1)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Alternatively, first we transform the integral and then evaluate the transformed integral with new limits.

Let $t = x^5 + 1$. Then $dt = 5\, x^4\, dx$.

Note that, when $x = -1,\ t = 0$ and when $x = 1,\ t = 2$

Thus, as x varies from -1 to 1, t varies from 0 to 2

Therefore $$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\, dx = \int_0^2 \sqrt{t}\, dt$$

$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Example 29 Evaluate $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\, dx$

Solution Let $t = \tan^{-1}x$, then $dt = \frac{1}{1+x^2}\, dx$. The new limits are, when $x = 0$, $t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$. Thus, as x varies from 0 to 1, t varies from 0 to $\frac{\pi}{4}$.

Therefore $$\int_0^1 \frac{\tan^{-1}x}{1+x^2}\, dx = \int_0^{\frac{\pi}{4}} t\, dt \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

1. $\int_0^1 \frac{x}{x^2+1}\, dx$ **2.** $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\, \cos^5\phi\, d\phi$ **3.** $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

4. $\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$) **5.** $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$

6. $\int_0^2 \frac{dx}{x+4-x^2}$ **7.** $\int_{-1}^1 \frac{dx}{x^2+2x+5}$ **8.** $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\, dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

10. If $f(x) = \int_0^x t \sin t\, dt$, then $f'(x)$ is

(A) $\cos x + x \sin x$ (B) $x \sin x$

(C) $x \cos x$ (D) $\sin x + x \cos x$

7.10 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

P$_0$: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P$_1$: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$

P$_2$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

P$_3$: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P$_4$: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(Note that P$_4$ is a particular case of P$_3$)

P$_5$: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P$_6$: $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$ and

0 if $f(2a-x) = -f(x)$

P$_7$: (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function, i.e., if $f(-x) = f(x)$.

(ii) $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.

We give the proofs of these properties one by one.

Proof of P$_0$ It follows directly by making the substitution $x = t$.

Proof of P$_1$ Let F be anti derivative of f. Then, by the second fundamental theorem of calculus, we have $\int_a^b f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\,dx$

Here, we observe that, if $a = b$, then $\int_a^a f(x)\,dx = 0$.

Proof of P$_2$ Let F be anti derivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{... (1)}$$

$$\int_a^c f(x)\,dx = F(c) - F(a) \quad \text{... (2)}$$

and $$\int_c^b f(x)\,dx = F(b) - F(c) \quad \text{... (3)}$$

Adding (2) and (3), we get $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) = \int_a^b f(x)\,dx$

This proves the property P_2.

Proof of P_3 Let $t = a + b - x$. Then $dt = -\,dx$. When $x = a$, $t = b$ and when $x = b$, $t = a$. Therefore

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \quad (\text{by } P_1)$$

$$= \int_a^b f(a+b-x)\,dx \text{ by } P_0$$

Proof of P_4 Put $t = a - x$. Then $dt = -\,dx$. When $x = 0$, $t = a$ and when $x = a$, $t = 0$. Now proceed as in P_3.

Proof of P_5 Using P_2, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$.

Let $t = 2a - x$ in the second integral on the right hand side. Then $dt = -\,dx$. When $x = a$, $t = a$ and when $x = 2a$, $t = 0$. Also $x = 2a - t$.

Therefore, the second integral becomes

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

Hence $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Proof of P_6 Using P_5, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ ... (1)

Now, if $f(2a - x) = f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx,$$

and if $f(2a - x) = -f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

Proof of P_7 Using P_2, we have

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \text{. Then}$$

Let $t = -x$ in the first integral on the right hand side. $dt = -\,dx$. When $x = -a$, $t = a$ and when $x = 0$, $t = 0$. Also $x = -t$.

Therefore $\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{(by } P_0\text{)} \quad \dots (1)$$

(i) Now, if f is an even function, then $f(-x) = f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(ii) If f is an odd function, then $f(-x) = -f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Example 30 Evaluate $\int_{-1}^{2} \left| x^3 - x \right| dx$

Solution We note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and that $x^3 - x \geq 0$ on $[1, 2]$. So by P_2 we write

$$\int_{-1}^{2} \left| x^3 - x \right| dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

Example 31 Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx$

Solution We observe that $\sin^2 x$ is an even function. Therefore, by P_7 (i), we get

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{(1-\cos 2x)}{2}\,dx = \int_0^{\frac{\pi}{4}} (1-\cos 2x)\,dx$$

$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{\pi}{4} - \frac{1}{2}$$

Example 32 Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

Solution Let I = $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$. Then, by P_4, we have

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\,dx}{1+\cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1+\cos^2 x} = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - I$$

or $$2\,I = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

or $$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$. Therefore, (by P_1) we get

$$I = \frac{-\pi}{2}\int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi\int_0^{1} \frac{dt}{1+t^2} \quad \text{(by } P_7\text{, since } \frac{1}{1+t^2} \text{ is even function)}$$

$$= \pi\left[\tan^{-1} t\right]_0^1 = \pi\left[\tan^{-1} 1 - \tan^{-1} 0\right] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$

Example 33 Evaluate $\int_{-1}^{1} \sin^5 x \cos^4 x\,dx$

Solution Let I = $\int_{-1}^{1} \sin^5 x \cos^4 x\,dx$. Let $f(x) = \sin^5 x \cos^4 x$. Then

$f(-x) = \sin^5(-x)\cos^4(-x) = -\sin^5 x \cos^4 x = -f(x)$, i.e., f is an odd function. Therefore, by P_7 (ii), I = 0

Example 34 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

Solution Let I = $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ ... (1)

Then, by P_4

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 (\frac{\pi}{2} - x)}{\sin^4 (\frac{\pi}{2} - x) + \cos^4 (\frac{\pi}{2} - x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x} dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Hence $\quad I = \frac{\pi}{4}$

Example 35 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}}$

Solution Let I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}}$ ... (1)

Then, by P_3

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}\, dx}{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \text{ Hence } I = \frac{\pi}{12}$$

Example 36 Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Solution Let $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Then, by P_4

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

Adding the two values of I, we get

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x \cos x + \log 2 - \log 2) \, dx \text{ (by adding and subtracting } \log 2)$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx \quad \text{(Why?)}$$

Put $2x = t$ in the first integral. Then $2\,dx = dt$, when $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$.

Therefore $$2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2}\log 2$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \log \sin t \, dt - \frac{\pi}{2}\log 2 \quad [\text{by } P_6 \text{ as } \sin(\pi - t) = \sin t]$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2}\log 2 \quad \text{(by changing variable } t \text{ to } x)$$

$$= I - \frac{\pi}{2}\log 2$$

Hence $$\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{-\pi}{2}\log 2.$$

EXERCISE 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1. $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

3. $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \, dx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x \, dx}{\sin^5 x+\cos^5 x}$

5. $\int_{-5}^{5} |x+2| \, dx$

6. $\int_2^8 |x-5| \, dx$

7. $\int_0^1 x(1-x)^n \, dx$

8. $\int_0^{\frac{\pi}{4}} \log(1+\tan x) \, dx$

9. $\int_0^2 x\sqrt{2-x} \, dx$

10. $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x) \, dx$

11. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$

12. $\int_0^{\pi} \frac{x \, dx}{1+\sin x}$

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$

14. $\int_0^{2\pi} \cos^5 x \, dx$

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$

16. $\int_0^{\pi} \log(1+\cos x) \, dx$

17. $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$

18. $\int_0^4 |x-1| \, dx$

19. Show that $\int_0^a f(x)g(x) \, dx = 2\int_0^a f(x) \, dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Choose the correct answer in Exercises 20 and 21.

20. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1) \, dx$ is

(A) 0 (B) 2 (C) π (D) 1

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(A) 2 (B) $\frac{3}{4}$ (C) 0 (D) -2

Miscellaneous Examples

Example 37 Find $\int \cos 6x \sqrt{1+\sin 6x}\, dx$

Solution Put $t = 1 + \sin 6x$, so that $dt = 6 \cos 6x\, dx$

Therefore $$\int \cos 6x\sqrt{1+\sin 6x}\, dx = \frac{1}{6}\int t^{\frac{1}{2}} dt$$

$$= \frac{1}{6}\times\frac{2}{3}(t)^{\frac{3}{2}} + \text{C} = \frac{1}{9}(1+\sin 6x)^{\frac{3}{2}} + \text{C}$$

Example 38 Find $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx$

Solution We have $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx = \int \frac{(1-\frac{1}{x^3})^{\frac{1}{4}}}{x^4} dx$

Put $1-\frac{1}{x^3} = 1 - x^{-3} = t$, so that $\frac{3}{x^4} dx = dt$

Therefore $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx = \frac{1}{3}\int t^{\frac{1}{4}}\, dt = \frac{1}{3}\times\frac{4}{5} t^{\frac{5}{4}} + \text{C} = \frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}} + \text{C}$

Example 39 Find $\int \frac{x^4\, dx}{(x-1)\,(x^2+1)}$

Solution We have

$$\frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{x^3 - x^2 + x - 1}$$

$$= (x+1) + \frac{1}{(x-1)\,(x^2+1)} \qquad \text{... (1)}$$

Now express $$\frac{1}{(x-1)(x^2+1)} = \frac{\text{A}}{(x-1)} + \frac{\text{B}x+\text{C}}{(x^2+1)} \qquad \text{... (2)}$$

So $$1 = \text{A}\,(x^2 + 1) + (\text{B}x + \text{C})\,(x - 1)$$
$$= (\text{A} + \text{B})\,x^2 + (\text{C} - \text{B})\,x + \text{A} - \text{C}$$

Equating coefficients on both sides, we get A + B = 0, C – B = 0 and A – C = 1, which give $\text{A} = \frac{1}{2}, \text{B} = \text{C} = -\frac{1}{2}$. Substituting values of A, B and C in (2), we get

$$\frac{1}{(x-1)\,(x^2+1)} = \frac{1}{2(x-1)} - \frac{1}{2}\,\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)} \qquad \text{... (3)}$$

Again, substituting (3) in (1), we have

$$\frac{x^4}{(x-1)\,(x^2+x+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2}\,\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)}$$

Therefore

$$\int \frac{x^4}{(x-1)\,(x^2+x+1)}\,dx = \frac{x^2}{2} + x + \frac{1}{2}\log|\,x-1\,| - \frac{1}{4}\log\,(x^2+1) - \frac{1}{2}\tan^{-1} x + \text{C}$$

Example 40 Find $\int \left[\log\,(\log x) + \frac{1}{(\log x)^2}\right] dx$

Solution Let $\text{I} = \int \left[\log\,(\log x) + \frac{1}{(\log x)^2}\right] dx$

$$= \int \log\,(\log x)\,dx + \int \frac{1}{(\log x)^2}\,dx$$

In the first integral, let us take 1 as the second function. Then integrating it by parts, we get

$$\text{I} = x \log\,(\log x) - \int \frac{1}{x \log x}\,x\,dx + \int \frac{dx}{(\log x)^2}$$

$$= x \log\,(\log x) - \int \frac{dx}{\log x} + \int \frac{dx}{(\log x)^2} \qquad \text{... (1)}$$

Again, consider $\int \frac{dx}{\log x}$, take 1 as the second function and integrate it by parts, we have $$\int \frac{dx}{\log x} = \left[\frac{x}{\log x} - \int x \left\{-\frac{1}{(\log x)^2}\left(\frac{1}{x}\right)\right\} dx\right] \qquad \text{... (2)}$$

Putting (2) in (1), we get

$$\mathrm{I} = x\log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2} + \int \frac{dx}{(\log x)^2} = x\log(\log x) - \frac{x}{\log x} + \mathrm{C}$$

Example 41 Find $\int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx$

Solution We have

$$\mathrm{I} = \int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx = \int \sqrt{\tan x}\,(1 + \cot x)\, dx$$

Put $\tan x = t^2$, so that $\sec^2 x\, dx = 2t\, dt$

or $$dx = \frac{2t\, dt}{1+t^4}$$

Then $$\mathrm{I} = \int t\left(1+\frac{1}{t^2}\right)\frac{2t}{(1+t^4)}\, dt$$

$$= 2\int \frac{(t^2+1)}{t^4+1}\, dt = 2\int \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t^2+\frac{1}{t^2}\right)} = 2\int \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+2}$$

Put $t - \frac{1}{t} = y$, so that $\left(1+\frac{1}{t^2}\right) dt = dy$. Then

$$\mathrm{I} = 2\int \frac{dy}{y^2 + \left(\sqrt{2}\right)^2} = \sqrt{2}\tan^{-1}\frac{y}{\sqrt{2}} + \mathrm{C} = \sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}} + \mathrm{C}$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right) + \mathrm{C} = \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \mathrm{C}$$

Example 42 Find $\int \frac{\sin 2x \cos 2x\, dx}{\sqrt{9-\cos^4(2x)}}$

Solution Let $\mathrm{I} = \int \frac{\sin 2x \cos 2x}{\sqrt{9-\cos^4 2x}}\, dx$

Put $\cos^2(2x) = t$ so that $4 \sin 2x \cos 2x \, dx = -dt$

Therefore $$\mathrm{I} = -\frac{1}{4}\int \frac{dt}{\sqrt{9-t^2}} = -\frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right) + \mathrm{C} = -\frac{1}{4}\sin^{-1}\left[\frac{1}{3}\cos^2 2x\right] + \mathrm{C}$$

Example 43 Evaluate $\int_{-1}^{\frac{3}{2}} |x \sin(\pi x)| dx$

Solution Here $f(x) = |x \sin \pi x| = \begin{cases} x \sin \pi x \text{ for } -1 \le x \le 1 \\ -x \sin \pi x \text{ for } 1 \le x \le \dfrac{3}{2} \end{cases}$

Therefore $$\int_{-1}^{\frac{3}{2}} |x \sin \pi x| dx = \int_{-1}^{1} x \sin \pi x \, dx + \int_{1}^{\frac{3}{2}} -x \sin \pi x \, dx$$

$$= \int_{-1}^{1} x \sin \pi x \, dx - \int_{1}^{\frac{3}{2}} x \sin \pi x \, dx$$

Integrating both integrals on righthand side, we get

$$\int_{-1}^{\frac{3}{2}} |x \sin \pi x| dx = \left[\frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{-1}^{1} - \left[\frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{1}^{\frac{3}{2}}$$

$$= \frac{2}{\pi} - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{3}{\pi} + \frac{1}{\pi^2}$$

Example 44 Evaluate $\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Solution Let $\mathrm{I} = \int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\pi} \frac{(\pi - x) \, dx}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)}$ (using P_4)

$$= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - \int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - \mathrm{I}$$

Thus $$2\mathrm{I} = \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

or $$I = \frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \text{ (using P}_6\text{)}$$

$$= \pi\left[\int_0^{\frac{\pi}{4}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right]$$

$$= \pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\operatorname{cosec}^2 x\,dx}{a^2\cot^2 x + b^2}\right]$$

$$= \pi\left[\int_0^1\frac{dt}{a^2 + b^2t^2} - \int_1^0\frac{du}{a^2u^2 + b^2}\right]\left(put \tan x = t\ and\ \cot x = u\right)$$

$$= \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^1 - \frac{\pi}{ab}\left[\tan^{-1}\frac{au}{b}\right]_1^0 = \frac{\pi}{ab}\left[\tan^{-1}\frac{b}{a} + \tan^{-1}\frac{a}{b}\right] = \frac{\pi^2}{2ab}$$

Miscellaneous Exercise on Chapter 7

Integrate the functions in Exercises 1 to 24.

1. $\frac{1}{x - x^3}$ **2.** $\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$ **3.** $\frac{1}{x\sqrt{ax - x^2}}$ [Hint: Put $x = \frac{a}{t}$]

4. $\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$ **5.** $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ [Hint: $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1 + x^{\frac{1}{6}}\right)}$, put $x = t^6$]

6. $\frac{5x}{(x+1)(x^2+9)}$ **7.** $\frac{\sin x}{\sin(x-a)}$ **8.** $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

9. $\frac{\cos x}{\sqrt{4 - \sin^2 x}}$ **10.** $\frac{\sin^8 - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}$ **11.** $\frac{1}{\cos(x+a)\cos(x+b)}$

12. $\frac{x^3}{\sqrt{1 - x^8}}$ **13.** $\frac{e^x}{(1+e^x)(2+e^x)}$ **14.** $\frac{1}{(x^2+1)(x^2+4)}$

15. $\cos^3 x\ e^{\log \sin x}$ **16.** $e^{3\log x}(x^4 + 1)^{-1}$ **17.** $f'(ax+b)\,[f(ax+b)]^n$

18. $\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$ **19.** $\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$, $x \in [0, 1]$

20. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$ **21.** $\dfrac{2+\sin 2x}{1+\cos 2x}e^x$ **22.** $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$

23. $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$ **24.** $\dfrac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}$

Evaluate the definite integrals in Exercises 25 to 33.

25. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$ **26.** $\displaystyle\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ **27.** $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos^2 x\,dx}{\cos^2 x+4\sin^2 x}$

28. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$ **29.** $\displaystyle\int_0^1\frac{dx}{\sqrt{1+x}-\sqrt{x}}$ **30.** $\displaystyle\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

31. $\displaystyle\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)\,dx$ **32.** $\displaystyle\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$

33. $\displaystyle\int_1^4[|x-1|+|x-2|+|x-3|]\,dx$

Prove the following (Exercises 34 to 39)

34. $\displaystyle\int_1^3\frac{dx}{x^2(x+1)}=\frac{2}{3}+\log\frac{2}{3}$ **35.** $\displaystyle\int_0^1 x\,e^x dx=1$

36. $\displaystyle\int_{-1}^1 x^{17}\cos^4 x\,dx=0$ **37.** $\displaystyle\int_0^{\frac{\pi}{2}}\sin^3 x\,dx=\frac{2}{3}$

38. $\displaystyle\int_0^{\frac{\pi}{4}}2\tan^3 x\,dx=1-\log 2$ **39.** $\displaystyle\int_0^1\sin^{-1}x\,dx=\frac{\pi}{2}-1$

40. Evaluate $\displaystyle\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Choose the correct answers in Exercises 41 to 44.

41. $\displaystyle\int\frac{dx}{e^x+e^{-x}}$ is equal to

(A) $\tan^{-1}(e^x)+C$ (B) $\tan^{-1}(e^{-x})+C$

(C) $\log(e^x-e^{-x})+C$ (D) $\log(e^x+e^{-x})+C$

42. $\displaystyle\int\frac{\cos 2x}{(\sin x+\cos x)^2}dx$ is equal to

(A) $\dfrac{-1}{\sin x+\cos x}+C$ (B) $\log|\sin x+\cos x|+C$

(C) $\log|\sin x-\cos x|+C$ (D) $\dfrac{1}{(\sin x+\cos x)^2}$

43. If $f(a+b-x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to

(A) $\dfrac{a+b}{2}\int_a^b f(b-x)\, dx$ (B) $\dfrac{a+b}{2}\int_a^b f(b+x)\, dx$

(C) $\dfrac{b-a}{2}\int_a^b f(x)\, dx$ (D) $\dfrac{a+b}{2}\int_a^b f(x)\, dx$

44. The value of $\int_0^1 \tan^{-1}\left(\dfrac{2x-1}{1+x-x^2}\right) dx$ is

(A) 1 (B) 0 (C) -1 (D) $\dfrac{\pi}{4}$

Summary

- Integration is the inverse process of differentiation. In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given. Thus, integration is a process which is the inverse of differentiation.

  Let $\dfrac{d}{dx}\mathrm{F}(x) = f(x)$. Then we write $\int f(x)\, dx = \mathrm{F}(x) + \mathrm{C}$. These integrals are called indefinite integrals or general integrals, C is called constant of integration. All these integrals differ by a constant.
- From the geometric point of view, an indefinite integral is collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards along the y-axis.
- Some properties of indefinite integrals are as follows:

  1. $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

  2. For any real number k, $\int k\, f(x)\, dx = k\int f(x)\, dx$

  More generally, if $f_1, f_2, f_3, \ldots, f_n$ are functions and $k_1, k_2, \ldots, k_n$ are real numbers. Then

$$\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\, dx$$

$$= k_1\int f_1(x)\, dx + k_2\int f_2(x)\, dx + \ldots + k_n\int f_n(x)\, dx$$

◆ Some standard integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$. Particularly, $\int dx = x + C$

(ii) $\int \cos x \, dx = \sin x + C$

(iii) $\int \sin x \, dx = -\cos x + C$

(iv) $\int \sec^2 x \, dx = \tan x + C$

(v) $\int \text{cosec}^2 x \, dx = -\cot x + C$

(vi) $\int \sec x \tan x \, dx = \sec x + C$

(vii) $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + C$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

(x) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(xi) $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\int e^x dx = e^x + C$

(xiii) $\int a^x dx = \frac{a^x}{\log a} + C$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\text{cosec}^{-1} x + C$

(xvi) $\int \frac{1}{x} dx = \log |x| + C$

◆ Integration by partial fractions

Recall that a rational function is ratio of two polynomials of the form $\frac{P(x)}{Q(x)}$, where P(x) and Q (x) are polynomials in x and Q (x) ≠ 0. If degree of the polynomial P (x) is greater than the degree of the polynomial Q (x), then we may divide P (x) by Q (x) so that $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T($x$) is a polynomial in x and degree of $P_1(x)$ is less than the degree of Q(x). T(x) being polynomial can be easily integrated. $\frac{P_1(x)}{Q(x)}$ can be integrated by expressing $\frac{P_1(x)}{Q(x)}$ as the sum of partial fractions of the following type:

1. $$\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a}+\frac{B}{x-b}, \; a \neq b$$

2. $$\frac{px+q}{(x-a)^2} = \frac{A}{x-a}+\frac{B}{(x-a)^2}$$

3. $$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$$

4. $$\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$$

5. $$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$$

where $x^2 + bx + c$ can not be factorised further.

◆ **Integration by substitution**

A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method in which we change the variable to some other variable is called the method of substitution. When the integrand involves some trigonometric functions, we use some well known identities to find the integrals. Using substitution technique, we obtain the following standard integrals.

(i) $\int \tan x \, dx = \log |\sec x| + C$ (ii) $\int \cot x \, dx = \log |\sin x| + C$

(iii) $\int \sec x \, dx = \log |\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + C$

◆ **Integrals of some special functions**

(i) $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

(ii) $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$ (iii) $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$ (v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

(vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}| + C$

- **Integration by parts**

  For given functions f_1 and f_2, we have

  $\int f_1(x) \cdot f_2(x)\, dx = f_1(x) \int f_2(x)\, dx - \int \left[\frac{d}{dx} f_1(x) \cdot \int f_2(x)\, dx\right] dx$, i.e., the integral of the product of two functions = first function × integral of the second function – integral of {differential coefficient of the first function × integral of the second function}. Care must be taken in choosing the first function and the second function. Obviously, we must take that function as the second function whose integral is well known to us.

- $\int e^x [f(x) + f'(x)]\, dx = \int e^x f(x)\, dx + C$

- **Some special types of integrals**

  (i) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

  (ii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

  (iii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

  (iv) Integrals of the types $\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing

  $$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

  (v) Integrals of the types $\int \frac{px + q\, dx}{ax^2 + bx + c}$ or $\int \frac{px + q\, dx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing

$$px+q = A\frac{d}{dx}(ax^2+bx+c)+B = A\,(2ax+b)+B$$, where A and B are determined by comparing coefficients on both sides.

- We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, $a \le x \le b$, the x-axis and the ordinates $x = a$ and $x = b$. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the **Area function** A (x). This concept of area function leads to the Fundamental Theorems of Integral Calculus.
- **First fundamental theorem of integral calculus**

  Let the area function be defined by $A(x) = \int_a^x f(x)\,dx$ for all $x \ge a$, where the function f is assumed to be continuous on $[a, b]$. Then $A'(x) = f(x)$ for all $x \in [a, b]$.
- **Second fundamental theorem of integral calculus**

  Let f be a continuous function of x defined on the closed interval $[a, b]$ and let F be another function such that $\frac{d}{dx}F(x) = f(x)$ for all x in the domain of f, then $\int_a^b f(x)\,dx = \left[F(x)+C\right]_a^b = F(b) - F(a)$.

  This is called the definite integral of f over the range $[a, b]$, where a and b are called the limits of integration, a being the lower limit and b the upper limit.

—❖—

Chapter 8

APPLICATION OF INTEGRALS

❖ *One should study Mathematics because it is only through Mathematics that nature can be conceived in harmonious form. – BIRKHOFF* ❖

8.1 Introduction

A.L. Cauchy (1789-1857)

In geometry, we have learnt formulae to calculate areas of various geometrical figures including triangles, rectangles, trapezias and circles. Such formulae are fundamental in the applications of mathematics to many real life problems. The formulae of elementary geometry allow us to calculate areas of many simple figures. However, they are inadequate for calculating the areas enclosed by curves. For that we shall need some concepts of Integral Calculus.

In the previous chapter, we have studied to find the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and x-axis, while calculating definite integral as the limit of a sum. Here, in this chapter, we shall study a specific application of integrals to find the area under simple curves, area between lines and arcs of circles, parabolas and ellipses (standard forms only). We shall also deal with finding the area bounded by the above said curves.

8.2 Area under Simple Curves

In the previous chapter, we have studied definite integral as the limit of a sum and how to evaluate definite integral using Fundamental Theorem of Calculus. Now, we consider the easy and intuitive way of finding the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$. From Fig 8.1, we can think of area under the curve as composed of large number of very thin vertical strips. Consider an arbitrary strip of height y and width dx, then dA (area of the elementary strip) $= ydx$, where, $y = f(x)$.

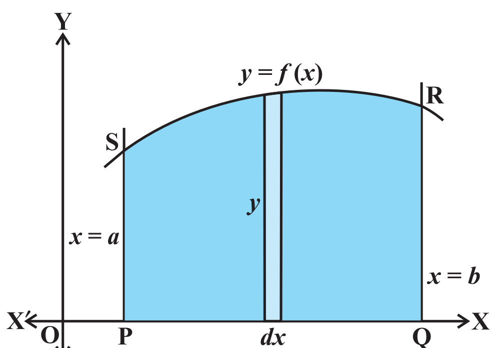

Fig 8.1

This area is called the *elementary area* which is located at an arbitrary position within the region which is specified by some value of x between a and b. We can think of the total area A of the region between x-axis, ordinates $x = a$, $x = b$ and the curve $y = f(x)$ as the result of adding up the elementary areas of thin strips across the region PQRSP. Symbolically, we express

$$A = \int_a^b dA = \int_a^b y dx = \int_a^b f(x)\, dx$$

The area A of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$, $y = d$ is given by

$$A = \int_c^d x dy = \int_c^d g(y)\, dy$$

Here, we consider horizontal strips as shown in the Fig 8.2

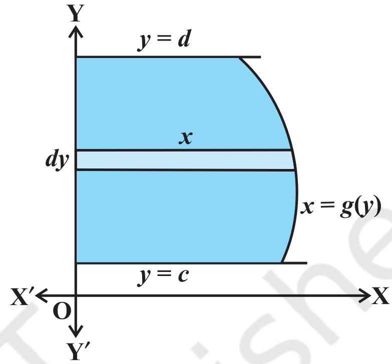

Fig 8.2

Remark If the position of the curve under consideration is below the x-axis, then since $f(x) < 0$ from $x = a$ to $x = b$, as shown in Fig 8.3, the area bounded by the curve, x-axis and the ordinates $x = a$, $x = b$ come out to be negative. But, it is only the numerical value of the area which is taken into consideration. Thus, if the area is negative, we take its absolute value, i.e., $\left|\int_a^b f(x)\, dx\right|$.

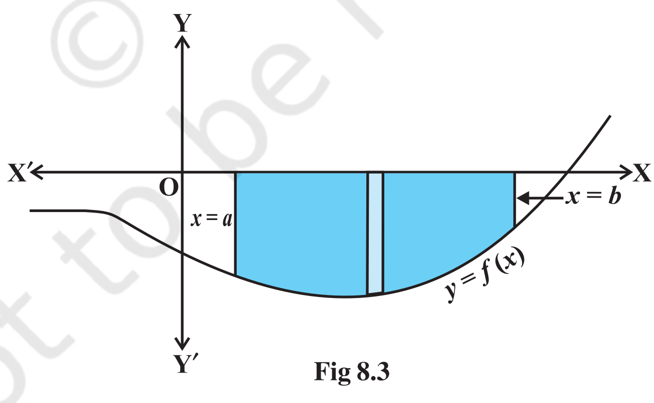

Fig 8.3

Generally, it may happen that some portion of the curve is above x-axis and some is below the x-axis as shown in the Fig 8.4. Here, $A_1 < 0$ and $A_2 > 0$. Therefore, the area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by $A = |A_1| + A_2$.

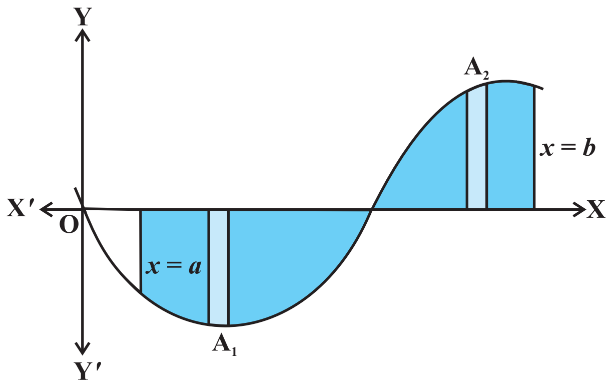

Fig 8.4

Example 1 Find the area enclosed by the circle $x^2 + y^2 = a^2$.

Solution From Fig 8.5, the whole area enclosed by the given circle

= 4 (area of the region AOBA bounded by the curve, x-axis and the ordinates $x = 0$ and $x = a$) [as the circle is symmetrical about both x-axis and y-axis]

$$= 4\int_0^a y\,dx \text{ (taking vertical strips)}$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

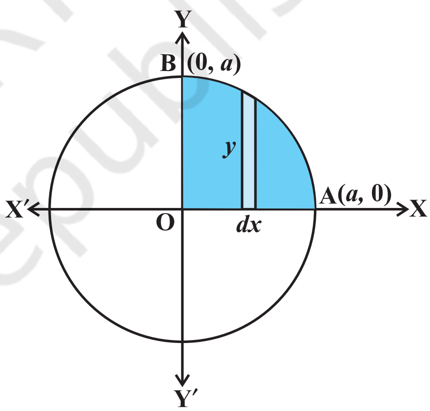

Fig 8.5

Since $x^2 + y^2 = a^2$ gives $\quad y = \pm\sqrt{a^2 - x^2}$

As the region AOBA lies in the first quadrant, y is taken as positive. Integrating, we get the whole area enclosed by the given circle

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right] = 4\left(\frac{a^2}{2}\right)\left(\frac{\pi}{2}\right) = \pi a^2$$

Alternatively, considering horizontal strips as shown in Fig 8.6, the whole area of the region enclosed by circle

$$= 4\int_0^a x\,dy = 4\int_0^a \sqrt{a^2 - y^2}\,dy \qquad \text{(Why?)}$$

$$= 4\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

$$= 4\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= 4\frac{a^2}{2}\frac{\pi}{2} = \pi a^2$$

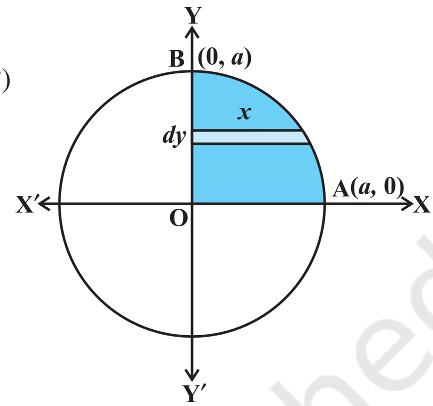

Fig 8.6

Example 2 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution From Fig 8.7, the area of the region ABA′B′A bounded by the ellipse

$$= 4\left(\begin{array}{l} \textit{area of the region AOBA in the first quadrant bounded} \\ \textit{by the curve, x-axis and the ordinates } x = 0, x = a \end{array}\right)$$

(as the ellipse is symmetrical about both x-axis and y-axis)

$$= 4\int_0^a y\,dx \quad \text{(taking vertical strips)}$$

Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$, but as the region AOBA lies in the first quadrant, y is taken as positive. So, the required area is

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a \quad \text{(Why?)}$$

$$= \frac{4b}{a}\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= \frac{4b}{a}\frac{a^2}{2}\frac{\pi}{2} = \pi ab$$

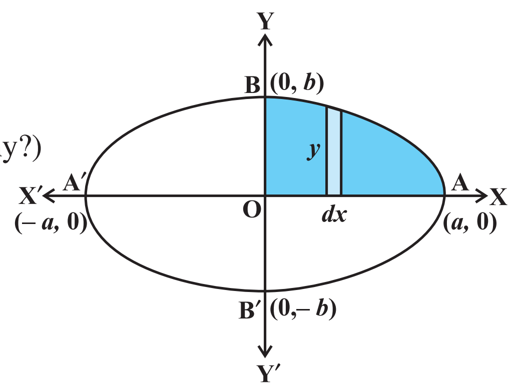

Fig 8.7

Alternatively, considering horizontal strips as shown in the Fig 8.8, the area of the ellipse is

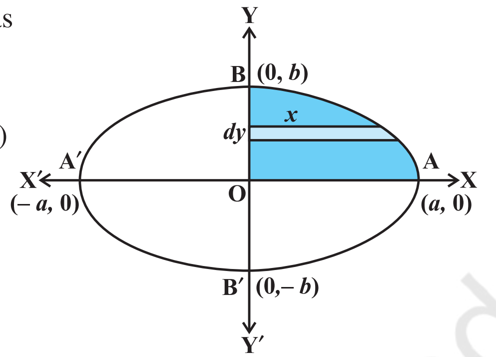

Fig 8.8

$$= 4\int_0^b x\,dy = 4\frac{a}{b}\int_0^b \sqrt{b^2 - y^2}\,dy \quad \text{(Why?)}$$

$$= \frac{4a}{b}\left[\frac{y}{2}\sqrt{b^2 - y^2} + \frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b$$

$$= \frac{4a}{b}\left[\left(\frac{b}{2}\times 0 + \frac{b^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= \frac{4a}{b}\frac{b^2}{2}\frac{\pi}{2} = \pi ab$$

8.2.1 *The area of the region bounded by a curve and a line*

In this subsection, we will find the area of the region bounded by a line and a circle, a line and a parabola, a line and an ellipse. Equations of above mentioned curves will be in their standard forms only as the cases in other forms go beyond the scope of this textbook.

Example 3 Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

Solution Since the given curve represented by the equation $y = x^2$ is a parabola symmetrical about y-axis only, therefore, from Fig 8.9, the required area of the region AOBA is given by

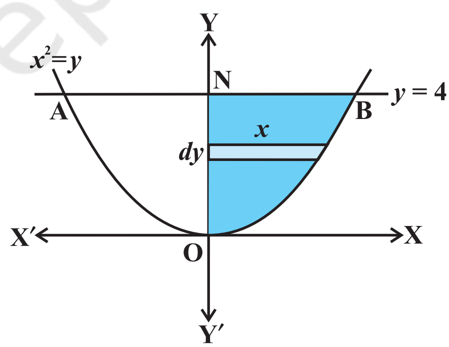

Fig 8.9

$$2\int_0^4 x\,dy =$$

$$2\begin{pmatrix}\text{area of the region BONB bounded by curve, } y-\text{axis} \\ \text{and the lines } y=0 \text{ and } y=4\end{pmatrix}$$

$$= 2\int_0^4 \sqrt{y}\,dy = 2\times\frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^4 = \frac{4}{3}\times 8 = \frac{32}{3} \quad \text{(Why?)}$$

Here, we have taken horizontal strips as indicated in the Fig 8.9.

Alternatively, we may consider the vertical strips like PQ as shown in the Fig 8.10 to obtain the area of the region AOBA. To this end, we solve the equations $x^2 = y$ and $y = 4$ which gives $x = -2$ and $x = 2$.

Thus, the region AOBA may be stated as the region bounded by the curve $y = x^2$, $y = 4$ and the ordinates $x = -2$ and $x = 2$.

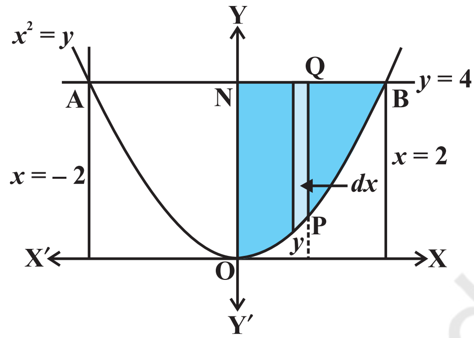

Fig 8.10

Therefore, the area of the region AOBA

$$= \int_{-2}^{2} y\,dx$$

[y = (y-coordinate of Q) – (y-coordinate of P) = $4 - x^2$]

$$= 2\int_{0}^{2}\left(4 - x^2\right)dx \qquad \text{(Why?)}$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = 2\left[4\times 2 - \frac{8}{3}\right] = \frac{32}{3}$$

Remark From the above examples, it is inferred that we can consider either vertical strips or horizontal strips for calculating the area of the region. Henceforth, we shall consider either of these two, most preferably vertical strips.

Example 4 Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$, and the circle $x^2 + y^2 = 32$.

Solution The given equations are

$$y = x \qquad \text{... (1)}$$

and $$x^2 + y^2 = 32 \qquad \text{... (2)}$$

Solving (1) and (2), we find that the line and the circle meet at B(4, 4) in the first quadrant (Fig 8.11). Draw perpendicular BM to the x-axis.

Therefore, the required area = area of the region OBMO + area of the region BMAB.

Now, the area of the region OBMO

$$= \int_0^4 y\,dx = \int_0^4 x\,dx \qquad \text{... (3)}$$

$$= \frac{1}{2}\left[x^2\right]_0^4 = 8$$

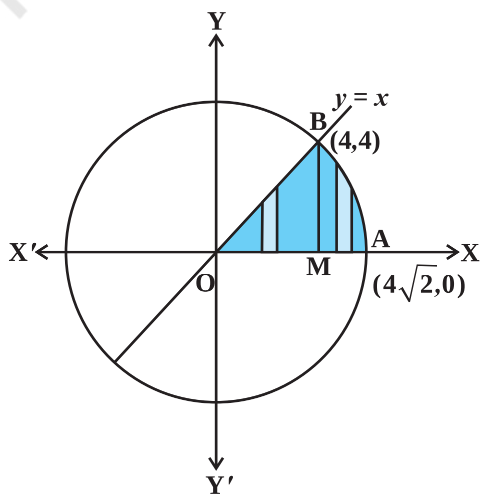

Fig 8.11

Again, the area of the region BMAB

$$= \int_4^{4\sqrt{2}} y\,dx = \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

$$= \left[\frac{1}{2} x\sqrt{32 - x^2} + \frac{1}{2} \times 32 \times \sin^{-1} \frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{1}{2} 4\sqrt{2} \times 0 + \frac{1}{2} \times 32 \times \sin^{-1} 1\right) - \left(\frac{4}{2}\sqrt{32 - 16} + \frac{1}{2} \times 32 \times \sin^{-1} \frac{1}{\sqrt{2}}\right)$$

$$= 8\pi - (8 + 4\pi) = 4\pi - 8 \quad \text{... (4)}$$

Adding (3) and (4), we get, the required area = 4π.

Example 5 Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and $x = ae$, where, $b^2 = a^2(1 - e^2)$ and $e < 1$.

Solution The required area (Fig 8.12) of the region BOB′RFSB is enclosed by the ellipse and the lines $x = 0$ and $x = ae$.

Note that the area of the region BOB′RFSB

$$= 2\int_0^{ae} y\,dx = 2\frac{b}{a}\int_0^{ae} \sqrt{a^2 - x^2}\,dx$$

$$= \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$$

$$= \frac{2b}{2a}\left[ae\sqrt{a^2 - a^2e^2} + a^2 \sin^{-1} e\right]$$

$$= ab\left[e\sqrt{1 - e^2} + \sin^{-1} e\right]$$

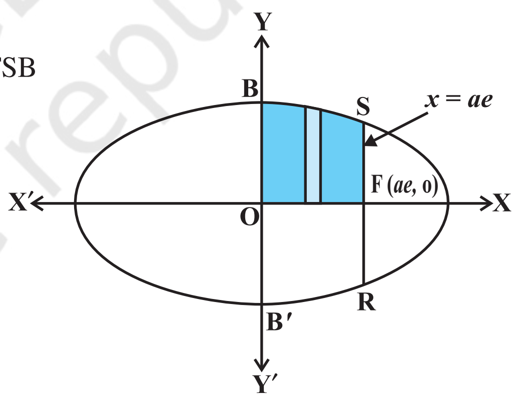

Fig 8.12

EXERCISE 8.1

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis in the first quadrant.
2. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

4. Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

5. Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

6. Find the area of the region in the first quadrant enclosed by x-axis, line $x=\sqrt{3}\,y$ and the circle $x^2 + y^2 = 4$.

7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$.

8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y – 2$.

11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Choose the correct answer in the following Exercises 12 and 13.

12. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

13. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

(A) 2 (B) $\frac{9}{4}$ (C) $\frac{9}{3}$ (D) $\frac{9}{2}$

8.3 Area between Two Curves

Intuitively, true in the sense of Leibnitz, integration is the act of calculating the area by cutting the region into a large number of small strips of elementary area and then adding up these elementary areas. Suppose we are given two curves represented by $y = f(x)$, $y = g(x)$, where $f(x) \geq g(x)$ in $[a, b]$ as shown in Fig 8.13. Here the points of intersection of these two curves are given by $x = a$ and $x = b$ obtained by taking common values of y from the given equation of two curves.

For setting up a formula for the integral, it is convenient to take elementary area in the form of vertical strips. As indicated in the Fig 8.13, elementary strip has height

$f(x) - g(x)$ and width dx so that the elementary area

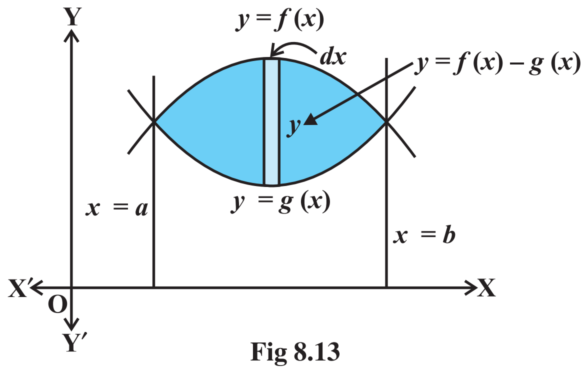

Fig 8.13

$dA = [f(x) - g(x)]\, dx$, and the total area A can be taken as

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Alternatively,

A = [area bounded by $y = f(x)$, x-axis and the lines $x = a$, $x = b$]

– [area bounded by $y = g(x)$, x-axis and the lines $x = a$, $x = b$]

$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b]$$

If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, where $a < c < b$ as shown in the Fig 8.14, then the area of the regions bounded by curves can be written as

Total Area = Area of the region ACBDA + Area of the region BPRQB

$$= \int_a^c [f(x) - g(x)]\, dx + \int_c^b [g(x) - f(x)]\, dx$$

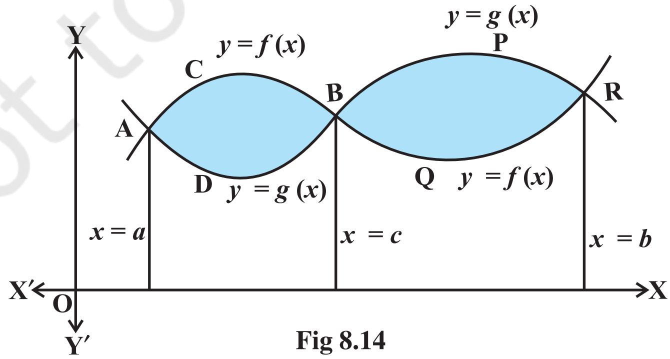

Fig 8.14

Example 6 Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Solution The point of intersection of these two parabolas are O (0, 0) and A (1, 1) as shown in the Fig 8.15.

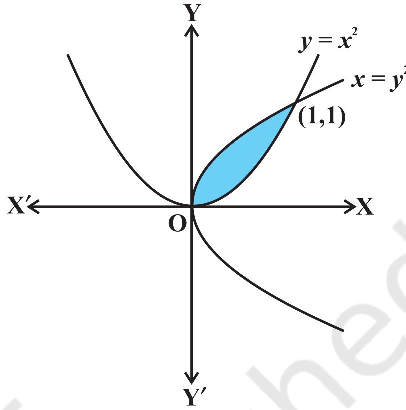

Fig 8.15

Here, we can set $y^2 = x$ or $y = \sqrt{x} = f(x)$ and $y = x^2 = g(x)$, where, $f(x) \geq g(x)$ in [0, 1].

Therefore, the required area of the shaded region

$$= \int_0^1 [f(x) - g(x)]\, dx$$

$$= \int_0^1 \left[\sqrt{x} - x^2\right] dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Example 7 Find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

Solution The given equation of the circle $x^2 + y^2 = 8x$ can be expressed as $(x - 4)^2 + y^2 = 16$. Thus, the centre of the circle is (4, 0) and radius is 4. Its intersection with the parabola $y^2 = 4x$ gives

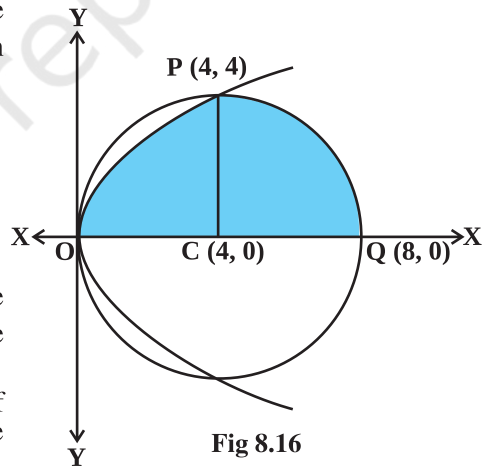

Fig 8.16

$$x^2 + 4x = 8x$$

or $$x^2 - 4x = 0$$

or $$x(x - 4) = 0$$

or $$x = 0, x = 4$$

Thus, the points of intersection of these two curves are O(0, 0) and P(4,4) above the x-axis.

From the Fig 8.16, the required area of the region OPQCO included between these two curves above x-axis is

= (area of the region OCPO) + (area of the region PCQP)

$$= \int_0^4 y\,dx + \int_4^8 y\,dx$$

$$= 2\int_0^4 \sqrt{x}\,dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\,dx \quad \text{(Why?)}$$

$$= 2 \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4 + \int_0^4 \sqrt{4^2 - t^2}\, dt, \text{ where, } x - 4 = t \text{ (Why?)}$$

$$= \frac{32}{3} + \left[\frac{t}{2}\sqrt{4^2 - t^2} + \frac{1}{2} \times 4^2 \times \sin^{-1}\frac{t}{4}\right]_0^4$$

$$= \frac{32}{3} + \left[\frac{4}{2} \times 0 + \frac{1}{2} \times 4^2 \times \sin^{-1} 1\right] = \frac{32}{3} + \left[0 + 8 \times \frac{\pi}{2}\right] = \frac{32}{3} + 4\pi = \frac{4}{3}(8 + 3\pi)$$

Example 8 In Fig 8.17, AOBA is the part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that OA = 2 and OB = 6. Find the area between the arc AB and the chord AB.

Solution Given equation of the ellipse $9x^2 + y^2 = 36$ can be expressed as $\frac{x^2}{4} + \frac{y^2}{36} = 1$ or $\frac{x^2}{2^2} + \frac{y^2}{6^2} = 1$ and hence, its shape is as given in Fig 8.17.

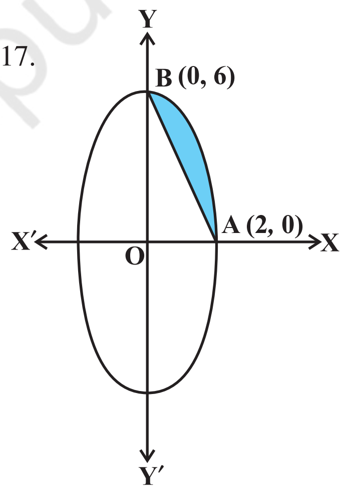

Fig 8.17

Accordingly, the equation of the chord AB is

$$y - 0 = \frac{6-0}{0-2}(x-2)$$

or $$y = -3(x-2)$$

or $$y = -3x + 6$$

Area of the shaded region as shown in the Fig 8.17.

$$= 3\int_0^2 \sqrt{4 - x^2}\, dx - \int_0^2 (6 - 3x)\, dx \qquad \text{(Why?)}$$

$$= 3\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[6x - \frac{3x^2}{2}\right]_0^2$$

$$= 3\left[\frac{2}{2} \times 0 + 2\sin^{-1}(1)\right] - \left[12 - \frac{12}{2}\right] = 3 \times 2 \times \frac{\pi}{2} - 6 = 3\pi - 6$$

Example 9 Using integration find the area of region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3, 1).

Solution Let A (1, 0), B (2, 2) and C (3, 1) be the vertices of a triangle ABC (Fig 8.18).

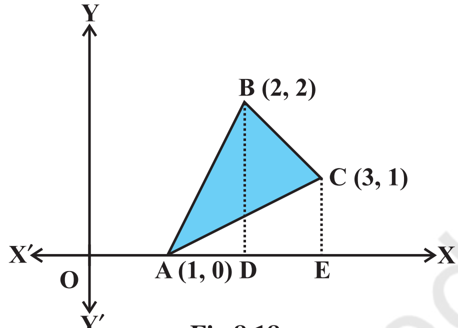

Fig 8.18

Area of ΔABC

$$= \text{Area of } \Delta\text{ABD} + \text{Area of trapezium BDEC} - \text{Area of } \Delta\text{AEC}$$

Now equation of the sides AB, BC and CA are given by

$$y = 2(x-1),\ y = 4 - x,\ y = \frac{1}{2}(x-1), \text{ respectively.}$$

Hence, area of Δ ABC $= \int_1^2 2(x-1)\,dx + \int_2^3 (4-x)\,dx - \int_1^3 \frac{x-1}{2}\,dx$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2\left[\left(\frac{2^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right] + \left[\left(4\times 3 - \frac{3^2}{2}\right) - \left(4\times 2 - \frac{2^2}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{3^2}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{3}{2}$$

Example 10 Find the area of the region enclosed between the two circles: $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

Solution Equations of the given circles are

$$x^2 + y^2 = 4 \quad \text{... (1)}$$

and $$(x-2)^2 + y^2 = 4 \quad \text{... (2)}$$

Equation (1) is a circle with centre O at the origin and radius 2. Equation (2) is a circle with centre C (2, 0) and radius 2. Solving equations (1) and (2), we have

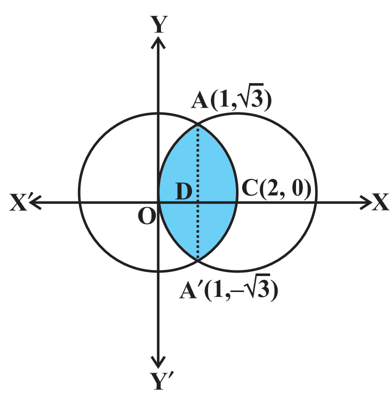

Fig 8.19

$$(x-2)^2 + y^2 = x^2 + y^2$$

or $$x^2 - 4x + 4 + y^2 = x^2 + y^2$$

or $$x = 1 \text{ which gives } y = \pm\sqrt{3}$$

Thus, the points of intersection of the given circles are A(1, $\sqrt{3}$) and A′(1, $-\sqrt{3}$) as shown in the Fig 8.19.

Required area of the enclosed region OACA′O between circles

$= 2$ [area of the region ODCAO] (Why?)

$= 2$ [area of the region ODAO + area of the region DCAD]

$$= 2\left[\int_0^1 y\,dx + \int_1^2 y\,dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right] \quad \text{(Why?)}$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+\ 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[\left(-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(-1)\right] + \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$

$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

EXERCISE 8.2

1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.
2. Find the area bounded by curves $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.
3. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.
4. Using integration find the area of region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$ and $(3, 2)$.
5. Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Choose the correct answer in the following exercises 6 and 7.

6. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(A) $2(\pi - 2)$ (B) $\pi - 2$ (C) $2\pi - 1$ (D) $2(\pi + 2)$

7. Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

Miscellaneous Examples

Example 11 Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.

Solution From Fig 8.20, the vertex of the parabola $y^2 = 4ax$ is at origin (0, 0). The equation of the latus rectum LSL′ is $x = a$. Also, parabola is symmetrical about the x-axis.

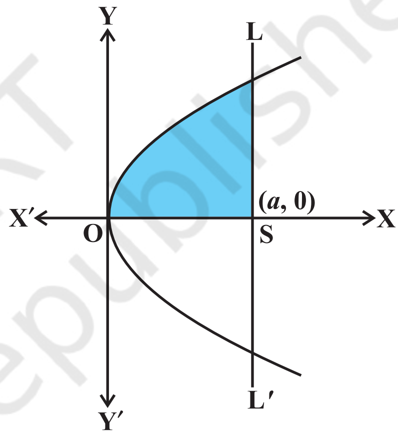

Fig 8.20

The required area of the region OLL′O

$= 2$ (area of the region OLSO)

$$= 2\int_0^a y\,dx = 2\int_0^a \sqrt{4ax}\,dx$$

$$= 2 \times 2\sqrt{a}\int_0^a \sqrt{x}\,dx$$

$$= 4\sqrt{a} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\left[a^{\frac{3}{2}}\right] = \frac{8}{3}a^2$$

Example 12 Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$.

Solution As shown in the Fig 8.21, the line $y = 3x + 2$ meets x-axis at $x = \frac{-2}{3}$ and its graph lies below x-axis for $x \in \left(-1, \frac{-2}{3}\right)$ and above x-axis for $x \in \left(\frac{-2}{3}, 1\right)$.

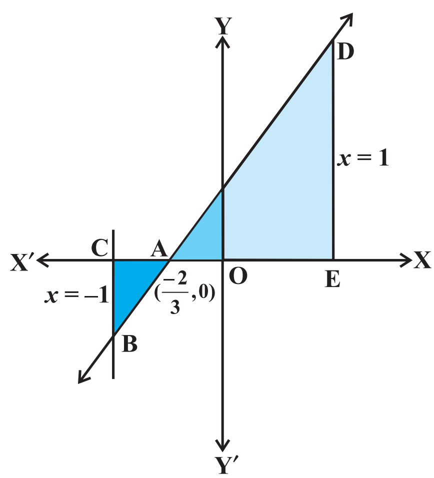

Fig 8.21

The required area = Area of the region ACBA + Area of the region ADEA

$$= \left|\int_{-1}^{\frac{-2}{3}}(3x+2)dx\right| + \int_{\frac{-2}{3}}^{1}(3x+2)dx$$

$$= \left|\left[\frac{3x^2}{2}+2x\right]_{-1}^{\frac{-2}{3}}\right| + \left[\frac{3x^2}{2}+2x\right]_{\frac{-2}{3}}^{1} = \frac{1}{6}+\frac{25}{6}=\frac{13}{3}$$

Example 13 Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.

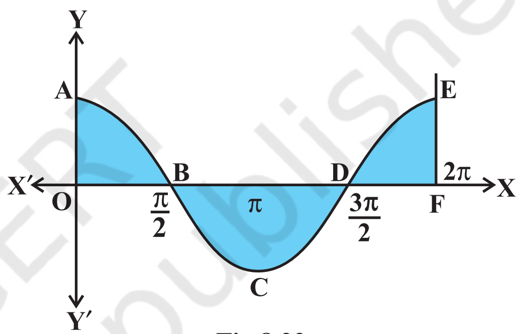

Fig 8.22

Solution From the Fig 8.22, the required area = area of the region OABO + area of the region BCDB + area of the region DEFD.

Thus, we have the required area

$$= \int_0^{\frac{\pi}{2}}\cos x\,dx + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos x\,dx\right| + \int_{\frac{3\pi}{2}}^{2\pi}\cos x\,dx$$

$$= [\sin x]_0^{\frac{\pi}{2}} + \left|[\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| + [\sin x]_{\frac{3\pi}{2}}^{2\pi}$$

$$= 1 + 2 + 1 = 4$$

Example 13 Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts.

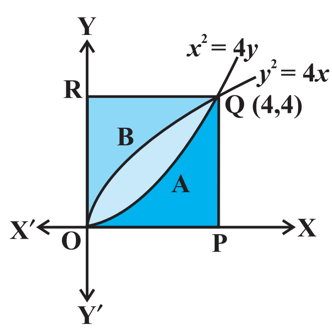

Fig 8.23

Solution Note that the point of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$ are (0, 0) and (4, 4) as shown in the Fig 8.23.

Now, the area of the region OAQBO bounded by curves $y^2 = 4x$ and $x^2 = 4y$.

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2 \times \frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \qquad \text{... (1)}$$

Again, the area of the region OPQAO bounded by the curves $x^2 = 4y$, $x = 0$, $x = 4$ and x-axis

$$= \int_0^4 \frac{x^2}{4} dx = \frac{1}{12}\left[x^3\right]_0^4 = \frac{16}{3} \qquad \text{... (2)}$$

Similarly, the area of the region OBQRO bounded by the curve $y^2 = 4x$, y-axis, $y = 0$ and $y = 4$

$$= \int_0^4 x dy = \int_0^4 \frac{y^2}{4} dy = \frac{1}{12}\left[y^3\right]_0^4 = \frac{16}{3} \qquad \text{... (3)}$$

From (1), (2) and (3), it is concluded that the area of the region OAQBO = area of the region OPQAO = area of the region OBQRO, i.e., area bounded by parabolas $y^2 = 4x$ and $x^2 = 4y$ divides the area of the square in three equal parts.

Example 14 Find the area of the region

$$\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$$

Solution Let us first sketch the region whose area is to be found out. This region is the intersection of the following regions.

$$A_1 = \{(x, y) : 0 \le y \le x^2 + 1\},$$

$$A_2 = \{(x, y) : 0 \le y \le x + 1\}$$

and

$$A_3 = \{(x, y) : 0 \le x \le 2\}$$

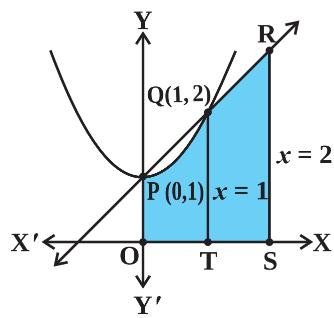

Fig 8.24

The points of intersection of $y = x^2 + 1$ and $y = x + 1$ are points P(0, 1) and Q(1, 2). From the Fig 8.24, the required region is the shaded region OPQRSTO whose area

= area of the region OTQPO + area of the region TSRQT

$$= \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx \qquad \text{(Why?)}$$

$$= \left[\left(\frac{x^3}{3}+x\right)\right]_0^1 + \left[\left(\frac{x^2}{2}+x\right)\right]_1^2$$

$$= \left[\left(\frac{1}{3}+1\right)-0\right] + \left[(2+2)-\left(\frac{1}{2}+1\right)\right] = \frac{23}{6}$$

Miscellaneous Exercise on Chapter 8

1. Find the area under the given curves and given lines:

(i) $y = x^2$, $x = 1$, $x = 2$ and x-axis

(ii) $y = x^4$, $x = 1$, $x = 5$ and x-axis

2. Find the area between the curves $y = x$ and $y = x^2$.

3. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.

4. Sketch the graph of $y = |x+3|$ and evaluate $\int_{-6}^{0} |x+3|\, dx$.

5. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

6. Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.

7. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

8. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$.

9. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$.

10. Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x + 2$ and the x-axis.

11. Using the method of integration find the area bounded by the curve $|x|+|y|=1$.

[Hint: The required region is bounded by lines $x + y = 1$, $x - y = 1$, $-x + y = 1$ and $-x - y = 1$].

12. Find the area bounded by curves $\{(x, y) : y \geq x^2 \text{ and } y = |x|\}$.

13. Using the method of integration find the area of the triangle ABC, coordinates of whose vertices are A(2, 0), B (4, 5) and C (6, 3).

14. Using the method of integration find the area of the region bounded by lines:

$2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$

15. Find the area of the region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$

Choose the correct answer in the following Exercises from 16 to 20.

16. Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

(A) -9 (B) $\frac{-15}{4}$ (C) $\frac{15}{4}$ (D) $\frac{17}{4}$

17. The area bounded by the curve $y = x\,|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

[Hint : $y = x^2$ if $x > 0$ and $y = -x^2$ if $x < 0$].

18. The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is

(A) $\frac{4}{3}(4\pi - \sqrt{3})$ (B) $\frac{4}{3}(4\pi + \sqrt{3})$ (C) $\frac{4}{3}(8\pi - \sqrt{3})$ (D) $\frac{4}{3}(8\pi + \sqrt{3})$

19. The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \leq x \leq \frac{\pi}{2}$ is

(A) $2(\sqrt{2} - 1)$ (B) $\sqrt{2} - 1$ (C) $\sqrt{2} + 1$ (D) $\sqrt{2}$

Summary

- The area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ ($b > a$) is given by the formula: $\text{Area} = \int_a^b y\,dx = \int_a^b f(x)\,dx$.
- The area of the region bounded by the curve $x = \phi(y)$, y-axis and the lines $y = c$, $y = d$ is given by the formula: $\text{Area} = \int_c^d x\,dy = \int_c^d \phi(y)\,dy$.

- The area of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by the formula,

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx, \text{ where, } f(x) \geq g(x) \text{ in } [a, b]$$

- If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then

$$\text{Area} = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx\,.$$

Historical Note

The origin of the Integral Calculus goes back to the early period of development of Mathematics and it is related to the method of exhaustion developed by the mathematicians of ancient Greece. This method arose in the solution of problems on calculating areas of plane figures, surface areas and volumes of solid bodies etc. In this sense, the method of exhaustion can be regarded as an early method of integration. The greatest development of method of exhaustion in the early period was obtained in the works of Eudoxus (440 B.C.) and Archimedes (300 B.C.)

Systematic approach to the theory of Calculus began in the 17th century. In 1665, Newton began his work on the Calculus described by him as the theory of fluxions and used his theory in finding the tangent and radius of curvature at any point on a curve. Newton introduced the basic notion of inverse function called the anti derivative (indefinite integral) or the inverse method of tangents.

During 1684-86, Leibnitz published an article in the *Acta Eruditorum* which he called *Calculas summatorius*, since it was connected with the summation of a number of infinitely small areas, whose sum, he indicated by the symbol '$\int$'. In 1696, he followed a suggestion made by J. Bernoulli and changed this article to Calculus integrali. This corresponded to Newton's inverse method of tangents.

Both Newton and Leibnitz adopted quite independent lines of approach which was radically different. However, respective theories accomplished results that were practically identical. Leibnitz used the notion of definite integral and what is quite certain is that he first clearly appreciated tie up between the antiderivative and the definite integral.

Conclusively, the fundamental concepts and theory of Integral Calculus and primarily its relationships with Differential Calculus were developed in the work of P.de Fermat, I. Newton and G. Leibnitz at the end of 17th century.

However, this justification by the concept of limit was only developed in the works of A.L. Cauchy in the early 19th century. Lastly, it is worth mentioning the following quotation by Lie Sophie's:

"It may be said that the conceptions of differential quotient and integral which in their origin certainly go back to Archimedes were introduced in Science by the investigations of Kepler, Descartes, Cavalieri, Fermat and Wallis .... The discovery that differentiation and integration are inverse operations belongs to Newton and Leibnitz".

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Chapter 9

DIFFERENTIAL EQUATIONS

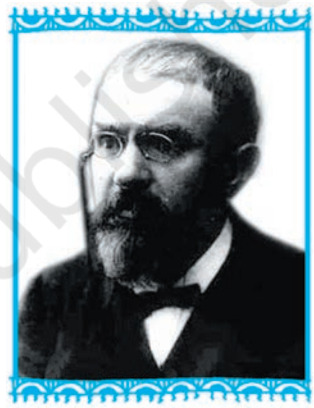
❖ *He who seeks for methods without having a definite problem in mind seeks for the most part in vain. – D. HILBERT* ❖

9.1 Introduction

In Class XI and in Chapter 5 of the present book, we discussed how to differentiate a given function f with respect to an independent variable, i.e., how to find $f'(x)$ for a given function f at each x in its domain of definition. Further, in the chapter on Integral Calculus, we discussed how to find a function f whose derivative is the function g, which may also be formulated as follows:

For a given function g, find a function f such that

$$\frac{dy}{dx} = g(x), \text{ where } y = f(x) \qquad \ldots (1)$$



Henri Poincare (1854-1912)

An equation of the form (1) is known as a *differential equation*. A formal definition will be given later.

These equations arise in a variety of applications, may it be in Physics, Chemistry, Biology, Anthropology, Geology, Economics etc. Hence, an indepth study of differential equations has assumed prime importance in all modern scientific investigations.

In this chapter, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation, formation of differential equations, some methods to solve a first order - first degree differential equation and some applications of differential equations in different areas.

9.2 Basic Concepts

We are already familiar with the equations of the type:

$$x^2 - 3x + 3 = 0 \qquad \ldots (1)$$

$$\sin x + \cos x = 0 \qquad \ldots (2)$$

$$x + y = 7 \qquad \ldots (3)$$

Let us consider the equation:

$$x\frac{dy}{dx} + y = 0 \qquad \text{... (4)}$$

We see that equations (1), (2) and (3) involve independent and/or dependent variable (variables) only but equation (4) involves variables as well as derivative of the dependent variable y with respect to the independent variable x. Such an equation is called a *differential equation*.

In general, an equation involving derivative (derivatives) of the dependent variable with respect to independent variable (variables) is called a differential equation.

A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation, e.g.,

$$2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0 \text{ is an ordinary differential equation} \qquad \text{.... (5)}$$

Of course, there are differential equations involving derivatives with respect to more than one independent variables, called partial differential equations but at this stage we shall confine ourselves to the study of ordinary differential equations only. Now onward, we will use the term 'differential equation' for 'ordinary differential equation'.

Note

1. We shall prefer to use the following notations for derivatives:

   $$\frac{dy}{dx} = y', \frac{d^2y}{dx^2} = y'', \frac{d^3y}{dx^3} = y'''$$

2. For derivatives of higher order, it will be inconvenient to use so many dashes as supersuffix therefore, we use the notation y_n for nth order derivative $\frac{d^ny}{dx^n}$.

9.2.1. *Order of a differential equation*

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Consider the following differential equations:

$$\frac{dy}{dx} = e^x \qquad \text{... (6)}$$

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{... (7)}$$

$$\left(\frac{d^3y}{dx^3}\right) + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0 \quad \text{... (8)}$$

The equations (6), (7) and (8) involve the highest derivative of first, second and third order respectively. Therefore, the order of these equations are 1, 2 and 3 respectively.

9.2.2 *Degree of a differential equation*

To study the degree of a differential equation, the key point is that the differential equation must be a polynomial equation in derivatives, i.e., y', y'', y''' etc. Consider the following differential equations:

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0 \quad \text{... (9)}$$

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin^2 y = 0 \quad \text{... (10)}$$

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0 \quad \text{... (11)}$$

We observe that equation (9) is a polynomial equation in y''', y'' and y', equation (10) is a polynomial equation in y' (not a polynomial in y though). Degree of such differential equations can be defined. But equation (11) is not a polynomial equation in y' and degree of such a differential equation can not be defined.

By the degree of a differential equation, when it is a polynomial equation in derivatives, we mean the highest power (positive integral index) of the highest order derivative involved in the given differential equation.

In view of the above definition, one may observe that differential equations (6), (7), (8) and (9) each are of degree one, equation (10) is of degree two while the degree of differential equation (11) is not defined.

Note Order and degree (if defined) of a differential equation are always positive integers.

Example 1 Find the order and degree, if defined, of each of the following differential equations:

(i) $\frac{dy}{dx} - \cos x = 0$ (ii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

(iii) $y''' + y^2 + e^{y'} = 0$

Solution

(i) The highest order derivative present in the differential equation is $\frac{dy}{dx}$, so its order is one. It is a polynomial equation in y' and the highest power raised to $\frac{dy}{dx}$ is one, so its degree is one.

(ii) The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$, so its order is two. It is a polynomial equation in $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and the highest power raised to $\frac{d^2y}{dx^2}$ is one, so its degree is one.

(iii) The highest order derivative present in the differential equation is y''', so its order is three. The given differential equation is not a polynomial equation in its derivatives and so its degree is not defined.

EXERCISE 9.1

Determine order and degree (if defined) of differential equations given in Exercises 1 to 10.

1. $\frac{d^4y}{dx^4} + \sin(y''') = 0$ **2.** $y' + 5y = 0$ **3.** $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

4. $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$ **5.** $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$

6. $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ **7.** $y''' + 2y'' + y' = 0$

8. $y' + y = e^x$ **9.** $y'' + (y')^2 + 2y = 0$ **10.** $y'' + 2y' + \sin y = 0$

11. The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0 \text{ is}$$

(A) 3 (B) 2 (C) 1 (D) not defined

12. The order of the differential equation

$$2x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0 \text{ is}$$

(A) 2 (B) 1 (C) 0 (D) not defined

9.3. General and Particular Solutions of a Differential Equation

In earlier Classes, we have solved the equations of the type:

$$x^2 + 1 = 0 \quad \text{... (1)}$$

$$\sin^2 x - \cos x = 0 \quad \text{... (2)}$$

Solution of equations (1) and (2) are numbers, real or complex, that will satisfy the given equation i.e., when that number is substituted for the unknown x in the given equation, L.H.S. becomes equal to the R.H.S..

Now consider the differential equation $\frac{d^2y}{dx^2} + y = 0$ ... (3)

In contrast to the first two equations, the solution of this differential equation is a function ϕ that will satisfy it i.e., when the function ϕ is substituted for the unknown y (dependent variable) in the given differential equation, L.H.S. becomes equal to R.H.S..

The curve $y = \phi(x)$ is called the solution curve (integral curve) of the given differential equation. Consider the function given by

$$y = \phi(x) = a \sin(x + b), \quad \text{... (4)}$$

where $a, b \in \mathbf{R}$. When this function and its derivative are substituted in equation (3), L.H.S. = R.H.S.. So it is a solution of the differential equation (3).

Let a and b be given some particular values say $a = 2$ and $b = \frac{\pi}{4}$, then we get a function

$$y = \phi_1(x) = 2\sin\left(x + \frac{\pi}{4}\right) \quad \text{... (5)}$$

When this function and its derivative are substituted in equation (3) again L.H.S. = R.H.S.. Therefore ϕ_1 is also a solution of equation (3).

Function ϕ consists of two arbitrary constants (parameters) a, b and it is called *general solution* of the given differential equation. Whereas function ϕ_1 contains no arbitrary constants but only the particular values of the parameters a and b and hence is called a *particular solution* of the given differential equation.

The solution which contains arbitrary constants is called the *general solution* (*primitive*) of the differential equation.

The solution free from arbitrary constants i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants is called a *particular solution* of the differential equation.

Example 2 Verify that the function $y = e^{-3x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Solution Given function is $y = e^{-3x}$. Differentiating both sides of equation with respect to x , we get

$$\frac{dy}{dx} = -3e^{-3x} \quad \text{... (1)}$$

Now, differentiating (1) with respect to x, we have

$$\frac{d^2y}{dx^2} = 9e^{-3x}$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we get

L.H.S. $= 9\ e^{-3x} + (-3e^{-3x}) - 6.e^{-3x} = 9\ e^{-3x} - 9\ e^{-3x} = 0 =$ R.H.S..

Therefore, the given function is a solution of the given differential equation.

Example 3 Verify that the function $y = a \cos x + b \sin x$, where, $a, b \in \mathbf{R}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

Solution The given function is

$$y = a \cos x + b \sin x \quad \text{... (1)}$$

Differentiating both sides of equation (1) with respect to x, successively, we get

$$\frac{dy}{dx} = -a \sin x + b \cos x$$

$$\frac{d^2y}{dx^2} = -a \cos x - b \sin x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

L.H.S. = $(-a\cos x - b\sin x) + (a\cos x + b\sin x) = 0$ = R.H.S.

Therefore, the given function is a solution of the given differential equation.

EXERCISE 9.2

In each of the Exercises 1 to 10 verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

1. $y = e^x + 1$: $y'' - y' = 0$

2. $y = x^2 + 2x + C$: $y' - 2x - 2 = 0$

3. $y = \cos x + C$: $y' + \sin x = 0$

4. $y = \sqrt{1+x^2}$: $y' = \frac{xy}{1+x^2}$

5. $y = Ax$: $xy' = y \ (x \neq 0)$

6. $y = x\sin x$: $xy' = y + x\sqrt{x^2 - y^2} \ (x \neq 0$ and $x > y$ or $x < -y)$

7. $xy = \log y + C$: $y' = \frac{y^2}{1-xy} \ (xy \neq 1)$

8. $y - \cos y = x$: $(y\sin y + \cos y + x)\, y' = y$

9. $x + y = \tan^{-1}y$: $y^2 y' + y^2 + 1 = 0$

10. $y = \sqrt{a^2 - x^2}\ x \in (-a, a)$: $x + y\frac{dy}{dx} = 0 \ (y \neq 0)$

11. The number of arbitrary constants in the general solution of a differential equation of fourth order are:

(A) 0 (B) 2 (C) 3 (D) 4

12. The number of arbitrary constants in the particular solution of a differential equation of third order are:

(A) 3 (B) 2 (C) 1 (D) 0

9.4 Formation of a Differential Equation whose General Solution is given

We know that the equation

$$x^2 + y^2 + 2x - 4y + 4 = 0 \quad \ldots (1)$$

represents a circle having centre at $(-1, 2)$ and radius 1 unit.

Differentiating equation (1) with respect to x, we get

$$\frac{dy}{dx} = \frac{x+1}{2-y} \quad (y \neq 2) \qquad \text{... (2)}$$

which is a differential equation. You will find later on [See (example 9 section 9.5.1.)] that this equation represents the family of circles and one member of the family is the circle given in equation (1).

Let us consider the equation

$$x^2 + y^2 = r^2 \qquad \text{... (3)}$$

By giving different values to r, we get different members of the family e.g. $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $x^2 + y^2 = 9$ etc. (see Fig 9.1). Thus, equation (3) represents a family of concentric circles centered at the origin and having different radii.

We are interested in finding a differential equation that is satisfied by each member of the family. The differential equation must be free from r because r is different for different members of the family. This equation is obtained by differentiating equation (3) with respect to x, i.e.,

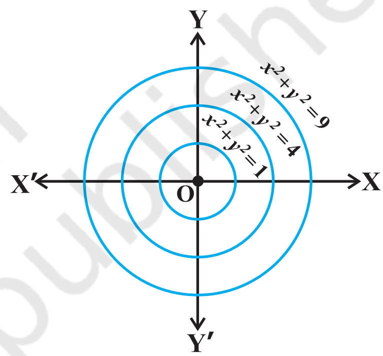


Fig 9.1

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad x + y\frac{dy}{dx} = 0 \qquad \text{... (4)}$$

which represents the family of concentric circles given by equation (3).

Again, let us consider the equation

$$y = mx + c \qquad \text{... (5)}$$

By giving different values to the parameters m and c, we get different members of the family, e.g.,

$y = x$ $\quad (m = 1, \; c = 0)$

$y = \sqrt{3}\,x$ $\quad (m = \sqrt{3}, \; c = 0)$

$y = x + 1$ $\quad (m = 1, \; c = 1)$

$y = -x$ $\quad (m = -1, \; c = 0)$

$y = -x - 1$ $\quad (m = -1, \; c = -1)$ etc. (see Fig 9.2).

Thus, equation (5) represents the family of straight lines, where m, c are parameters.

We are now interested in finding a differential equation that is satisfied by each member of the family. Further, the equation must be free from m and c because m and

c are different for different members of the family. This is obtained by differentiating equation (5) with respect to x, successively we get

$$\frac{dy}{dx} = m, \text{ and } \frac{d^2y}{dx^2} = 0 \qquad \ldots (6)$$

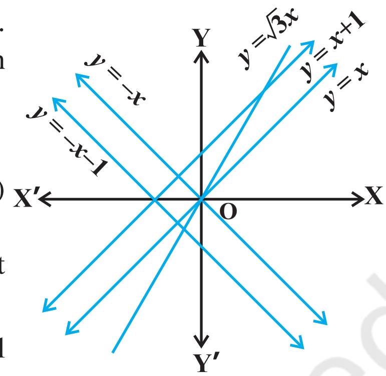


Fig 9.2

The equation (6) represents the family of straight lines given by equation (5).

Note that equations (3) and (5) are the general solutions of equations (4) and (6) respectively.

9.4.1 *Procedure to form a differential equation that will represent a given family of curves*

(a) If the given family F_1 of curves depends on only one parameter then it is represented by an equation of the form

$$F_1 (x, y, a) = 0 \qquad \ldots (1)$$

For example, the family of parabolas $y^2 = ax$ can be represented by an equation of the form $f(x, y, a) : y^2 = ax$.

Differentiating equation (1) with respect to x, we get an equation involving y', y, x, and a, i.e.,

$$g (x, y, y', a) = 0 \qquad \ldots (2)$$

The required differential equation is then obtained by eliminating a from equations (1) and (2) as

$$F(x, y, y') = 0 \qquad \ldots (3)$$

(b) If the given family F_2 of curves depends on the parameters a, b (say) then it is represented by an equation of the from

$$F_2 (x, y, a, b) = 0 \qquad \ldots (4)$$

Differentiating equation (4) with respect to x, we get an equation involving y', x, y, a, b, i.e.,

$$g (x, y, y', a, b) = 0 \qquad \ldots (5)$$

But it is not possible to eliminate two parameters a and b from the two equations and so, we need a third equation. This equation is obtained by differentiating equation (5), with respect to x, to obtain a relation of the form

$$h (x, y, y', y'', a, b) = 0 \qquad \ldots (6)$$

The required differential equation is then obtained by eliminating a and b from equations (4), (5) and (6) as

$$F(x, y, y', y'') = 0 \quad \ldots (7)$$

Note The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

Example 4 Form the differential equation representing the family of curves $y = mx$, where, m is arbitrary constant.

Solution We have

$$y = mx \quad \ldots (1)$$

Differentiating both sides of equation (1) with respect to x, we get

$$\frac{dy}{dx} = m$$

Substituting the value of m in equation (1) we get $y = \frac{dy}{dx} \cdot x$

or $$x\frac{dy}{dx} - y = 0$$

which is free from the parameter m and hence this is the required differential equation.

Example 5 Form the differential equation representing the family of curves $y = a \sin (x + b)$, where a, b are arbitrary constants.

Solution We have

$$y = a \sin (x + b) \quad \ldots (1)$$

Differentiating both sides of equation (1) with respect to x, successively we get

$$\frac{dy}{dx} = a \cos (x + b) \quad \ldots (2)$$

$$\frac{d^2y}{dx^2} = -a \sin (x + b) \quad \ldots (3)$$

Eliminating a and b from equations (1), (2) and (3), we get

$$\frac{d^2y}{dx^2} + y = 0 \quad \ldots (4)$$

which is free from the arbitrary constants a and b and hence this the required differential equation.

Example 6 Form the differential equation representing the family of ellipses having foci on x-axis and centre at the origin.

Solution We know that the equation of said family of ellipses (see Fig 9.3) is

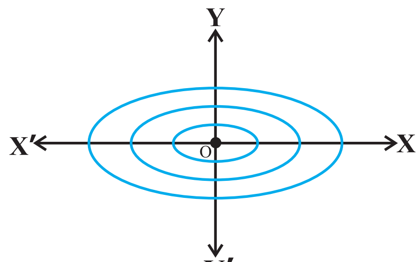


Fig 9.3

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad \text{... (1)}$$

Differentiating equation (1) with respect to x, we get $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

or $$\frac{y}{x}\left(\frac{dy}{dx}\right) = \frac{-b^2}{a^2} \qquad \text{... (2)}$$

Differentiating both sides of equation (2) with respect to x, we get

$$\left(\frac{y}{x}\right)\left(\frac{d^2y}{dx^2}\right)+\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)\frac{dy}{dx} = 0$$

or $$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx} = 0 \qquad \text{... (3)}$$

which is the required differential equation.

Example 7 Form the differential equation of the family of circles touching the x-axis at origin.

Solution Let C denote the family of circles touching x-axis at origin. Let $(0, a)$ be the coordinates of the centre of any member of the family (see Fig 9.4). Therefore, equation of family C is

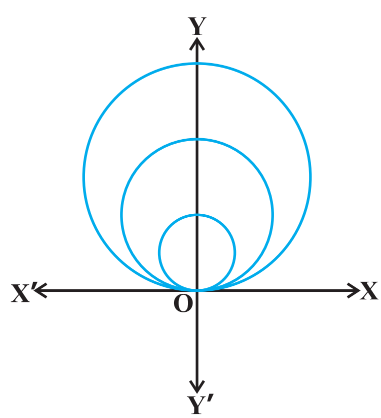


Fig 9.4

$$x^2 + (y - a)^2 = a^2 \text{ or } x^2 + y^2 = 2ay \qquad \text{... (1)}$$

where, a is an arbitrary constant. Differentiating both sides of equation (1) with respect to x, we get

$$2x+2y\frac{dy}{dx} = 2a\frac{dy}{dx}$$

or $$x + y\frac{dy}{dx} = a\frac{dy}{dx} \text{ or } a = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}} \quad \text{... (2)}$$

Substituting the value of a from equation (2) in equation (1), we get

$$x^2 + y^2 = 2y\frac{\left[x + y\frac{dy}{dx}\right]}{\frac{dy}{dx}}$$

or $$\frac{dy}{dx}(x^2 + y^2) = 2xy + 2y^2\frac{dy}{dx}$$

or $$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

This is the required differential equation of the given family of circles.

Example 8 Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

Solution Let P denote the family of above said parabolas (see Fig 9.5) and let $(a, 0)$ be the focus of a member of the given family, where a is an arbitrary constant. Therefore, equation of family P is

$$y^2 = 4ax \quad \text{... (1)}$$

Differentiating both sides of equation (1) with respect to x, we get

$$2y\frac{dy}{dx} = 4a \quad \text{... (2)}$$

Substituting the value of $4a$ from equation (2) in equation (1), we get

$$y^2 = \left(2y\frac{dy}{dx}\right)(x)$$

or $$y^2 - 2xy\frac{dy}{dx} = 0$$

which is the differential equation of the given family of parabolas.

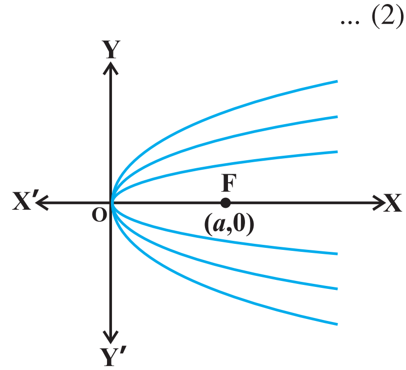


Fig 9.5

EXERCISE 9.3

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

1. $\frac{x}{a}+\frac{y}{b}=1$ **2.** $y^2 = a\ (b^2 – x^2)$ **3.** $y = a\ e^{3x} + b\ e^{-2x}$

4. $y = e^{2x}\ (a + bx)$ **5.** $y = e^x\ (a \cos x + b \sin x)$

6. Form the differential equation of the family of circles touching the y-axis at origin.

7. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

8. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

9. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

10. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

11. Which of the following differential equations has $y = c_1\, e^x + c_2\, e^{-x}$ as the general solution?

(A) $\frac{d^2y}{dx^2}+y=0$ (B) $\frac{d^2y}{dx^2}-y=0$ (C) $\frac{d^2y}{dx^2}+1=0$ (D) $\frac{d^2y}{dx^2}-1=0$

12. Which of the following differential equations has $y = x$ as one of its particular solution?

(A) $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=x$ (B) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+xy=x$

(C) $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=0$ (D) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+xy=0$

9.5. Methods of Solving First Order, First Degree Differential Equations

In this section we shall discuss three methods of solving first order first degree differential equations.

9.5.1 *Differential equations with variables separable*

A first order-first degree differential equation is of the form

$$\frac{dy}{dx} = F(x, y) \quad \ldots (1)$$

If F(x, y) can be expressed as a product g (x) $h(y)$, where, $g(x)$ is a function of x and $h(y)$ is a function of y, then the differential equation (1) is said to be of variable separable type. The differential equation (1) then has the form

$$\frac{dy}{dx} = h(y) \cdot g(x) \qquad \text{... (2)}$$

If $h(y) \neq 0$, separating the variables, (2) can be rewritten as

$$\frac{1}{h(y)}\, dy = g(x)\, dx \qquad \text{... (3)}$$

Integrating both sides of (3), we get

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx \qquad \text{... (4)}$$

Thus, (4) provides the solutions of given differential equation in the form

$$\text{H}(y) = \text{G}(x) + \text{C}$$

Here, H (y) and G (x) are the anti derivatives of $\frac{1}{h(y)}$ and $g(x)$ respectively and C is the arbitrary constant.

Example 9 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}, (y \neq 2)$

Solution We have

$$\frac{dy}{dx} = \frac{x+1}{2-y} \qquad \text{... (1)}$$

Separating the variables in equation (1), we get

$$(2 - y)\, dy = (x + 1)\, dx \qquad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\int (2-y)\, dy = \int (x+1)\, dx$$

or $$2y - \frac{y^2}{2} = \frac{x^2}{2} + x + \text{C}_1$$

or $$x^2 + y^2 + 2x - 4y + 2\,\text{C}_1 = 0$$

or $$x^2 + y^2 + 2x - 4y + \text{C} = 0, \text{ where C} = 2\text{C}_1$$

which is the general solution of equation (1).

Example 10 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.

Solution Since $1 + y^2 \neq 0$, therefore separating the variables, the given differential equation can be written as

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or $$\tan^{-1} y = \tan^{-1} x + C$$

which is the general solution of equation (1).

Example 11 Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

Solution If $y \neq 0$, the given differential equation can be written as

$$\frac{dy}{y^2} = -4x\, dx \quad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{y^2} = -4\int x\, dx$$

or $$-\frac{1}{y} = -2x^2 + C$$

or $$y = \frac{1}{2x^2 - C} \quad \text{... (2)}$$

Substituting $y = 1$ and $x = 0$ in equation (2), we get, $C = -1$.

Now substituting the value of C in equation (2), we get the particular solution of the given differential equation as $y = \frac{1}{2x^2+1}$.

Example 12 Find the equation of the curve passing through the point (1, 1) whose differential equation is $x\, dy = (2x^2 + 1)\, dx$ $(x \neq 0)$.

Solution The given differential equation can be expressed as

$$dy^* = \left(\frac{2x^2+1}{x}\right)dx^*$$

or $$dy = \left(2x+\frac{1}{x}\right)dx \quad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int dy = \int\left(2x+\frac{1}{x}\right)dx$$

or $$y = x^2 + \log|x| + C \quad \text{... (2)}$$

Equation (2) represents the family of solution curves of the given differential equation but we are interested in finding the equation of a particular member of the family which passes through the point (1, 1). Therefore substituting $x = 1$, $y = 1$ in equation (2), we get C = 0.

Now substituting the value of C in equation (2) we get the equation of the required curve as $y = x^2 + \log|x|$.

Example 13 Find the equation of a curve passing through the point (–2, 3), given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$.

Solution We know that the slope of the tangent to a curve is given by $\frac{dy}{dx}$.

so, $$\frac{dy}{dx} = \frac{2x}{y^2} \quad \text{... (1)}$$

Separating the variables, equation (1) can be written as

$$y^2\,dy = 2x\,dx \quad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\int y^2 dy = \int 2x\,dx$$

or $$\frac{y^3}{3} = x^2 + C \quad \text{... (3)}$$

---

* The notation $\frac{dy}{dx}$ due to Leibnitz is extremely flexible and useful in many calculation and formal transformations, where, we can deal with symbols dy and dx exactly as if they were ordinary numbers. By treating dx and dy like separate entities, we can give neater expressions to many calculations.

Refer: Introduction to Calculus and Analysis, volume-I page 172, By Richard Courant, Fritz John Spinger – Verlog New York.

Substituting $x = -2$, $y = 3$ in equation (3), we get C = 5.

Substituting the value of C in equation (3), we get the equation of the required curve as

$$\frac{y^3}{3} = x^2 + 5 \quad \text{or} \quad y = (3x^2 + 15)^{\frac{1}{3}}$$

Example 14 In a bank, principal increases continuously at the rate of 5% per year. In how many years Rs 1000 double itself?

Solution Let P be the principal at any time t. According to the given problem,

$$\frac{dp}{dt} = \left(\frac{5}{100}\right) \times \text{P}$$

or $$\frac{dp}{dt} = \frac{\text{P}}{20} \quad \text{... (1)}$$

separating the variables in equation (1), we get

$$\frac{dp}{\text{P}} = \frac{dt}{20} \quad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\log \text{P} = \frac{t}{20} + \text{C}_1$$

or $$\text{P} = e^{\frac{t}{20}} \cdot e^{\text{C}_1}$$

or $$\text{P} = \text{C}\, e^{\frac{t}{20}} \quad (\text{where } e^{\text{C}_1} = \text{C}) \quad \text{... (3)}$$

Now $\text{P} = 1000$, when $t = 0$

Substituting the values of P and t in (3), we get C = 1000. Therefore, equation (3), gives

$$\text{P} = 1000\, e^{\frac{t}{20}}$$

Let t years be the time required to double the principal. Then

$$2000 = 1000\, e^{\frac{t}{20}} \Rightarrow t = 20 \log_e 2$$

EXERCISE 9.4

For each of the differential equations in Exercises 1 to 10, find the general solution:

1. $\dfrac{dy}{dx} = \dfrac{1 - \cos x}{1 + \cos x}$

2. $\dfrac{dy}{dx} = \sqrt{4 - y^2} \quad (-2 < y < 2)$

3. $\dfrac{dy}{dx} + y = 1 \; (y \neq 1)$

4. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

5. $(e^x + e^{-x}) \, dy - (e^x - e^{-x}) \, dx = 0$

6. $\dfrac{dy}{dx} = (1 + x^2)(1 + y^2)$

7. $y \log y \, dx - x \, dy = 0$

8. $x^5 \dfrac{dy}{dx} = -y^5$

9. $\dfrac{dy}{dx} = \sin^{-1} x$

10. $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

For each of the differential equations in Exercises 11 to 14, find a particular solution satisfying the given condition:

11. $(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x$; $y = 1$ when $x = 0$

12. $x(x^2 - 1)\dfrac{dy}{dx} = 1$; $y = 0$ when $x = 2$

13. $\cos\left(\dfrac{dy}{dx}\right) = a \; (a \in \mathbf{R})$; $y = 1$ when $x = 0$

14. $\dfrac{dy}{dx} = y \tan x$; $y = 1$ when $x = 0$

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

16. For the differential equation $xy\dfrac{dy}{dx} = (x + 2)(y + 2)$, find the solution curve passing through the point (1, –1).

17. Find the equation of a curve passing through the point (0, –2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

18. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, –3). Find the equation of the curve given that it passes through (–2, 1).

19. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

20. In a bank, principal increases continuously at the rate of $r\%$ per year. Find the value of r if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$).

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

22. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

9.5.2 *Homogeneous differential equations*

Consider the following functions in x and y

$F_1(x, y) = y^2 + 2xy$, $F_2(x, y) = 2x - 3y$,

$F_3(x, y) = \cos\left(\frac{y}{x}\right)$, $F_4(x, y) = \sin x + \cos y$

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant λ, we get

$F_1(\lambda x, \lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$

$F_2(\lambda x, \lambda y) = \lambda (2x - 3y) = \lambda F_2(x, y)$

$F_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right) = \lambda^0 F_3(x, y)$

$F_4(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n F_4(x, y)$, for any $n \in \mathbf{N}$

Here, we observe that the functions F_1, F_2, F_3 can be written in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ but F_4 can not be written in this form. This leads to the following definition:

A function $F(x, y)$ is said to be *homogeneous function of degree n* if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.

We note that in the above examples, F_1, F_2, F_3 are homogeneous functions of degree 2, 1, 0 respectively but F_4 is not a homogeneous function.

We also observe that

$$F_1(x, y) = x^2\left(\frac{y^2}{x^2}+\frac{2y}{x}\right) = x^2 h_1\left(\frac{y}{x}\right)$$

or $$F_1(x, y) = y^2\left(1+\frac{2x}{y}\right) = y^2 h_2\left(\frac{x}{y}\right)$$

$$F_2(x, y) = x^1\left(2-\frac{3y}{x}\right) = x^1 h_3\left(\frac{y}{x}\right)$$

or $$F_2(x, y) = y^1\left(2\frac{x}{y}-3\right) = y^1 h_4\left(\frac{x}{y}\right)$$

$$F_3(x, y) = x^0 \cos\left(\frac{y}{x}\right) = x^0\, h_5\left(\frac{y}{x}\right)$$

$$F_4(x, y) \neq x^n\, h_6\left(\frac{y}{x}\right), \text{ for any } n \in \mathbf{N}$$

or $$F_4(x, y) \neq y^n\, h_7\left(\frac{x}{y}\right), \text{ for any } n \in \mathbf{N}$$

Therefore, a function F (x, y) is a homogeneous function of degree n if

$$F(x, y) = x^n g\left(\frac{y}{x}\right) \quad \text{or} \quad y^n h\left(\frac{x}{y}\right)$$

A differential equation of the form $\frac{dy}{dx}$ = F (x, y) is said to be *homogenous* if F(x, y) is a homogenous function of degree zero.

To solve a homogeneous differential equation of the type

$$\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right) \qquad \text{... (1)}$$

We make the substitution $y = v \, . \, x$ ... (2)

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get

$$v + x\frac{dv}{dx} = g(v)$$

or $$x\frac{dv}{dx} = g(v) - v \qquad \dots (4)$$

Separating the variables in equation (4), we get

$$\frac{dv}{g(v) - v} = \frac{dx}{x} \qquad \dots (5)$$

Integrating both sides of equation (5), we get

$$\int \frac{dv}{g(v) - v} = \int \frac{1}{x}dx + C \qquad \dots (6)$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace v by $\frac{y}{x}$.

Note If the homogeneous differential equation is in the form $\frac{dx}{dy} = F(x, y)$ where, $F(x, y)$ is homogenous function of degree zero, then we make substitution $\frac{x}{y} = v$ i.e., $x = vy$ and we proceed further to find the general solution as discussed above by writing $\frac{dx}{dy} = F(x, y) = h\left(\frac{x}{y}\right)$.

Example 15 Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Solution The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x + 2y}{x - y} \qquad \dots (1)$$

Let $$F(x, y) = \frac{x + 2y}{x - y}$$

Now $$F(\lambda x, \lambda y) = \frac{\lambda(x + 2y)}{\lambda(x - y)} = \lambda^0 \cdot f(x, y)$$

Therefore, $F(x, y)$ is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

Alternatively,

$$\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \qquad \text{... (2)}$$

R.H.S. of differential equation (2) is of the form $g\left(\frac{y}{x}\right)$ and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (3)}$$

Differentiating equation (3) with respect to, x we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (4)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1) we get

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

or $$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

or $$x\frac{dv}{dx} = \frac{v^2+v+1}{1-v}$$

or $$\frac{v-1}{v^2+v+1}dv = \frac{-dx}{x}$$

Integrating both sides of equation (5), we get

$$\int \frac{v-1}{v^2+v+1}dv = -\int \frac{dx}{x}$$

or $$\frac{1}{2}\int \frac{2v+1-3}{v^2+v+1}dv = -\log|x| + C_1$$

or $$\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv-\frac{3}{2}\int\frac{1}{v^2+v+1}dv=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\int\frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dv=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|+\frac{1}{2}\log x^2=\sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)+C_1 \qquad \text{(Why?)}$$

Replacing v by $\frac{y}{x}$, we get

or $$\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|+\frac{1}{2}\log x^2=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right|=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\log\left|(y^2+xy+x^2)\right|=2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+2C_1$$

or $$\log\left|(x^2+xy+y^2)\right|=2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)+C$$

which is the general solution of the differential equation (1)

Example 16 Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$ is homogeneous and solve it.

Solution The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)} \qquad \dots (1)$$

It is a differential equation of the form $\frac{dy}{dx} = F(x, y)$.

Here $$F(x, y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

Replacing x by λx and y by λy, we get

$$F(\lambda x, \lambda y) = \frac{\lambda\left[y\cos\left(\frac{y}{x}\right) + x\right]}{\lambda\left(x\cos\frac{y}{x}\right)} = \lambda^0 [F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (2)}$$

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

or $$x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

or $$x\frac{dv}{dx} = \frac{1}{\cos v}$$

or $$\cos v\, dv = \frac{dx}{x}$$

Therefore $$\int \cos v\, dv = \int \frac{1}{x}\, dx$$

or $\sin v = \log|x| + \log|C|$

or $\sin v = \log|Cx|$

Replacing v by $\frac{y}{x}$, we get

$$\sin\left(\frac{y}{x}\right) = \log|Cx|$$

which is the general solution of the differential equation (1).

Example 17 Show that the differential equation $2y\, e^{\frac{x}{y}} dx + \left(y - 2x\, e^{\frac{x}{y}}\right) dy = 0$ is homogeneous and find its particular solution, given that, $x = 0$ when $y = 1$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2x\, e^{\frac{x}{y}} - y}{2y\, e^{\frac{x}{y}}} \quad \text{... (1)}$$

Let $$F(x, y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$$

Then $$F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^0 [F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$$x = vy \quad \text{... (2)}$$

Differentiating equation (2) with respect to y, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in equation (1), we get

$$v + y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v}$$

or $$y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v} - v$$

or $$y\frac{dv}{dy} = -\frac{1}{2e^v}$$

or $$2e^v\,dv = \frac{-dy}{y}$$

or $$\int 2e^v \cdot dv = -\int \frac{dy}{y}$$

or $$2\,e^v = -\log|y| + C$$

and replacing v by $\frac{x}{y}$, we get

$$2\,e^{\frac{x}{y}} + \log|y| = C \qquad \ldots (3)$$

Substituting $x = 0$ and $y = 1$ in equation (3), we get

$$2\,e^0 + \log|1| = C \Rightarrow C = 2$$

Substituting the value of C in equation (3), we get

$$2\,e^{\frac{x}{y}} + \log|y| = 2$$

which is the particular solution of the given differential equation.

Example 18 Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Solution We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

or $$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \qquad \text{... (1)}$$

Clearly, (1) is a homogenous differential equation. To solve it we make substitution

$$y = vx$$

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

or $$\frac{2v}{v^2-1}dv = -\frac{dx}{x}$$

Therefore $$\int\frac{2v}{v^2-1}dv = -\int\frac{1}{x}dx$$

or $$\log|v^2 - 1| = -\log|x| + \log|C_1|$$

or $$\log|(v^2 - 1)(x)| = \log|C_1|$$

or $$(v^2 - 1)\, x = \pm C_1$$

Replacing v by $\frac{y}{x}$, we get

$$\left(\frac{y^2}{x^2}-1\right)x = \pm C_1$$

or $$(y^2 - x^2) = \pm C_1\, x \text{ or } x^2 - y^2 = Cx$$

EXERCISE 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$

2. $y' = \dfrac{x+y}{x}$

3. $(x - y)\, dy - (x + y)\, dx = 0$

4. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

5. $x^2 \dfrac{dy}{dx} = x^2 - 2y^2 + xy$

6. $x\, dy - y\, dx = \sqrt{x^2 + y^2}\, dx$

7. $\left\{x\cos\left(\dfrac{y}{x}\right) + y\sin\left(\dfrac{y}{x}\right)\right\} y\, dx = \left\{y\sin\left(\dfrac{y}{x}\right) - x\cos\left(\dfrac{y}{x}\right)\right\} x\, dy$

8. $x\dfrac{dy}{dx} - y + x\sin\left(\dfrac{y}{x}\right) = 0$

9. $y\, dx + x\log\left(\dfrac{y}{x}\right) dy - 2x\, dy = 0$

10. $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right) dy = 0$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. $(x + y)\, dy + (x - y)\, dx = 0;\ y = 1$ when $x = 1$

12. $x^2\, dy + (xy + y^2)\, dx = 0;\ y = 1$ when $x = 1$

13. $\left[x\sin^2\left(\dfrac{y}{x}\right) - y\right] dx + x\, dy = 0;\ y = \dfrac{\pi}{4}$ when $x = 1$

14. $\dfrac{dy}{dx} - \dfrac{y}{x} + \operatorname{cosec}\left(\dfrac{y}{x}\right) = 0;\ y = 0$ when $x = 1$

15. $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0;\ y = 2$ when $x = 1$

16. A homogeneous differential equation of the from $\dfrac{dx}{dy} = h\left(\dfrac{x}{y}\right)$ can be solved by making the substitution.

(A) $y = vx$ (B) $v = yx$ (C) $x = vy$ (D) $x = v$

17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$

(C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

9.5.3 *Linear differential equations*

A differential equation of the from

$$\frac{dy}{dx} + \text{P}y = \text{Q}$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x\log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + \text{P}_1 x = \text{Q}_1$$

where, P_1 and Q_1 are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} + \text{P}y = \text{Q} \qquad \ldots (1)$$

Multiply both sides of the equation by a function of x say $g(x)$ to get

$$g(x)\frac{dy}{dx} + \text{P}.(g(x))\, y = \text{Q}\,.\, g(x) \qquad \ldots (2)$$

Choose $g(x)$ in such a way that R.H.S. becomes a derivative of $y \,.\, g(x)$.

i.e. $$g(x)\frac{dy}{dx} + \text{P}.\,g(x)\,y = \frac{d}{dx}\,[y \,.\, g(x)]$$

or $$g(x)\frac{dy}{dx} + \text{P}.\,g(x)\,y = g(x)\frac{dy}{dx} + y\,g'(x)$$

$\Rightarrow$ $$\text{P}.\,g(x) = g'(x)$$

or $$\text{P} = \frac{g'(x)}{g(x)}$$

Integrating both sides with respect to x, we get

$$\int \text{P}dx = \int \frac{g'(x)}{g(x)}\,dx$$

or $$\int \text{P}\cdot dx = \log(g(x))$$

or $$g(x) = e^{\int \text{P}\,dx}$$

On multiplying the equation (1) by $g(x) = e^{\int \text{P}\,dx}$, the L.H.S. becomes the derivative of some function of x and y. This function $g(x) = e^{\int \text{P}\,dx}$ is called *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of $g(x)$ in equation (2), we get

$$e^{\int \text{P}\,dx}\frac{dy}{dx} + \text{P}\,e^{\int \text{P}\,dx}\,y = \text{Q}\cdot e^{\int \text{P}\,dx}$$

or $$\frac{d}{dx}\left(y e^{\int \text{P}dx}\right) = \text{Q}\,e^{\int \text{P}dx}$$

Integrating both sides with respect to x, we get

$$y\cdot e^{\int \text{P}dx} = \int\left(\text{Q}.e^{\int \text{P}\,dx}\right)dx$$

or $$y = e^{-\int \text{P}\,dx}.\int\left(\text{Q}.e^{\int \text{P}\,dx}\right)dx + \text{C}$$

which is the general solution of the differential equation.

Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form $\frac{dy}{dx} + Py = Q$ where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) = $e^{\int P\,dx}$.

(iii) Write the solution of the given differential equation as

$$y \text{ (I.F)} = \int (Q \times \text{I.F})\, dx + C$$

In case, the first order linear differential equation is in the form $\frac{dx}{dy} + P_1 x = Q_1$, where, P_1 and Q_1 are constants or functions of y only. Then I.F = $e^{\int P_1 dy}$ and the solution of the differential equation is given by

$$x \,.\, \text{(I.F)} = \int (Q_1 \times \text{I.F})\, dy + C$$

Example 19 Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$.

Solution Given differential equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -1 \text{ and } Q = \cos x$$

Therefore $\quad \text{I.F} = e^{\int -1\, dx} = e^{-x}$

Multiplying both sides of equation by I.F, we get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$$

or $$\frac{dy}{dx}\left(ye^{-x}\right) = e^{-x}\cos x$$

On integrating both sides with respect to x, we get

$$ye^{-x} = \int e^{-x}\cos x\, dx + C \qquad \text{... (1)}$$

Let $$I = \int e^{-x}\cos x\, dx$$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int (-\sin x)\,(-e^{-x})\, dx$$

$$= -\cos x\, e^{-x} - \int \sin x\, e^{-x}\, dx$$

$$= -\cos x\, e^{-x} - \left[\sin x(-e^{-x}) - \int \cos x\,(-e^{-x})\, dx\right]$$

$$= -\cos x\, e^{-x} + \sin x\, e^{-x} - \int \cos x\, e^{-x}\, dx$$

or $\quad \mathrm{I} = -\, e^{-x} \cos x + \sin x\, e^{-x} - \mathrm{I}$

or $\quad 2\mathrm{I} = (\sin x - \cos x)\, e^{-x}$

or $\quad \mathrm{I} = \dfrac{(\sin x - \cos x)e^{-x}}{2}$

Substituting the value of I in equation (1), we get

$$ye^{-x} = \left(\frac{\sin x - \cos x}{2}\right)e^{-x} + \mathrm{C}$$

or $\quad y = \left(\dfrac{\sin x - \cos x}{2}\right) + \mathrm{C}\, e^{x}$

which is the general solution of the given differential equation.

Example 20 Find the general solution of the differential equation $x\dfrac{dy}{dx} + 2y = x^2\ (x \neq 0)$.

Solution The given differential equation is

$$x\frac{dy}{dx} + 2y = x^2 \qquad \text{... (1)}$$

Dividing both sides of equation (1) by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

which is a linear differential equation of the type $\dfrac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$, where $\mathrm{P} = \dfrac{2}{x}$ and $\mathrm{Q} = x$.

So $\quad \mathrm{I.F} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2\ [as\ e^{\log f(x)} = f(x)]$

Therefore, solution of the given equation is given by

$$y \,.\, x^2 = \int (x)\,(x^2)\, dx + \mathrm{C} = \int x^3 dx + \mathrm{C}$$

or $\quad y = \dfrac{x^2}{4} + \mathrm{C}\, x^{-2}$

which is the general solution of the given differential equation.

Example 21 Find the general solution of the differential equation $y\,dx - (x + 2y^2)\,dy = 0$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is a linear differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$, where $P_1 = -\frac{1}{y}$ and $Q_1 = 2y$. Therefore $\text{I.F} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log (y)^{-1}} = \frac{1}{y}$

Hence, the solution of the given differential equation is

$$x\frac{1}{y} = \int (2y)\left(\frac{1}{y}\right) dy + C$$

or $$\frac{x}{y} = \int (2dy) + C$$

or $$\frac{x}{y} = 2y + C$$

or $$x = 2y^2 + Cy$$

which is a general solution of the given differential equation.

Example 22 Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\cot x = 2x + x^2 \cot x \ (x \neq 0)$$

given that $y = 0$ when $x = \frac{\pi}{2}$.

Solution The given equation is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = \cot x$ and $Q = 2x + x^2 \cot x$. Therefore

$$\text{I.F} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

Hence, the solution of the differential equation is given by

$$y \,.\, \sin x = \int (2x + x^2 \cot x) \sin x\, dx + C$$

or $$y \sin x = \int 2x \sin x\, dx + \int x^2 \cos x\, dx + C$$

or $$y \sin x = \sin x\left(\frac{2x^2}{2}\right) - \int \cos x\left(\frac{2x^2}{2}\right)dx + \int x^2 \cos x\, dx + C$$

or $$y \sin x = x^2 \sin x - \int x^2 \cos x\, dx + \int x^2 \cos x\, dx + C$$

or $$y \sin x = x^2 \sin x + C \qquad \text{... (1)}$$

Substituting $y = 0$ and $x = \frac{\pi}{2}$ in equation (1), we get

$$0 = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + C$$

or $$C = \frac{-\pi^2}{4}$$

Substituting the value of C in equation (1), we get

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

or $$y = x^2 - \frac{\pi^2}{4 \sin x} \ (\sin x \neq 0)$$

which is the particular solution of the given differential equation.

Example 23 Find the equation of a curve passing through the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.

Solution We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = x + xy$$

or $$\frac{dy}{dx} - xy = x \qquad \text{... (1)}$$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where P = – x and Q = x.

Therefore, $$\text{I.F} = e^{\int -x\, dx} = e^{\frac{-x^2}{2}}$$

Hence, the solution of equation is given by

$$y \cdot e^{\frac{-x^2}{2}} = \int (x)\left(e^{\frac{-x^2}{2}}\right) dx + \text{C} \qquad \text{... (2)}$$

Let $$\text{I} = \int (x)\, e^{\frac{-x^2}{2}}\, dx$$

Let $\frac{-x^2}{2} = t$, then $-\, x\, dx = dt$ or $x\, dx = -\, dt$.

Therefore, $$\text{I} = -\int e^t\, dt = -e^t = -\, e^{\frac{-x^2}{2}}$$

Substituting the value of I in equation (2), we get

$$y\, e^{\frac{-x^2}{2}} = -e^{\frac{-x^2}{2}} + \text{C}$$

or $$y = -1 + \text{C}\, e^{\frac{x^2}{2}} \qquad \text{... (3)}$$

Now (3) represents the equation of family of curves. But we are interested in finding a particular member of the family passing through (0, 1). Substituting $x = 0$ and $y = 1$ in equation (3) we get

$$1 = -\, 1 + \text{C}\, .\, e^0 \quad \text{or} \quad \text{C} = 2$$

Substituting the value of C in equation (3), we get

$$y = -1 + 2\, e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

EXERCISE 9.6

For each of the differential equations given in Exercises 1 to 12, find the general solution:

1. $\frac{dy}{dx} + 2y = \sin x$ **2.** $\frac{dy}{dx} + 3y = e^{-2x}$ **3.** $\frac{dy}{dx} + \frac{y}{x} = x^2$

4. $\frac{dy}{dx} + (\sec x)\, y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$ **5.** $\cos^2 x \frac{dy}{dx} + y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$

6. $x\frac{dy}{dx} + 2y = x^2 \log x$ **7.** $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$

8. $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx\ (x \ne 0)$

9. $x\frac{dy}{dx}+y-x+xy\cot x=0\ (x\neq 0)$ **10.** $(x+y)\frac{dy}{dx}=1$

11. $y\,dx + (x - y^2)\,dy = 0$ **12.** $(x+3y^2)\frac{dy}{dx}=y\ (y>0)$.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\frac{dy}{dx}+2y\tan x=\sin x;\ y=0$ when $x=\frac{\pi}{3}$

14. $(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2};\ y=0$ when $x=1$

15. $\frac{dy}{dx}-3y\cot x=\sin 2x;\ y=2$ when $x=\frac{\pi}{2}$

16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

17. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

18. The Integrating Factor of the differential equation $x\frac{dy}{dx}-y=2x^2$ is

(A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x

19. The Integrating Factor of the differential equation

$(1-y^2)\frac{dx}{dy}+yx = ay\,(-1<y<1)$ is

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Miscellaneous Examples

Example 24 Verify that the function $y = c_1\, e^{ax} \cos bx + c_2\, e^{ax} \sin bx$, where c_1, c_2 are arbitrary constants is a solution of the differential equation

$$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+\left(a^2+b^2\right)y=0$$

Solution The given function is

$$y = e^{ax} [c_1 \cos bx + c_2 \sin bx] \qquad \ldots (1)$$

Differentiating both sides of equation (1) with respect to x, we get

$$\frac{dy}{dx} = e^{ax}\left[-bc_1 \sin bx + b c_2 \cos bx\right] + \left[c_1 \cos bx + c_2 \sin bx\right] e^{ax} \cdot a$$

or
$$\frac{dy}{dx} = e^{ax}[(b c_2 + a c_1) \cos bx + (a c_2 - b c_1) \sin bx] \qquad \ldots (2)$$

Differentiating both sides of equation (2) with respect to x, we get

$$\frac{d^2 y}{dx^2} = e^{ax}[(b c_2 + a c_1)(-b \sin bx) + (a c_2 - b c_1)(b \cos bx)]$$

$$+ [(b c_2 + a c_1) \cos bx + (a c_2 - b c_1) \sin bx]\, e^{ax}.a$$

$$= e^{ax}[(a^2 c_2 - 2ab c_1 - b^2 c_2) \sin bx + (a^2 c_1 + 2ab c_2 - b^2 c_1) \cos bx]$$

Substituting the values of $\frac{d^2 y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we get

$$\text{L.H.S.} = e^{ax}[a^2 c_2 - 2abc_1 - b^2 c_2) \sin bx + (a^2 c_1 + 2abc_2 - b^2 c_1) \cos bx]$$

$$-2ae^{ax}[(bc_2 + ac_1) \cos bx + (ac_2 - bc_1) \sin bx]$$

$$+(a^2 + b^2) e^{ax}[c_1 \cos bx + c_2 \sin bx]$$

$$= e^{ax}\left[\begin{array}{l}\left(a^2 c_2 - 2abc_1 - b^2 c_2 - 2a^2 c_2 + 2abc_1 + a^2 c_2 + b^2 c_2\right) \sin bx \\ +(a^2 c_1 + 2abc_2 - b^2 c_1 - 2abc_2 - 2a^2 c_1 + a^2 c_1 + b^2 c_1) \cos bx\end{array}\right]$$

$$= e^{ax}[0 \times \sin bx + 0 \cos bx] = e^{ax} \times 0 = 0 = \text{R.H.S.}$$

Hence, the given function is a solution of the given differential equation.

Example 25 Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Solution Let C denote the family of circles in the second quadrant and touching the coordinate axes. Let $(-a, a)$ be the coordinate of the centre of any member of this family (see Fig 9.6).

Equation representing the family C is

$$(x + a)^2 + (y - a)^2 = a^2 \qquad \ldots (1)$$

or $\quad x^2 + y^2 + 2ax - 2ay + a^2 = 0 \qquad \ldots (2)$

Differentiating equation (2) with respect to x, we get

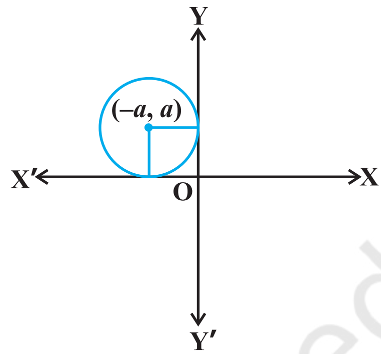


Fig 9.6

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

or $$x + y\frac{dy}{dx} = a\left(\frac{dy}{dx} - 1\right)$$

or $$a = \frac{x + y\,y'}{y' - 1}$$

Substituting the value of a in equation (1), we get

$$\left[x + \frac{x + y\,y'}{y' - 1}\right]^2 + \left[y - \frac{x + y\,y'}{y' - 1}\right]^2 = \left[\frac{x + y\,y'}{y' - 1}\right]^2$$

or $\quad [xy' - x + x + y\,y']^2 + [y\,y' - y - x - y\,y']^2 = [x + y\,y']^2$

or $\quad (x + y)^2\,y'^2 + [x + y]^2 = [x + y\,y']^2$

or $\quad (x + y)^2\,[(y')^2 + 1] = [x + y\,y']^2$

which is the differential equation representing the given family of circles.

Example 26 Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ given that $y = 0$ when $x = 0$.

Solution The given differential equation can be written as

$$\frac{dy}{dx} = e^{(3x + 4y)}$$

or $$\frac{dy}{dx} = e^{3x} \,.\, e^{4y} \qquad \ldots (1)$$

Separating the variables, we get

$$\frac{dy}{e^{4y}} = e^{3x}\,dx$$

Therefore $$\int e^{-4y}\,dy = \int e^{3x}\,dx$$

or $$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

or $$4\,e^{3x} + 3\,e^{-4y} + 12\,C = 0 \qquad \text{... (2)}$$

Substituting $x = 0$ and $y = 0$ in (2), we get

$$4 + 3 + 12\,C = 0 \text{ or } C = \frac{-7}{12}$$

Substituting the value of C in equation (2), we get

$$4\,e^{3x} + 3\,e^{-4y} - 7 = 0,$$

which is a particular solution of the given differential equation.

Example 27 Solve the differential equation

$$(x\,dy - y\,dx)\,y\sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\,x\cos\left(\frac{y}{x}\right).$$

Solution The given differential equation can be written as

$$\left[x\,y\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)\right]dy = \left[xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right]dx$$

or $$\frac{dy}{dx} = \frac{xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)}$$

Dividing numerator and denominator on RHS by x^2, we get

$$\frac{dy}{dx} = \frac{\frac{y}{x}\cos\left(\frac{y}{x}\right) + \left(\frac{y^2}{x^2}\right)\sin\left(\frac{y}{x}\right)}{\frac{y}{x}\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

Clearly, equation (1) is a homogeneous differential equation of the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

To solve it, we make the substitution

$$y = vx \qquad \text{... (2)}$$

or $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v} \qquad \text{(using (1) and (2))}$$

or $$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

or $$\left(\frac{v\sin v - \cos v}{v\cos v}\right) dv = \frac{2\,dx}{x}$$

Therefore $$\int \left(\frac{v\sin v - \cos v}{v\cos v}\right) dv = 2\int \frac{1}{x}\,dx$$

or $$\int \tan v\,dv - \int \frac{1}{v}\,dv = 2\int \frac{1}{x}\,dx$$

or $$\log|\sec v| - \log|v| = 2\log|x| + \log|C_1|$$

or $$\log\left|\frac{\sec v}{v\,x^2}\right| = \log|C_1|$$

or $$\frac{\sec v}{v\,x^2} = \pm C_1 \qquad \text{... (3)}$$

Replacing v by $\frac{y}{x}$ in equation (3), we get

$$\frac{\sec\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)(x^2)} = C \text{ where, } C = \pm C_1$$

or $$\sec\left(\frac{y}{x}\right) = C\,xy$$

which is the general solution of the given differential equation.

Example 28 Solve the differential equation

$$(\tan^{-1}y - x)\,dy = (1 + y^2)\,dx.$$

Solution The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \qquad \text{... (1)}$$

Now (1) is a linear differential equation of the form $\frac{dx}{dy} + P_1 x = Q_1$,

where, $P_1 = \frac{1}{1+y^2}$ and $Q_1 = \frac{\tan^{-1} y}{1+y^2}$.

Therefore, $I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Thus, the solution of the given differential equation is

$$x e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2} \right) e^{\tan^{-1} y} dy + C \qquad \text{... (2)}$$

Let $$I = \int \left(\frac{\tan^{-1} y}{1+y^2} \right) e^{\tan^{-1} y} dy$$

Substituting $\tan^{-1} y = t$ so that $\left(\frac{1}{1+y^2} \right) dy = dt$, we get

$$I = \int t\, e^t dt = t\, e^t - \int 1 \,.\, e^t\, dt = t\, e^t - e^t = e^t\, (t - 1)$$

or $$I = e^{\tan^{-1} y} (\tan^{-1} y - 1)$$

Substituting the value of I in equation (2), we get

$$x \,.\, e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + C$$

or $$x = (\tan^{-1} y - 1) + C\, e^{-\tan^{-1} y}$$

which is the general solution of the given differential equation.

Miscellaneous Exercise on Chapter 9

1. For each of the differential equations given below, indicate its order and degree (if defined).

(i) $\frac{d^2 y}{dx^2} + 5x \left(\frac{dy}{dx} \right)^2 - 6y = \log x$ (ii) $\left(\frac{dy}{dx} \right)^3 - 4 \left(\frac{dy}{dx} \right)^2 + 7y = \sin x$

(iii) $\frac{d^4 y}{dx^4} - \sin \left(\frac{d^3 y}{dx^3} \right) = 0$

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

(i) $xy = a\,e^x + b\,e^{-x} + x^2$: $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0$

(ii) $y = e^x (a \cos x + b \sin x)$: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$

(iii) $y = x \sin 3x$: $\dfrac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

(iv) $x^2 = 2y^2 \log y$: $(x^2 + y^2)\dfrac{dy}{dx} - xy = 0$

3. Form the differential equation representing the family of curves given by $(x - a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

4. Prove that $x^2 - y^2 = c\,(x^2 + y^2)^2$ is the general solution of differential equation $(x^3 - 3x\,y^2)\,dx = (y^3 - 3x^2y)\,dy$, where c is a parameter.

5. Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

6. Find the general solution of the differential equation $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$.

7. Show that the general solution of the differential equation $\dfrac{dy}{dx} + \dfrac{y^2 + y + 1}{x^2 + x + 1} = 0$ is given by $(x + y + 1) = \text{A}\,(1 - x - y - 2xy)$, where A is parameter.

8. Find the equation of the curve passing through the point $\left(0, \dfrac{\pi}{4}\right)$ whose differential equation is $\sin x \cos y\,dx + \cos x \sin y\,dy = 0$.

9. Find the particular solution of the differential equation $(1 + e^{2x})\,dy + (1 + y^2)\,e^x\,dx = 0$, given that $y = 1$ when $x = 0$.

10. Solve the differential equation $y\,e^{\frac{x}{y}}\,dx = \left(x\,e^{\frac{x}{y}} + y^2\right)dy\ (y \neq 0)$.

11. Find a particular solution of the differential equation $(x - y)\,(dx + dy) = dx - dy$, given that $y = -1$, when $x = 0$. (Hint: put $x - y = t$)

12. Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1\,(x \neq 0)$.

13. Find a particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$ $(x \neq 0)$, given that $y = 0$ when $x = \frac{\pi}{2}$.

14. Find a particular solution of the differential equation $(x + 1)\frac{dy}{dx} = 2\,e^{-y} - 1$, given that $y = 0$ when $x = 0$.

15. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20, 000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

16. The general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$ is

(A) $xy = C$ (B) $x = Cy^2$ (C) $y = Cx$ (D) $y = Cx^2$

17. The general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is

(A) $y\,e^{\int P_1\,dy} = \int\left(Q_1 e^{\int P_1\,dy}\right)dy + C$

(B) $y\,.\,e^{\int P_1\,dx} = \int\left(Q_1 e^{\int P_1\,dx}\right)dx + C$

(C) $x\,e^{\int P_1\,dy} = \int\left(Q_1 e^{\int P_1\,dy}\right)dy + C$

(D) $x\,e^{\int P_1\,dx} = \int\left(Q_1 e^{\int P_1\,dx}\right)dx + C$

18. The general solution of the differential equation $e^x\,dy + (y\,e^x + 2x)\,dx = 0$ is

(A) $x\,e^y + x^2 = C$ (B) $x\,e^y + y^2 = C$

(C) $y\,e^x + x^2 = C$ (D) $y\,e^y + x^2 = C$

Summary

- An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.
- Order of a differential equation is the order of the highest order derivative occurring in the differential equation.
- Degree of a differential equation is defined if it is a polynomial equation in its derivatives.
- Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
- A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.
- To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.
- Variable separable method is used to solve such an equation in which variables can be separated completely i.e. terms containing y should remain with dy and terms containing x should remain with dx.
- A differential equation which can be expressed in the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$ where, $f(x, y)$ and $g(x, y)$ are homogenous functions of degree zero is called a homogeneous differential equation.
- A differential equation of the form $\frac{dy}{dx} + \text{P}y = \text{Q}$, where P and Q are constants or functions of x only is called a first order linear differential equation.

Historical Note

One of the principal languages of Science is that of differential equations. Interestingly, the date of birth of differential equations is taken to be November, 11,1675, when Gottfried Wilthelm Freiherr Leibnitz (1646 - 1716) first put in black and white the identity $\int y\, dy = \frac{1}{2} y^2$, thereby introducing both the symbols $\int$ and dy.

Leibnitz was actually interested in the problem of finding a curve whose tangents were prescribed. This led him to discover the '*method of separation of variables*' 1691. A year later he formulated the '*method of solving the homogeneous differential equations of the first order*'. He went further in a very short time to the discovery of the '*method of solving a linear differential equation of the first-order*'. How surprising is it that all these methods came from a single man and that too within 25 years of the birth of differential equations!

In the old days, what we now call the 'solution' of a differential equation, was used to be referred to as 'integral' of the differential equation, the word being coined by James Bernoulli (1654 - 1705) in 1690. The word 'solution was first used by Joseph Louis Lagrange (1736 - 1813) in 1774, which was almost hundred years since the birth of differential equations. It was Jules Henri Poincare (1854 - 1912) who strongly advocated the use of the word 'solution' and thus the word 'solution' has found its deserved place in modern terminology. The name of the '*method of separation of variables*' is due to John Bernoulli (1667 - 1748), a younger brother of James Bernoulli.

Application to geometric problems were also considered. It was again John Bernoulli who first brought into light the intricate nature of differential equations. In a letter to Leibnitz, dated May 20, 1715, he revealed the solutions of the differential equation

$$x^2 y'' = 2y,$$

which led to three types of curves, viz., parabolas, hyperbolas and a class of cubic curves. This shows how varied the solutions of such innocent looking differential equation can be. From the second half of the twentieth century attention has been drawn to the investigation of this complicated nature of the solutions of differential equations, under the heading '*qualitative analysis of differential equations*'. Now-a-days, this has acquired prime importance being absolutely necessary in almost all investigations.

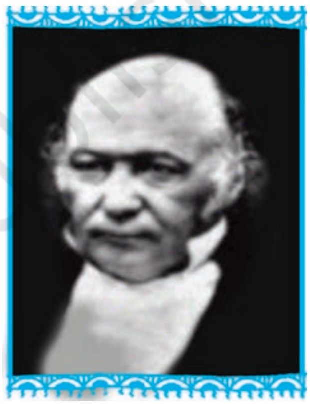
Chapter 10

VECTOR ALGEBRA

❖ *In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure. – HERMAN HANKEL* ❖

10.1 Introduction

In our day to day life, we come across many queries such as – What is your height? How should a football player hit the ball to give a pass to another player of his team? Observe that a possible answer to the first query may be 1.6 meters, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called ***scalars***. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called ***vectors.*** In mathematics, physics and engineering, we frequently come across with both types of quantities, namely, scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density, resistance etc. and vector quantities like displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc.



W.R. Hamilton (1805-1865)

In this chapter, we will study some of the basic concepts about vectors, various operations on vectors, and their algebraic and geometric properties. These two type of properties, when considered together give a full realisation to the concept of vectors, and lead to their vital applicability in various areas as mentioned above.

10.2 Some Basic Concepts

Let '*l*' be any straight line in plane or three dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a ***directed line*** (Fig 10.1 (i), (ii)).

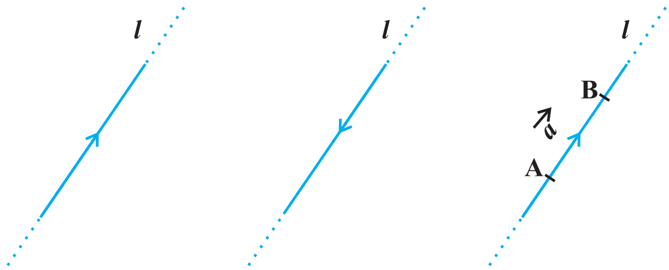


Fig 10.1

Now observe that if we restrict the line ***l*** to the line segment AB, then a magnitude is prescribed on the line ***l*** with one of the two directions, so that we obtain a ***directed line segment*** (Fig 10.1(iii)). Thus, a directed line segment has magnitude as well as direction.

Definition 1 A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector (Fig 10.1(iii)), denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

The point A from where the vector $\overrightarrow{AB}$ starts is called its ***initial point***, and the point B where it ends is called its ***terminal point***. The distance between initial and terminal points of a vector is called the ***magnitude*** (or length) of the vector, denoted as $|\overrightarrow{AB}|$, or $|\vec{a}|$, or ***a***. The arrow indicates the direction of the vector.

Note Since the length is never negative, the notation $|\vec{a}| < 0$ has no meaning.

Position Vector

From Class XI, recall the three dimensional right handed rectangular coordinate system (Fig 10.2(i)). Consider a point P in space, having coordinates (***x***, ***y***, ***z***) with respect to the origin O(0, 0, 0). Then, the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the ***position vector*** of the point P with respect to O. Using distance formula (from Class XI), the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

In practice, the position vectors of points A, B, C, etc., with respect to the origin O are denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$, etc., respectively (Fig 10.2 (ii)).

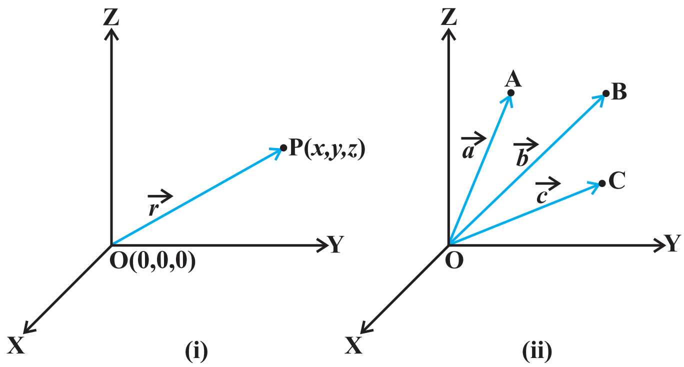


Fig 10.2

Direction Cosines

Consider the position vector $\overrightarrow{OP}\,(\text{or } \vec{r})$ of a point P($\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$) as in Fig 10.3. The angles α, β, γ made by the vector $\vec{r}$ with the positive directions of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$-axes respectively, are called its ***direction angles***. The cosine values of these angles, i.e., $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called ***direction cosines*** of the vector $\vec{r}$, and usually denoted by ***l***, ***m*** and ***n***, respectively.

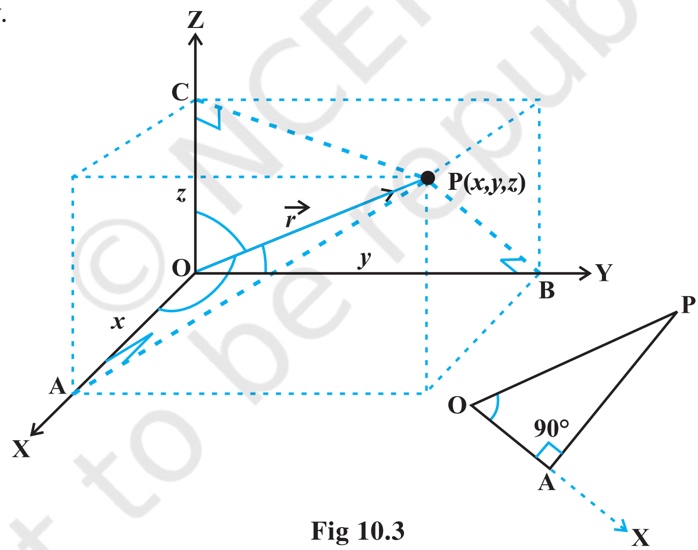


Fig 10.3

From Fig 10.3, one may note that the triangle OAP is right angled, and in it, we have $\cos\alpha = \dfrac{x}{r}$ (r stands for $|\vec{r}|$). Similarly, from the right angled triangles OBP and OCP, we may write $\cos\beta = \dfrac{\boldsymbol{y}}{\boldsymbol{r}}$ and $\cos\gamma = \dfrac{\boldsymbol{z}}{\boldsymbol{r}}$. Thus, the coordinates of the point P may also be expressed as (***lr***, ***mr***,***nr***). The numbers ***lr***, ***mr*** and ***nr***, proportional to the direction cosines are called as ***direction ratios*** of vector $\vec{r}$, and denoted as ***a***, ***b*** and ***c***, respectively.

Note One may note that $\boldsymbol{l}^2 + \boldsymbol{m}^2 + \boldsymbol{n}^2 = 1$ but $\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2 \neq 1$, in general.

10.3 Types of Vectors

Zero Vector A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{0}$. Zero vector can not be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vectors $\overrightarrow{AA}, \overrightarrow{BB}$ represent the zero vector,

Unit Vector A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{\boldsymbol{a}}$.

Coinitial Vectors Two or more vectors having the same initial point are called coinitial vectors.

Collinear Vectors Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

Equal Vectors Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.

Negative of a Vector A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but direction is opposite to that of it, is called ***negative*** of the given vector. For example, vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$, and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Remark The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called ***free vectors***. Throughout this chapter, we will be dealing with free vectors only.

Example 1 Represent graphically a displacement of 40 km, 30° west of south.

Solution The vector $\overrightarrow{OP}$ represents the required displacement (Fig 10.4).

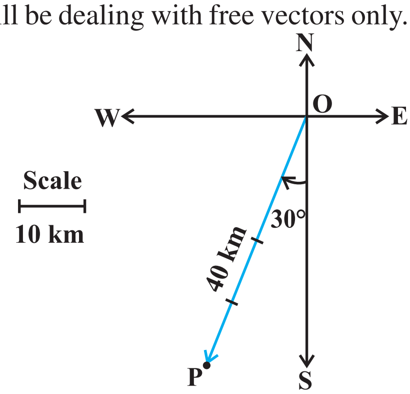


Fig 10.4

Example 2 Classify the following measures as scalars and vectors.

(i) 5 seconds

(ii) 1000 cm^3

(iii) 10 Newton (iv) 30 km/hr (v) 10 g/cm^3

(vi) 20 m/s towards north

Solution

(i) Time-scalar (ii) Volume-scalar (iii) Force-vector

(iv) Speed-scalar (v) Density-scalar (vi) Velocity-vector

Example 3 In Fig 10.5, which of the vectors are:

(i) Collinear (ii) Equal (iii) Coinitial

Solution

(i) Collinear vectors : $\vec{a}$, $\vec{c}$ and $\vec{d}$.

(ii) Equal vectors : $\vec{a}$ and $\vec{c}$.

(iii) Coinitial vectors : $\vec{b}$, $\vec{c}$ and $\vec{d}$.

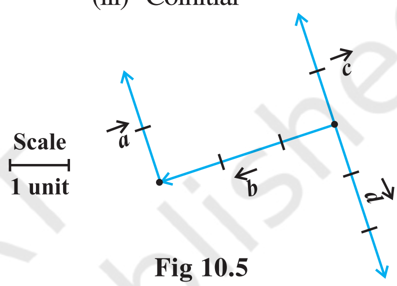


Fig 10.5

EXERCISE 10.1

1. Represent graphically a displacement of 40 km, 30° east of north.

2. Classify the following measures as scalars and vectors.

(i) 10 kg (ii) 2 meters north-west (iii) 40°

(iv) 40 watt (v) 10^{-19} coulomb (vi) 20 m/s^2

3. Classify the following as scalar and vector quantities.

(i) time period (ii) distance (iii) force

(iv) velocity (v) work done

4. In Fig 10.6 (a square), identify the following vectors.

(i) Coinitial (ii) Equal

(iii) Collinear but not equal

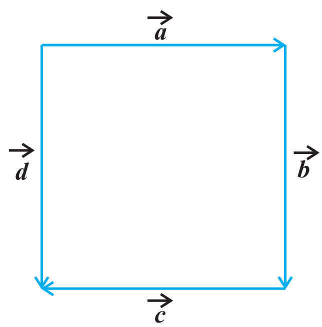


Fig 10.6

5. Answer the following as true or false.

(i) $\vec{a}$ and $-\vec{a}$ are collinear.

(ii) Two collinear vectors are always equal in magnitude.

(iii) Two vectors having same magnitude are collinear.

(iv) Two collinear vectors having the same magnitude are equal.

10.4 Addition of Vectors

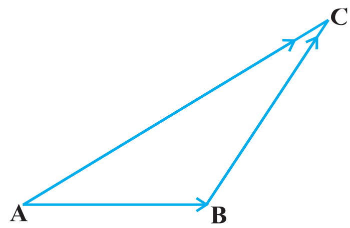


Fig 10.7

A vector $\overrightarrow{AB}$ simply means the displacement from a point A to the point B. Now consider a situation that a girl moves from A to B and then from B to C (Fig 10.7). The net displacement made by the girl from point A to the point C, is given by the vector $\overrightarrow{AC}$ and expressed as

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

This is known as the ***triangle law of vector addition.***

In general, if we have two vectors $\vec{a}$ and $\vec{b}$ (Fig 10.8 (i)), then to add them, they are positioned so that the initial point of one coincides with the terminal point of the other (Fig 10.8(ii)).

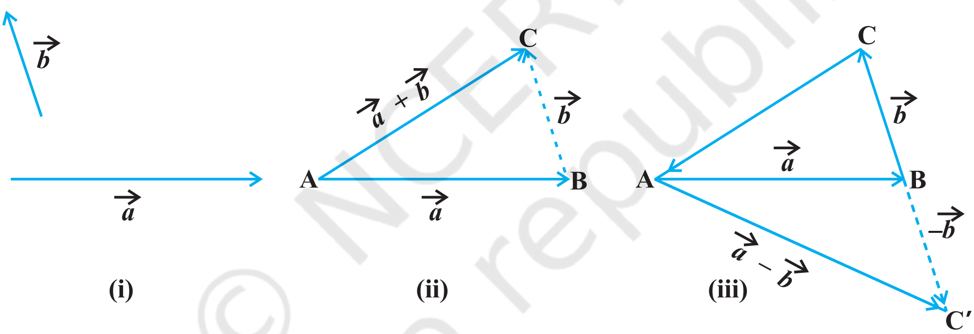


Fig 10.8

For example, in Fig 10.8 (ii), we have shifted vector $\vec{b}$ without changing its magnitude and direction, so that it's initial point coincides with the terminal point of $\vec{a}$. Then, the vector $\vec{a} + \vec{b}$, represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ i.e., in triangle ABC (Fig 10.8 (ii)), we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Now again, since $\overrightarrow{AC} = -\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided (Fig 10.8(iii)).

Now, construct a vector $\overrightarrow{BC'}$ so that its magnitude is same as the vector $\overrightarrow{BC}$, but the direction opposite to that of it (Fig 10.8 (iii)), i.e.,

$$\overrightarrow{BC'} = -\overrightarrow{BC}$$

Then, on applying triangle law from the Fig 10.8 (iii), we have

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

The vector $\overrightarrow{AC'}$ is said to represent the ***difference of*** $\vec{a}$ and $\vec{b}$.

Now, consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors–one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat in actual starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.

If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (Fig 10.9), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the ***parallelogram law of vector addition.***

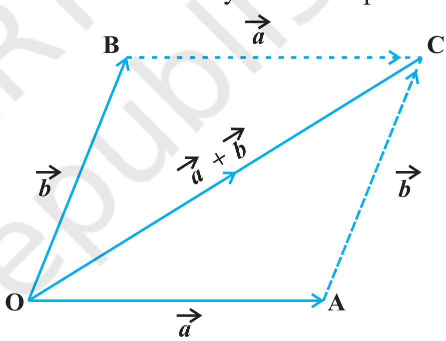


Fig 10.9

Note From Fig 10.9, using the triangle law, one may note that

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

or $$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC} \quad (\text{since } \overrightarrow{AC} = \overrightarrow{OB})$$

which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

Properties of vector addition

Property 1 For any two vectors $\vec{a}$ and $\vec{b}$,

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(Commutative property)}$$

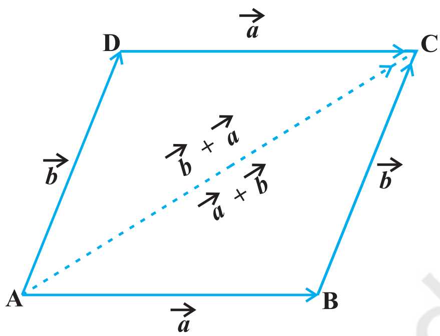


Fig 10.10

Proof Consider the parallelogram ABCD (Fig 10.10). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, then using the triangle law, from triangle ABC, we have

$$\overrightarrow{AC} = \vec{a} + \vec{b}$$

Now, since the opposite sides of a parallelogram are equal and parallel, from Fig 10.10, we have, $\overrightarrow{AD} = \overrightarrow{BC} = \vec{b}$ and $\overrightarrow{DC} = \overrightarrow{AB} = \vec{a}$. Again using triangle law, from triangle ADC, we have

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$$

Hence $\quad \vec{a} + \vec{b} = \vec{b} + \vec{a}$

Property 2 For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(Associative property)}$$

Proof Let the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$, respectively, as shown in Fig 10.11(i) and (ii).

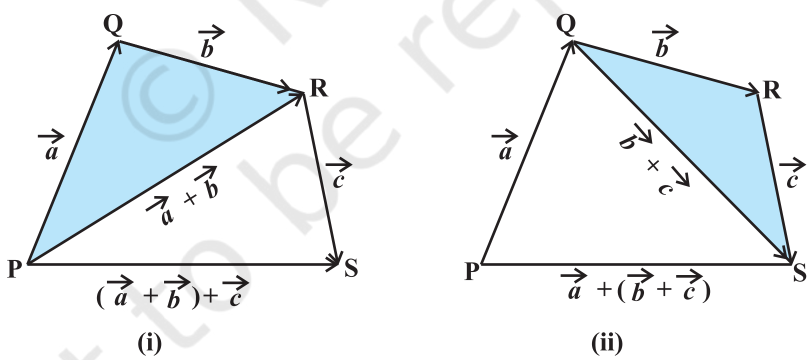


Fig 10.11

Then $\quad \vec{a} + \vec{b} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

and $\quad \vec{b} + \vec{c} = \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{QS}$

So $\quad (\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$

and $\qquad \vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$

Hence $\qquad (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

Remark The associative property of vector addition enables us to write the sum of three vectors $\vec{a}, \vec{b}, \vec{c}$ as $\vec{a} + \vec{b} + \vec{c}$ without using brackets.

Note that for any vector $\vec{a}$, we have

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

Here, the zero vector $\vec{0}$ is called the ***additive identity*** for the vector addition.

10.5 Multiplication of a Vector by a Scalar

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of the vector $\vec{a}$ by the scalar λ, denoted as $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ. Note that, $\lambda\vec{a}$ is also a vector, collinear to the vector $\vec{a}$. The vector $\lambda\vec{a}$ has the direction same (or opposite) to that of vector $\vec{a}$ according as the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of the vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualisation of multiplication of a vector by a scalar is given in Fig 10.12.

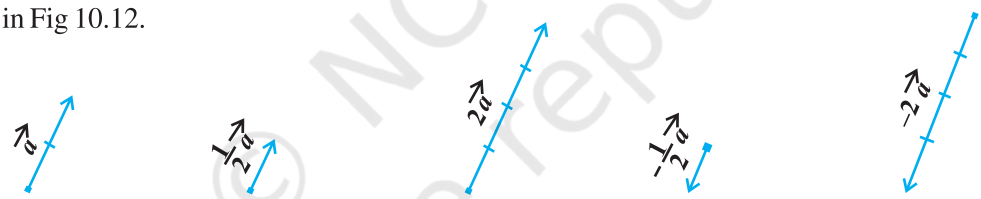


Fig 10.12

When $\lambda = -1$, then $\lambda\vec{a} = -\vec{a}$, which is a vector having magnitude equal to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$. The vector $-\vec{a}$ is called the ***negative*** (or ***additive inverse***) ***of vector*** $\vec{a}$ and we always have

$$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$$

Also, if $\lambda = \frac{1}{|\vec{a}|}$, provided $\vec{a} \neq 0$ i.e. $\vec{a}$ is not a null vector, then

$$|\lambda\vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

So, $\lambda \vec{a}$ ***represents the unit vector in the direction of*** $\vec{a}$. We write it as

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

Note For any scalar k, $k\vec{0} = \vec{0}$.

10.5.1 *Components of a vector*

Let us take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the x-axis, y-axis and z-axis, respectively. Then, clearly

$$|\overrightarrow{OA}| = 1, |\overrightarrow{OB}| = 1 \text{ and } |\overrightarrow{OC}| = 1$$

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called ***unit vectors along the axes*** OX, OY and OZ, respectively, and denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively (Fig 10.13).

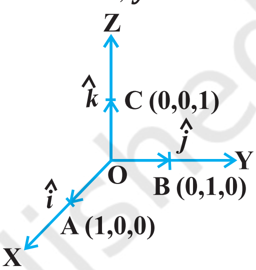


Fig 10.13

Now, consider the position vector $\overrightarrow{OP}$ of a point P(x, y, z) as in Fig 10.14. Let P_1 be the foot of the perpendicular from P on the plane XOY.

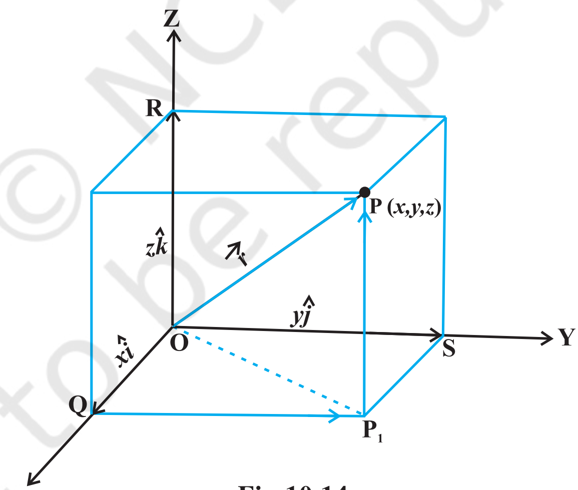


Fig 10.14

We, thus, see that P_1P is parallel to z-axis. As $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the x, y and z-axes, respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. Similarly, $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\hat{i} + y\hat{j}$

and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the position vector of P with reference to O is given by

$$\overrightarrow{OP} \text{ (or } \vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

This form of any vector is called its ***component form.*** Here, ***x***, ***y*** and ***z*** are called as the ***scalar components*** of $\vec{r}$, and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the ***vector components*** of $\vec{r}$ along the respective axes. Sometimes ***x***, ***y*** and ***z*** are also termed as ***rectangular components***.

The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, is readily determined by applying the Pythagoras theorem twice. We note that in the right angle triangle OQP_1 (Fig 10.14)

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QP_1}|^2} = \sqrt{x^2 + y^2},$$

and in the right angle triangle OP_1P, we have

$$\overrightarrow{OP} = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Hence, the length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, respectively, then

(i) the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(ii) the difference of the vector $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(iii) the vectors $\vec{a}$ and $\vec{b}$ are equal if and only if

$$a_1 = b_1,\ a_2 = b_2 \quad \text{and} \quad a_3 = b_3$$

(iv) the multiplication of vector $\vec{a}$ by any scalar λ is given by

$$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let $\vec{a}$ and $\vec{b}$ be any two vectors, and ***k*** and ***m*** be any scalars. Then

(i) $k\vec{a} + m\vec{a} = (k+m)\vec{a}$

(ii) $k(m\vec{a}) = (km)\vec{a}$

(iii) $k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$

Remarks

(i) One may observe that whatever be the value of λ, the vector $\lambda\vec{a}$ is always collinear to the vector $\vec{a}$. In fact, two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a nonzero scalar λ such that $\vec{b} = \lambda\vec{a}$. If the vectors $\vec{a}$ and $\vec{b}$ are given in the component form, i.e. $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then the two vectors are collinear if and only if

$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\Leftrightarrow \quad b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

$$\Leftrightarrow \quad b_1 = \lambda a_1,\ b_2 = \lambda a_2,\ b_3 = \lambda a_3$$

$$\Leftrightarrow \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then a_1, a_2, a_3 are also called direction ratios of $\vec{a}$.

(iii) In case if it is given that ***l***, ***m***, ***n*** are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k}$ $= (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector, where α, β and γ are the angles which the vector makes with ***x***, ***y*** and ***z*** axes respectively.

Example 4 Find the values of ***x***, ***y*** and ***z*** so that the vectors $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal.

Solution Note that two vectors are equal if and only if their corresponding components are equal. Thus, the given vectors $\vec{a}$ and $\vec{b}$ will be equal if and only if

$$x = 2,\ y = 2,\ z = 1$$

Example 5 Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal?

Solution We have $|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $|\vec{b}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

So, $|\vec{a}| = |\vec{b}|$. But, the two vectors are not equal since their corresponding components are distinct.

Example 6 Find unit vector in the direction of vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

Solution The unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$.

Now $$|\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

Therefore $$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{14}}(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = \frac{2}{\sqrt{14}}\hat{\boldsymbol{i}} + \frac{3}{\sqrt{14}}\hat{\boldsymbol{j}} + \frac{1}{\sqrt{14}}\hat{\boldsymbol{k}}$$

Example 7 Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

Solution The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{5}}(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}) = \frac{1}{\sqrt{5}}\hat{\boldsymbol{i}} - \frac{2}{\sqrt{5}}\hat{\boldsymbol{j}}$$

Therefore, the vector having magnitude equal to 7 and in the direction of $\vec{\boldsymbol{a}}$ is

$$7\hat{\boldsymbol{a}} = 7\left(\frac{1}{\sqrt{5}}\hat{\boldsymbol{i}} - \frac{2}{\sqrt{5}}\hat{\boldsymbol{j}}\right) = \frac{7}{\sqrt{5}}\hat{\boldsymbol{i}} - \frac{14}{\sqrt{5}}\hat{\boldsymbol{j}}$$

Example 8 Find the unit vector in the direction of the sum of the vectors, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$.

Solution The sum of the given vectors is

$$\vec{a} + \vec{b}\ (= \vec{c}, \text{say}) = 4\hat{i} + 3\hat{j} - 2\hat{k}$$

and $$|\vec{c}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$$

Thus, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{29}}(4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}) = \frac{4}{\sqrt{29}}\hat{\boldsymbol{i}} + \frac{3}{\sqrt{29}}\hat{\boldsymbol{j}} - \frac{2}{\sqrt{29}}\hat{\boldsymbol{k}}$$

Example 9 Write the direction ratio's of the vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and hence calculate its direction cosines.

Solution Note that the direction ratio's $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are just the respective components $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ of the vector. So, for the given vector, we have $\boldsymbol{a} = 1$, $\boldsymbol{b} = 1$ and $\boldsymbol{c} = -2$. Further, if $\boldsymbol{l}$, $\boldsymbol{m}$ and $\boldsymbol{n}$ are the direction cosines of the given vector, then

$$l = \frac{a}{|\vec{r}|} = \frac{1}{\sqrt{6}}, \quad m = \frac{b}{|\vec{r}|} = \frac{1}{\sqrt{6}}, \quad n = \frac{c}{|\vec{r}|} = \frac{-2}{\sqrt{6}} \text{ as } |\vec{r}| = \sqrt{6}$$

Thus, the direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

10.5.2 *Vector joining two points*

If $P_1(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}_1)$ and $P_2(\boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{z}_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$ (Fig 10.15).

Joining the points P_1 and P_2 with the origin O, and applying triangle law, from the triangle OP_1P_2, we have

$$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$$

Using the properties of vector addition, the above equation becomes

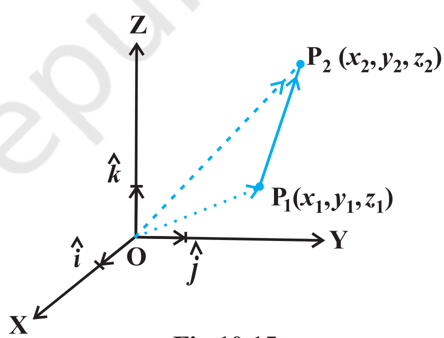


Fig 10.15

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

i.e.

$$\overrightarrow{P_1P_2} = (\boldsymbol{x}_2\hat{\boldsymbol{i}} + \boldsymbol{y}_2\hat{\boldsymbol{j}} + \boldsymbol{z}_2\hat{\boldsymbol{k}}) - (\boldsymbol{x}_1\hat{\boldsymbol{i}} + \boldsymbol{y}_1\hat{\boldsymbol{j}} + \boldsymbol{z}_1\hat{\boldsymbol{k}})$$

$$= (\boldsymbol{x}_2 - \boldsymbol{x}_1)\hat{\boldsymbol{i}} + (\boldsymbol{y}_2 - \boldsymbol{y}_1)\hat{\boldsymbol{j}} + (\boldsymbol{z}_2 - \boldsymbol{z}_1)\hat{\boldsymbol{k}}$$

The magnitude of vector $\overrightarrow{P_1P_2}$ is given by

$$|\overrightarrow{P_1P_2}| = \sqrt{(\boldsymbol{x}_2 - \boldsymbol{x}_1)^2 + (\boldsymbol{y}_2 - \boldsymbol{y}_1)^2 + (\boldsymbol{z}_2 - \boldsymbol{z}_1)^2}$$

Example 10 Find the vector joining the points P(2, 3, 0) and Q(– 1, – 2, – 4) directed from P to Q.

Solution Since the vector is to be directed from P to Q, clearly P is the initial point and Q is the terminal point. So, the required vector joining P and Q is the vector $\overrightarrow{PQ}$, given by

$$\overrightarrow{PQ} = (-1-2)\hat{i} + (-2-3)\hat{j} + (-4-0)\hat{k}$$

i.e. $$\overrightarrow{PQ} = -3\hat{i} - 5\hat{j} - 4\hat{k}.$$

10.5.3 *Section formula*

Let P and Q be two points represented by the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, respectively, with respect to the origin O. Then the line segment joining the points P and Q may be divided by a third point, say R, in two ways – internally (Fig 10.16) and externally (Fig 10.17). Here, we intend to find the position vector $\overrightarrow{OR}$ for the point R with respect to the origin O. We take the two cases one by one.

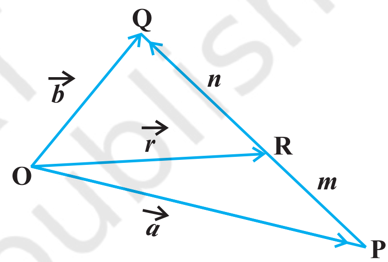


Fig 10.16

Case I When R divides PQ internally (Fig 10.16).

If R divides $\overrightarrow{PQ}$ such that $m\,\overrightarrow{RQ} = n\,\overrightarrow{PR}$,

where ***m*** and ***n*** are positive scalars, we say that the point R divides $\overrightarrow{PQ}$ internally in the ratio of ***m*** : ***n***. Now from triangles ORQ and OPR, we have

$$\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR} = \vec{b} - \vec{r}$$

and $$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \vec{r} - \vec{a},$$

Therefore, we have $$m(\vec{b} - \vec{r}) = n(\vec{r} - \vec{a}) \quad \text{(Why?)}$$

or $$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n} \quad \text{(on simplification)}$$

Hence, the position vector of the point R which divides P and Q internally in the ratio of ***m*** : ***n*** is given by

$$\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Case II When R divides PQ externally (Fig 10.17). We leave it to the reader as an exercise to verify that the position vector of the point R which divides the line segment PQ externally in the ratio $\boldsymbol{m} : \boldsymbol{n}$ i.e. $\dfrac{\text{PR}}{\text{QR}} = \dfrac{\boldsymbol{m}}{\boldsymbol{n}}$ is given by

$$\overrightarrow{\text{OR}} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

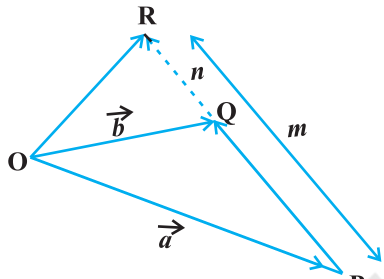


Fig 10.17

Remark If R is the midpoint of PQ , then $\boldsymbol{m} = \boldsymbol{n}$. And therefore, from Case I, the midpoint R of $\overrightarrow{\text{PQ}}$, will have its position vector as

$$\overrightarrow{\text{OR}} = \frac{\vec{a} + \vec{b}}{2}$$

Example 11 Consider two points P and Q with position vectors $\overrightarrow{\text{OP}} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{\text{OQ}} = \vec{a} + \vec{b}$. Find the position vector of a point R which divides the line joining P and Q in the ratio 2:1, (i) internally, and (ii) externally.

Solution

(i) The position vector of the point R dividing the join of P and Q internally in the ratio 2:1 is

$$\overrightarrow{\text{OR}} = \frac{2(\vec{a} + \vec{b}) + (3\vec{a} - 2\vec{b})}{2 + 1} = \frac{5\vec{a}}{3}$$

(ii) The position vector of the point R dividing the join of P and Q externally in the ratio 2:1 is

$$\overrightarrow{\text{OR}} = \frac{2(\vec{a} + \vec{b}) - (3\vec{a} - 2\vec{b})}{2 - 1} = 4\vec{b} - \vec{a}$$

Example 12 Show that the points $\text{A}(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$, $\text{B}(\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}})$, $\text{C}(3\hat{\boldsymbol{i}} - 4\boldsymbol{j} - 4\hat{\boldsymbol{k}})$ are the vertices of a right angled triangle.

Solution We have

$$\overrightarrow{\text{AB}} = (1-2)\hat{\boldsymbol{i}} + (-3+1)\hat{\boldsymbol{j}} + (-5-1)\hat{\boldsymbol{k}} = -\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$$

$$\overrightarrow{\text{BC}} = (3-1)\hat{\boldsymbol{i}} + (-4+3)\hat{\boldsymbol{j}} + (-4+5)\hat{\boldsymbol{k}} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$$

and

$$\overrightarrow{\text{CA}} = (2-3)\hat{\boldsymbol{i}} + (-1+4)\hat{\boldsymbol{j}} + (1+4)\hat{\boldsymbol{k}} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$$

Further, note that

$$|\overrightarrow{AB}|^2 = 41 = 6 + 35 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$$

Hence, the triangle is a right angled triangle.

EXERCISE 10.2

1. Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + k; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \quad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

2. Write two different vectors having same magnitude.

3. Write two different vectors having same direction.

4. Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.

5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (– 5, 7).

6. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

7. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.

9. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.

10. Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.

11. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

12. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

13. Find the direction cosines of the vector joining the points A (1, 2, –3) and B(–1, –2, 1), directed from A to B.

14. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

15. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1

(i) internally (ii) externally

16. Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

17. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right angled triangle.

18. In triangle ABC (Fig 10.18), which of the following is not true:

(A) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(B) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(C) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(D) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

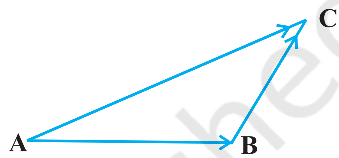


Fig 10.18

19. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

(A) $\vec{b} = \lambda\vec{a}$, for some scalar λ

(B) $\vec{a} = \pm\vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are not proportional

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes.

10.6 Product of Two Vectors

So far we have studied about addition and subtraction of vectors. An other algebraic operation which we intend to discuss regarding vectors is their product. We may recall that product of two numbers is a number, product of two matrices is again a matrix. But in case of functions, we may multiply them in two ways, namely, multiplication of two functions pointwise and composition of two functions. Similarly, multiplication of two vectors is also defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is a vector. Based upon these two types of products for vectors, they have found various applications in geometry, mechanics and engineering. In this section, we will discuss these two types of products.

10.6.1 *Scalar (or dot) product of two vectors*

Definition 2 The scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as $$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta,$$

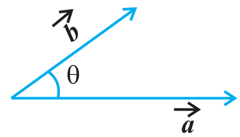


Fig 10.19

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ (Fig 10.19).

If either $\vec{a} = 0$ or $\vec{b} = 0$ then θ is not defined, and in this case, we define $\vec{a} \cdot \vec{b} = 0$

Observations

1. $\vec{a} \cdot \vec{b}$ is a real number.
2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors, then $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other. i.e.
   $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
3. If θ = 0, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$

   In particular, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, as θ in this case is 0.
4. If θ = π, then $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = -|\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|$

   In particular, $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$, as θ in this case is π.
5. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$, we have
   $$\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1,$$
   $$\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{i}} = 0$$
6. The angle between two nonzero vectors $\vec{a}$ and $\vec{b}$ is given by
   $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}, \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$
7. The scalar product is commutative. i.e.
   $$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad \text{(Why?)}$$

Two important properties of scalar product

Property 1 (Distributivity of scalar product over addition) Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Property 2 Let $\vec{a}$ and $\vec{b}$ be any two vectors, and l be any scalar. Then

$$(\lambda\vec{a})\cdot\vec{b} = (\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$$

If two vectors $\vec{a}$ and $\vec{b}$ are given in component form as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then their scalar product is given as

$$\begin{aligned}
\vec{a}\cdot\vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\
&= a_1\hat{i}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\
&= a_1b_1(\hat{i}\cdot\hat{i}) + a_1b_2(\hat{i}\cdot\hat{j}) + a_1b_3(\hat{i}\cdot\hat{k}) + a_2b_1(\hat{j}\cdot\hat{i}) + a_2b_2(\hat{j}\cdot\hat{j}) + a_2b_3(\hat{j}\cdot\hat{k}) \\
&\quad + a_3b_1(\hat{k}\cdot\hat{i}) + a_3b_2(\hat{k}\cdot\hat{j}) + a_3b_3(\hat{k}\cdot\hat{k}) \quad \text{(Using the above Properties 1 and 2)} \\
&= a_1b_1 + a_2b_2 + a_3b_3 \quad \text{(Using Observation 5)}
\end{aligned}$$

Thus $\quad \vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

10.6.2 *Projection of a vector on a line*

Suppose a vector $\overrightarrow{AB}$ makes an angle θ with a given directed line ***l*** (say), in the ***anticlockwise direction*** (Fig 10.20). Then the projection of $\overrightarrow{AB}$ on ***l*** is a vector $\vec{p}$ (say) with magnitude $|\overrightarrow{AB}|\,|\cos\theta|$, and the direction of $\vec{p}$ being the same (or opposite) to that of the line ***l***, depending upon whether cos θ is positive or negative. The vector $\vec{p}$

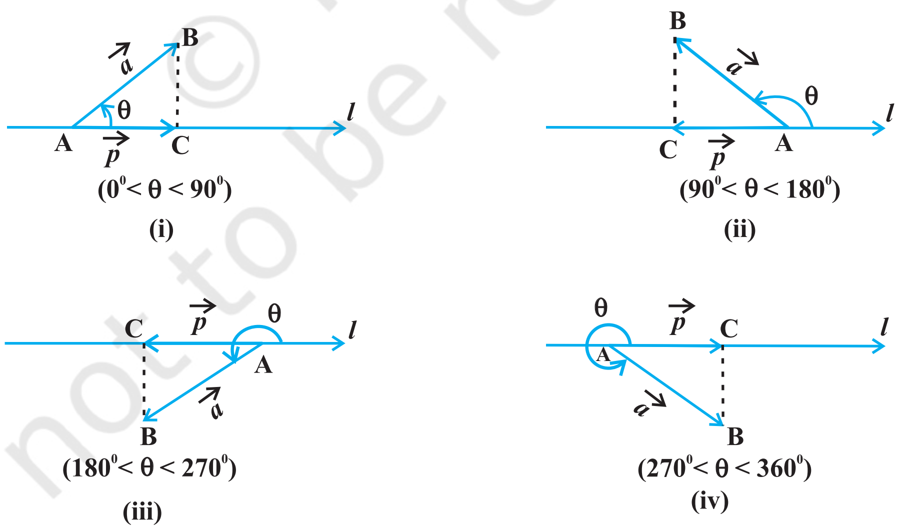


Fig 10.20

is called the ***projection vector***, and its magnitude $|\vec{p}|$ is simply called as the ***projection*** of the vector $\overrightarrow{AB}$ on the directed line $\boldsymbol{l}$.

For example, in each of the following figures (Fig 10.20 (i) to (iv)), projection vector of $\overrightarrow{AB}$ along the line $\boldsymbol{l}$ is vector $\overrightarrow{AC}$.

Observations

1. If $\hat{\boldsymbol{p}}$ is the unit vector along a line $\boldsymbol{l}$, then the projection of a vector $\vec{a}$ on the line $\boldsymbol{l}$ is given by $\vec{a}\cdot\hat{\boldsymbol{p}}$.
2. Projection of a vector $\vec{a}$ on other vector $\vec{\boldsymbol{b}}$, is given by

$$\vec{a}\cdot\hat{\boldsymbol{b}}, \quad \text{or} \quad \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right), \quad \text{or} \quad \frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$$

3. If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.
4. If $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

Remark If α, β and γ are the direction angles of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its direction cosines may be given as

$$\cos\alpha = \frac{\vec{a}\cdot\hat{i}}{|\vec{a}||\hat{i}|} = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|}, \quad \text{and} \quad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

Also, note that $|\vec{a}|\cos\alpha$, $|\vec{a}|\cos\beta$ and $|\vec{a}|\cos\gamma$ are respectively the projections of $\vec{a}$ along OX, OY and OZ. i.e., the scalar components $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ of the vector $\vec{a}$, are precisely the projections of $\vec{a}$ along $\boldsymbol{x}$-axis, $\boldsymbol{y}$-axis and $\boldsymbol{z}$-axis, respectively. Further, if $\vec{a}$ is a unit vector, then it may be expressed in terms of its direction cosines as

$$\vec{a} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$$

Example 13 Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $\vec{a}\cdot\vec{b} = 1$.

Solution Given $\vec{a}\cdot\vec{b} = 1, |\vec{a}| = 1$ and $|\vec{b}| = 2$. We have

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Example 14 Find angle 'θ' between the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Solution The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Now $\quad \vec{a}\cdot\vec{b} = (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})\cdot(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 1 - 1 - 1 = -1.$

Therefore, we have $\quad \cos\theta = \frac{-1}{3}$

hence the required angle is $\quad \theta = \cos^{-1}\left(-\frac{1}{3}\right)$

Example 15 If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.

Solution We know that two nonzero vectors are perpendicular if their scalar product is zero.

Here $\quad \vec{a} + \vec{b} = (5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) + (\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}) = 6\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 8\hat{\boldsymbol{k}}$

and $\quad \vec{a} - \vec{b} = (5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) - (\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}) = 4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

So $\quad (\vec{a} + \vec{b})\,.\,(\vec{a} - \vec{b}) = (6\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 8\hat{\boldsymbol{k}})\cdot(4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = 24 - 8 - 16 = 0.$

Hence $\quad \vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors.

Example 16 Find the projection of the vector $\vec{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ on the vector $\vec{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

Solution The projection of vector $\vec{\boldsymbol{a}}$ on the vector $\vec{\boldsymbol{b}}$ is given by

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b}) = \frac{(2\times1 + 3\times2 + 2\times1)}{\sqrt{(1)^2 + (2)^2 + (1)^2}} = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}$$

Example 17 Find $|\vec{a} - \vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 4$.

Solution We have

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$$

$$= \vec{a}.\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 - 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$$

$$= (2)^2 - 2(4) + (3)^2$$

Therefore $\qquad |\vec{a}-\vec{b}| = \sqrt{5}$

Example 18 If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$, then find $|\vec{x}|$.

Solution Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$. Also,

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$$

or $\qquad \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{a} - \vec{a}\cdot\vec{x} - \vec{a}\cdot\vec{a} = 8$

or $\qquad |\vec{x}|^2 - 1 = 8$ i.e. $|\vec{x}|^2 = 9$

Therefore $\qquad |\vec{x}| = 3$ (as magnitude of a vector is non negative).

Example 19 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$ (Cauchy-Schwartz inequality).

Solution The inequality holds trivially when either $\vec{\boldsymbol{a}} = \vec{0}$ or $\vec{\boldsymbol{b}} = \vec{0}$. Actually, in such a situation we have $|\vec{a}\cdot\vec{b}| = 0 = |\vec{a}||\vec{b}|$. So, let us assume that $|\vec{a}| \ne 0 \ne |\vec{b}|$.
Then, we have

$$\frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \le 1$$

Therefore $\qquad |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$

Example 20 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ (triangle inequality).

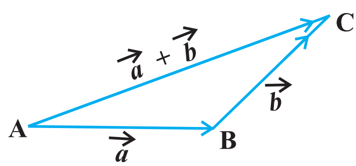


Fig 10.21

Solution The inequality holds trivially in case either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ (How?). So, let $|\vec{a}| \ne \vec{0} \ne |\vec{b}|$. Then,

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \qquad \text{(scalar product is commutative)}$$

$$\le |\vec{a}|^2 + 2|\vec{a}\cdot\vec{b}| + |\vec{b}|^2 \qquad (\text{since } \boldsymbol{x} \le |\boldsymbol{x}|\ \forall \boldsymbol{x} \in \mathbf{R})$$

$$\le |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 \qquad \text{(from Example 19)}$$

$$= (|\vec{a}| + |\vec{b}|)^2$$

Hence $$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$$

Remark If the equality holds in triangle inequality (in the above Example 20), i.e.

$$|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|,$$

then $$|\overrightarrow{AC}| = |\overrightarrow{AB}|+|\overrightarrow{BC}|$$

showing that the points A, B and C are collinear.

Example 21 Show that the points $A(-2\hat{i}+3\hat{j}+5\hat{k})$, $B(\hat{i}+2\hat{j}+3\hat{k})$ and $C(7\hat{i}-\hat{k})$ are collinear.

Solution We have

$$\overrightarrow{AB} = (1+2)\hat{i}+(2-3)\hat{j}+(3-5)\hat{k} = 3\hat{i}-\hat{j}-2\hat{k},$$

$$\overrightarrow{BC} = (7-1)\hat{i}+(0-2)\hat{j}+(-1-3)\hat{k} = 6\hat{i}-2\hat{j}-4\hat{k},$$

$$\overrightarrow{AC} = (7+2)\hat{i}+(0-3)\hat{j}+(-1-5)\hat{k} = 9\hat{i}-3\hat{j}-6\hat{k}$$

$$|\overrightarrow{AB}| = \sqrt{14},\ |\overrightarrow{BC}| = 2\sqrt{14} \text{ and } |\overrightarrow{AC}| = 3\sqrt{14}$$

Therefore $$|\overrightarrow{AC}| = |\overrightarrow{AB}|+|\overrightarrow{BC}|$$

Hence the points A, B and C are collinear.

Note In Example 21, one may note that although $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ but the points A, B and C do not form the vertices of a triangle.

EXERCISE 10.3

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$.
2. Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$
3. Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.
4. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.
5. Show that each of the given three vectors is a unit vector:

   $$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}),\ \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}),\ \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$$

   Also, show that they are mutually perpendicular to each other.

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$.

7. Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$.

10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$.

12. If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?

13. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

14. If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$].

16. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, –1) are collinear.

17. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

18. If $\vec{a}$ is a nonzero vector of magnitude '*a*' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = 1/|\lambda|$

10.6.3 *Vector (or cross) product of two vectors*

In Section 10.2, we have discussed on the three dimensional right handed rectangular coordinate system. In this system, when the positive ***x***-axis is rotated counterclockwise into the positive ***y***-axis, a right handed (standard) screw would advance in the direction of the positive ***z***-axis (Fig 10.22(i)).

In a right handed coordinate system, the thumb of the right hand points in the direction of the positive ***z***-axis when the fingers are curled in the direction away from the positive ***x***-axis toward the positive ***y***-axis (Fig 10.22(ii)).

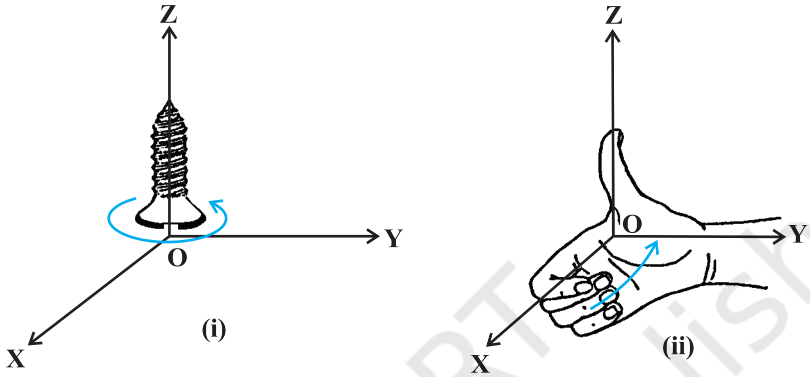


Fig 10.22 (i), (ii)

Definition 3 The vector product of two nonzero vectors $\vec{a}$ and $\vec{b}$, is denoted by $\vec{a} \times \vec{b}$ and defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n},$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system (Fig 10.23). i.e., the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{\boldsymbol{n}}$.

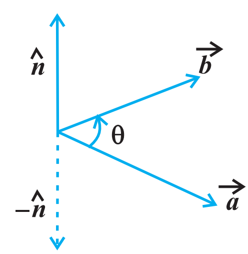


Fig 10.23

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined and in this case, we define $\vec{a} \times \vec{b} = \vec{0}$.

Observations

1. $\vec{a} \times \vec{b}$ is a vector.
2. Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are parallel (or collinear) to each other, i.e.,

$$\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$$

In particular, $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{a} \times (-\vec{a}) = \vec{0}$, since in the first situation, θ = 0 and in the second one, θ = π, making the value of sin θ to be 0.

3. If $\theta = \dfrac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$.

4. In view of the Observations 2 and 3, for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ (Fig 10.24), we have

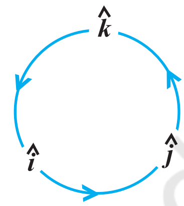


Fig 10.24

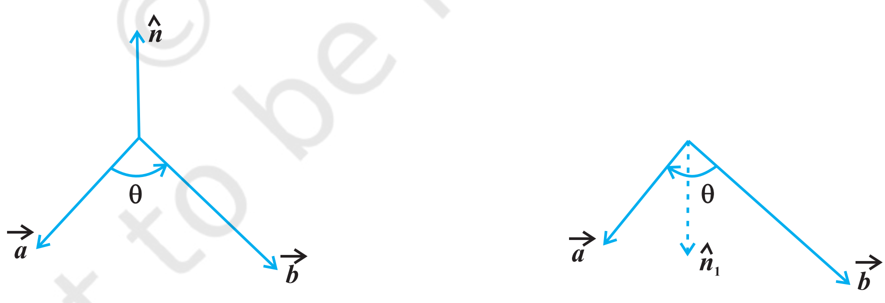
$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

5. In terms of vector product, the angle between two vectors $\vec{a}$ and $\vec{b}$ may be given as

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

6. It is always true that the vector product is not commutative, as $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. Indeed, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system, i.e., θ is traversed from $\vec{a}$ to $\vec{b}$, Fig 10.25 (i). While, $\vec{b} \times \vec{a} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}_1$, where $\vec{b}, \vec{a}$ and $\hat{n}_1$ form a right handed system i.e. θ is traversed from $\vec{b}$ to $\vec{a}$, Fig 10.25(ii).



(i) (ii)

Fig 10.25 (i), (ii)

Thus, if we assume $\vec{a}$ and $\vec{b}$ to lie in the plane of the paper, then $\hat{n}$ and $\hat{n}_1$ both will be perpendicular to the plane of the paper. But, $\hat{n}$ being directed above the paper while $\hat{n}_1$ directed below the paper. i.e. $\hat{n}_1 = -\hat{n}$.

Hence $\qquad \vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$$= -|\vec{a}||\vec{b}|\sin\theta\hat{n}_1 = -\vec{b}\times\vec{a}$$

7. In view of the Observations 4 and 6, we have

$$\hat{\boldsymbol{j}}\times\hat{\boldsymbol{i}} = -\hat{\boldsymbol{k}},\quad \hat{\boldsymbol{k}}\times\hat{\boldsymbol{j}} = -\hat{\boldsymbol{i}} \text{ and } \hat{\boldsymbol{i}}\times\hat{\boldsymbol{k}} = -\hat{\boldsymbol{j}}.$$

8. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is given as $\frac{1}{2}|\vec{a}\times\vec{b}|$.

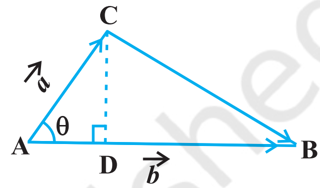


Fig 10.26

By definition of the area of a triangle, we have from Fig 10.26,

Area of triangle ABC $= \dfrac{1}{2}\text{AB}\cdot\text{CD}.$

But $\text{AB} = |\vec{b}|$ (as given), and $\text{CD} = |\vec{a}|\sin\theta$.

Thus, Area of triangle ABC $= \dfrac{1}{2}|\vec{b}||\vec{a}|\sin\theta = \dfrac{1}{2}|\vec{a}\times\vec{b}|.$

9. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a}\times\vec{b}|$.

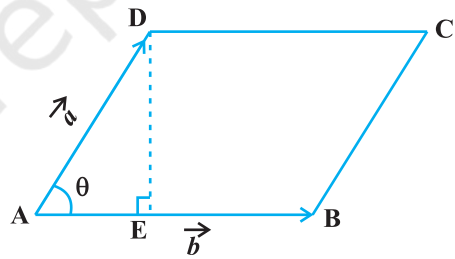


Fig 10.27

From Fig 10.27, we have

Area of parallelogram ABCD = AB. DE.

But $\text{AB} = |\vec{b}|$ (as given), and

$\text{DE} = |\vec{a}|\sin\theta$.

Thus,

Area of parallelogram ABCD $= |\vec{b}||\vec{a}|\sin\theta = |\vec{a}\times\vec{b}|.$

We now state two important properties of vector product.

Property 3 (Distributivity of vector product over addition): If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors and λ be a scalar, then

(i) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

(ii) $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$

Let $\vec{a}$ and $\vec{b}$ be two vectors given in component form as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, respectively. Then their cross product may be given by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Explanation We have

$$\begin{aligned}
\vec{a}\times\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \\
&= a_1b_1(\hat{i}\times\hat{i})+a_1b_2(\hat{i}\times\hat{j})+a_1b_3(\hat{i}\times\hat{k})+a_2b_1(\hat{j}\times\hat{i}) \\
&\quad + a_2b_2(\hat{j}\times\hat{j})+a_2b_3(\hat{j}\times\hat{k}) \\
&\quad + a_3b_1(\hat{k}\times\hat{i})+a_3b_2(\hat{k}\times\hat{j})+a_3b_3(\hat{k}\times\hat{k}) \qquad \text{(by Property 1)} \\
&= a_1b_2(\hat{i}\times\hat{j})-a_1b_3(\hat{k}\times\hat{i})-a_2b_1(\hat{i}\times\hat{j}) \\
&\quad + a_2b_3(\hat{j}\times\hat{k})+a_3b_1(\hat{k}\times\hat{i})-a_3b_2(\hat{j}\times\hat{k})
\end{aligned}$$

(as $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ and $\hat{i}\times\hat{k}=-\hat{k}\times\hat{i}$, $\hat{j}\times\hat{i}=-\hat{i}\times\hat{j}$ and $\hat{k}\times\hat{j}=-\hat{j}\times\hat{k}$)

$$= a_1b_2\hat{k}-a_1b_3\hat{j}-a_2b_1\hat{k}+a_2b_3\hat{i}+a_3b_1\hat{j}-a_3b_2\hat{i}$$

(as $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\times\hat{i}=\hat{j}$)

$$= (a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 22 Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$

Solution We have

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$$= \hat{i}(-2-15)-(-4-9)\hat{j}+(10-3)\hat{k} = -17\hat{i}+13\hat{j}+7\hat{k}$$

Hence $\quad |\vec{a}\times\vec{b}| = \sqrt{(-17)^2+(13)^2+(7)^2} = \sqrt{507}$

Example 23 Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.

Solution We have $\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{a}-\vec{b}=-\hat{j}-2\hat{k}$

A vector which is perpendicular to both $\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{a}}-\vec{\boldsymbol{b}}$ is given by

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i}+4\hat{j}-2\hat{k} \quad (=\vec{c},\ \text{say})$$

Now $$|\vec{c}| = \sqrt{4+16+4}=\sqrt{24}=2\sqrt{6}$$

Therefore, the required unit vector is

$$\frac{\vec{\boldsymbol{c}}}{|\vec{\boldsymbol{c}}|} = \frac{-1}{\sqrt{6}}\hat{\boldsymbol{i}}+\frac{2}{\sqrt{6}}\hat{\boldsymbol{j}}-\frac{1}{\sqrt{6}}\hat{\boldsymbol{k}}$$

Note There are two perpendicular directions to any plane. Thus, another unit vector perpendicular to $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ will be $\frac{1}{\sqrt{6}}\hat{\boldsymbol{i}}-\frac{2}{\sqrt{6}}\hat{\boldsymbol{j}}+\frac{1}{\sqrt{6}}\hat{\boldsymbol{k}}$. But that will be a consequence of $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$.

Example 24 Find the area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

Solution We have $\overrightarrow{AB}=\hat{j}+2\hat{k}$ and $\overrightarrow{AC}=\hat{i}+2\hat{j}$. The area of the given triangle is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$.

Now, $$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}}$$

Therefore $$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{16+4+1}=\sqrt{21}$$

Thus, the required area is $\frac{1}{2}\sqrt{21}$

Example 25 Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Solution The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a} \times \vec{b}|$.

Now
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

Therefore
$$|\vec{a} \times \vec{b}| = \sqrt{25+1+16} = \sqrt{42}$$

and hence, the required area is $\sqrt{42}$.

EXERCISE 10.4

1. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.
2. Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.
3. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{\mathbf{i}}$, $\frac{\pi}{4}$ with $\hat{\mathbf{j}}$ and an acute angle θ with $\hat{\mathbf{k}}$, then find θ and hence, the components of $\vec{a}$.
4. Show that
$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$$
5. Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.
6. Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?
7. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, $c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$. Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
8. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.
9. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{i}} - \frac{1}{2}\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $-\hat{\boldsymbol{i}} - \frac{1}{2}\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$, respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Miscellaneous Examples

Example 26 Write all the unit vectors in XY-plane.

Solution Let $\vec{r} = x\hat{i} + y\hat{j}$ be a unit vector in XY-plane (Fig 10.28). Then, from the figure, we have $\boldsymbol{x} = \cos\theta$ and $\boldsymbol{y} = \sin\theta$ (since $|\vec{r}| = 1$). So, we may write the vector $\vec{r}$ as

$$\vec{r}\,(= \overrightarrow{OP}) = \cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}} \qquad \ldots (1)$$

Clearly, $$|\vec{r}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

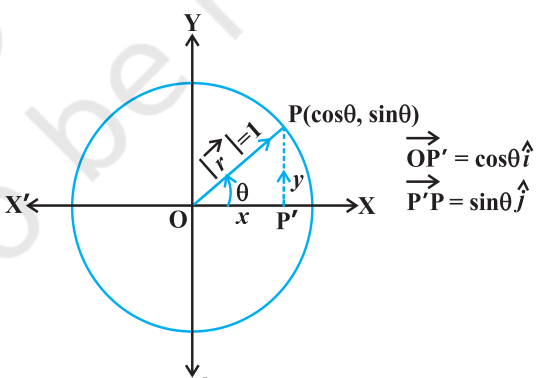


Fig 10.28

Also, as θ varies from 0 to 2π, the point P (Fig 10.28) traces the circle $x^2 + y^2 = 1$ counterclockwise, and this covers all possible directions. So, (1) gives every unit vector in the XY-plane.

Example 27 If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Solution Note that if θ is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\overrightarrow{AB}$ = Position vector of B – Position vector of A

$$= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$$

Therefore $$|\overrightarrow{AB}| = \sqrt{(1)^2+(4)^2+(-1)^2} = 3\sqrt{2}$$

Similarly $$\overrightarrow{CD} = -2\hat{i}-8\hat{j}+2\hat{k} \text{ and } |\overrightarrow{CD}| = 6\sqrt{2}$$

Thus $$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}$$

$$= \frac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \frac{-36}{36} = -1$$

Since $0 \le \theta \le \pi$, it follows that $\theta = \pi$. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Alternatively, $\overrightarrow{AB} = -\frac{1}{2}\overrightarrow{CD}$ which implies that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors.

Example 28 Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

Solution Given $\vec{a}\cdot(\vec{b}+\vec{c}) = 0,\ \vec{b}\cdot(\vec{c}+\vec{a}) = 0,\ \vec{c}\cdot(\vec{a}+\vec{b}) = 0.$

Now $$|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot\vec{b}+\vec{b}\cdot(\vec{a}+\vec{c})$$

$$+\ \vec{c}.(\vec{a}+\vec{b})+\vec{c}.\vec{c}$$

$$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$$

$$= 9 + 16 + 25 = 50$$

Therefore $$|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$$

Example 29 Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{c}|=2$.

Solution Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have

$$\vec{a}+\vec{b}+\vec{c}=\vec{0} \quad = 0$$

or
$$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} \quad = 0$$

Therefore
$$\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=-|\vec{a}|^2=-9 \qquad \text{... (1)}$$

Again,
$$\vec{b}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right) \quad = 0$$

or
$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}=-\left|\vec{b}\right|^2=-16 \qquad \text{... (2)}$$

Similarly
$$\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c} \quad = -4. \qquad \text{... (3)}$$

Adding (1), (2) and (3), we have

$$2\,(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{c}) \quad = -29$$

or
$$2\mu = -29, \text{ i.e., } \mu = \frac{-29}{2}$$

Example 30 If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, $\vec{\alpha}=3\hat{i}-\hat{j}$, $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Solution Let $\vec{\beta}_1=\lambda\vec{\alpha}$, λ is a scalar, i.e., $\vec{\beta}_1=3\lambda\hat{i}-\lambda\hat{j}$.

Now
$$\vec{\beta}_2=\vec{\beta}-\vec{\beta}_1= (2-3\lambda)\hat{\boldsymbol{i}}+(1+\lambda)\hat{\boldsymbol{j}}-3\hat{\boldsymbol{k}}.$$

Now, since $\vec{\beta}_2$ is to be perpendicular to $\vec{\alpha}$, we should have $\vec{\alpha}\cdot\vec{\beta}_2=0$. i.e.,

$$3(2-3\lambda)-(1+\lambda) \quad = 0$$

or
$$\lambda = \frac{1}{2}$$

Therefore
$$\vec{\beta}_1 = \frac{3}{2}\hat{\boldsymbol{i}}-\frac{1}{2}\hat{\boldsymbol{j}} \quad \text{and} \quad \vec{\beta}_2=\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$$

Miscellaneous Exercise on Chapter 10

1. Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

2. Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

3. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

4. If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

5. Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

7. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

8. Show that the points A (1, – 2, – 8), B (5, 0, –2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

9. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1 : 2. Also, show that P is the mid point of the line segment RQ.

10. The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

12. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 15$.

13. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

14. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{c} \cdot \vec{d} = 15$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

15. Prove that $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Choose the correct answer in Exercises 16 to 19.

16. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$ (B) $0 \leq \theta \leq \frac{\pi}{2}$

(C) $0 < \theta < \pi$ (D) $0 \leq \theta \leq \pi$

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$

18. The value of $\hat{\boldsymbol{i}}.(\hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}}) + \hat{\boldsymbol{j}}\cdot(\hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}}) + \hat{\boldsymbol{k}}\cdot(\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}})$ is

(A) 0 (B) –1 (C) 1 (D) 3

19. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Summary

- Position vector of a point P(x, y, z) is given as $\overrightarrow{\text{OP}}(=\vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$, and its magnitude by $\sqrt{x^2 + y^2 + z^2}$.
- The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- The magnitude (r), direction ratios (a, b, c) and direction cosines (l, m, n) of any vector are related as:

$$l = \frac{a}{r}, \quad m = \frac{b}{r}, \quad n = \frac{c}{r}$$

- The vector sum of the three sides of a triangle taken in order is $\vec{0}$.
- The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- The multiplication of a given vector by a scalar λ, changes the magnitude of the vector by the multiple $|\lambda|$, and keeps the direction same (or makes it opposite) according as the value of λ is positive (or negative).
- For a given vector $\vec{a}$, the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$.
- The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio $\boldsymbol{m} : \boldsymbol{n}$

  (i) internally, is given by $\frac{n\vec{a} + m\vec{b}}{m+n}$.

  (ii) externally, is given by $\frac{m\vec{b} - n\vec{a}}{m-n}$.
- The scalar product of two given vectors $\vec{a}$ and $\vec{b}$ having angle θ between them is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta .$$

  Also, when $\vec{a} \cdot \vec{b}$ is given, the angle 'θ' between the vectors $\vec{a}$ and $\vec{b}$ may be determined by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

- If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then their cross product is given as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

  where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Such that $\vec{a}, \vec{b}, \hat{n}$ form right handed system of coordinate axes.
- If we have two vectors $\vec{a}$ and $\vec{b}$, given in component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ any scalar,

then $\vec{a}+\vec{b} = (\boldsymbol{a}_1+\boldsymbol{b}_1)\hat{\boldsymbol{i}} + (\boldsymbol{a}_2+\boldsymbol{b}_2)\hat{\boldsymbol{j}} + (\boldsymbol{a}_3+\boldsymbol{b}_3)\hat{\boldsymbol{k}}$;

$\lambda\vec{a} = (\lambda\boldsymbol{a}_1)\hat{\boldsymbol{i}} + (\lambda\boldsymbol{a}_2)\hat{\boldsymbol{j}} + (\lambda\boldsymbol{a}_3)\hat{\boldsymbol{k}}$;

$\vec{a}\cdot\vec{b} = \boldsymbol{a}_1\boldsymbol{b}_1 + \boldsymbol{a}_2\boldsymbol{b}_2 + \boldsymbol{a}_3\boldsymbol{b}_3$;

and $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \boldsymbol{a}_1 & \boldsymbol{b}_1 & \boldsymbol{c}_1 \\ \boldsymbol{a}_2 & \boldsymbol{b}_2 & \boldsymbol{c}_2 \end{vmatrix}$.

Historical Note

The word ***vector*** has been derived from a Latin word ***vectus***, which means "to carry". The germinal ideas of modern vector theory date from around 1800 when Caspar Wessel (1745-1818) and Jean Robert Argand (1768-1822) described that how a complex number ***a + ib*** could be given a geometric interpretation with the help of a directed line segment in a coordinate plane. William Rowen Hamilton (1805-1865) an Irish mathematician was the first to use the term vector for a directed line segment in his book ***Lectures on Quaternions*** (1853). Hamilton's method of quaternions (an ordered set of four real numbers given as: $\boldsymbol{a} + \boldsymbol{b}\hat{\boldsymbol{i}} + \boldsymbol{c}\hat{\boldsymbol{j}} + \boldsymbol{d}\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{k}}$ following certain algebraic rules) was a solution to the problem of multiplying vectors in three dimensional space. Though, we must mention here that in practice, the idea of vector concept and their addition was known much earlier ever since the time of Aristotle (384-322 B.C.), a Greek philosopher, and pupil of Plato (427-348 B.C.). That time it was supposed to be known that the combined action of two or more forces could be seen by adding them according to parallelogram law. The correct law for the composition of forces, that forces add vectorially, had been discovered in the case of perpendicular forces by Stevin-Simon (1548-1620). In 1586 A.D., he analysed the principle of geometric addition of forces in his treatise ***DeBeghinselen der Weeghconst*** ("Principles of the Art of Weighing"), which caused a major breakthrough in the development of mechanics. But it took another 200 years for the general concept of vectors to form.

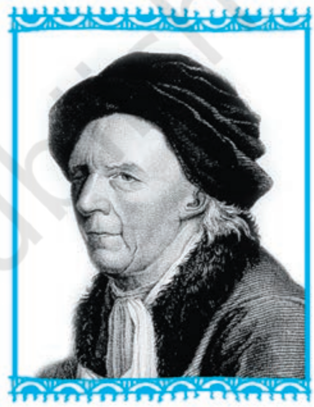
In the 1880, Josaih Willard Gibbs (1839-1903), an American physicist and mathematician, and Oliver Heaviside (1850-1925), an English engineer, created what we now know as ***vector analysis***, essentially by separating the real (***scalar***) part of quaternion from its imaginary (***vector***) part. In 1881 and 1884, Gibbs printed a treatise entitled ***Element of Vector Analysis***. This book gave a systematic and concise account of vectors. However, much of the credit for demonstrating the applications of vectors is due to the D. Heaviside and P.G. Tait (1831-1901) who contributed significantly to this subject.

Chapter 11

THREE DIMENSIONAL GEOMETRY

❖ *The moving power of mathematical invention is not reasoning but imagination. – A. DEMORGAN* ❖

11.1 Introduction



Leonhard Euler
(1707-1783)

In Class XI, while studying Analytical Geometry in two dimensions, and the introduction to three dimensional geometry, we confined to the Cartesian methods only. In the previous chapter of this book, we have studied some basic concepts of vectors. We will now use vector algebra to three dimensional geometry. The purpose of this approach to 3-dimensional geometry is that it makes the study simple and elegant*.

In this chapter, we shall study the direction cosines and direction ratios of a line joining two points and also discuss about the equations of lines and planes in space under different conditions, angle between two lines, two planes, a line and a plane, shortest distance between two skew lines and distance of a point from a plane. Most of the above results are obtained in vector form. Nevertheless, we shall also translate these results in the Cartesian form which, at times, presents a more clear geometric and analytic picture of the situation.

11.2 Direction Cosines and Direction Ratios of a Line

From Chapter 10, recall that if a directed line L passing through the origin makes angles α, β and γ with x, y and z-axes, respectively, called direction angles, then cosine of these angles, namely, cos α, cos β and cos γ are called direction cosines of the directed line L.

If we reverse the direction of L, then the direction angles are replaced by their supplements, i.e., $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Thus, the signs of the direction cosines are reversed.

* For various activities in three dimensional geometry, one may refer to the Book ***"A Hand Book for designing Mathematics Laboratory in Schools"***, NCERT, 2005

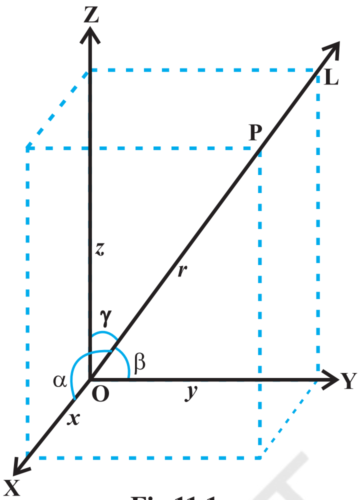


Fig 11.1

Note that a given line in space can be extended in two opposite directions and so it has two sets of direction cosines. In order to have a unique set of direction cosines for a given line in space, we must take the given line as a directed line. These unique direction cosines are denoted by l, m and n.

Remark If the given line in space does not pass through the origin, then, in order to find its direction cosines, we draw a line through the origin and parallel to the given line. Now take one of the directed lines from the origin and find its direction cosines as two parallel line have same set of direction cosines.

Any three numbers which are proportional to the direction cosines of a line are called the *direction ratios* of the line. If l, m, n are direction cosines and a, b, c are direction ratios of a line, then $a = \lambda l$, $b=\lambda m$ and $c = \lambda n$, for any nonzero $\lambda \in \mathbf{R}$.

Note Some authors also call direction ratios as direction numbers.

Let a, b, c be direction ratios of a line and let l, m and n be the direction cosines ($d.c$'s) of the line. Then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ (say), } k \text{ being a constant.}$$

Therefore $\qquad l = ak,\ m = bk,\ n = ck \qquad \text{... (1)}$

But $\qquad l^2 + m^2 + n^2 = 1$

Therefore $\qquad k^2 (a^2 + b^2 + c^2) = 1$

or
$$k = \pm \frac{1}{\sqrt{a^2+b^2+c^2}}$$

Hence, from (1), the *d.c.*'*s* of the line are

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

where, depending on the desired sign of k, either a positive or a negative sign is to be taken for l, m and n.

For any line, if a, b, c are direction ratios of a line, then ka, kb, kc; $k \neq 0$ is also a set of direction ratios. So, any two sets of direction ratios of a line are also proportional. Also, for any line there are infinitely many sets of direction ratios.

11.2.1 *Relation between the direction cosines of a line*

Consider a line RS with direction cosines l, m, n. Through the origin draw a line parallel to the given line and take a point P(x, y, z) on this line. From P draw a perpendicular PA on the x-axis (Fig. 11.2).

Let OP = r. Then $\cos\alpha = \frac{\text{OA}}{\text{OP}} = \frac{x}{r}$. This gives $x = lr$.

Similarly, $y = mr$ and $z = nr$

Thus $x^2 + y^2 + z^2 = r^2\,(l^2 + m^2 + n^2)$

But $x^2 + y^2 + z^2 = r^2$

Hence $\boldsymbol{l^2 + m^2 + n^2 = 1}$

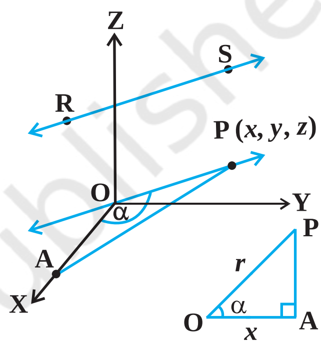


Fig 11.2

11.2.2 *Direction cosines of a line passing through two points*

Since one and only one line passes through two given points, we can determine the direction cosines of a line passing through the given points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) as follows (Fig 11.3 (a)).

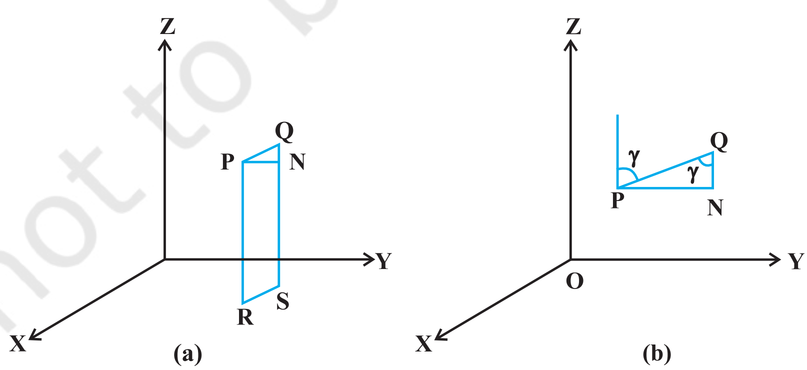


Fig 11.3

Let l, m, n be the direction cosines of the line PQ and let it makes angles α, β and γ with the x, y and z-axis, respectively.

Draw perpendiculars from P and Q to XY-plane to meet at R and S. Draw a perpendicular from P to QS to meet at N. Now, in right angle triangle PNQ, $\angle$PQN= γ (Fig 11.3 (b).

Therefore, $$\cos\gamma = \frac{NQ}{PQ} = \frac{z_2 - z_1}{PQ}$$

Similarly $$\cos\alpha = \frac{x_2 - x_1}{PQ} \text{ and } \cos\beta = \frac{y_2 - y_1}{PQ}$$

Hence, the direction cosines of the line segment joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Note The direction ratios of the line segment joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) may be taken as

$$x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \text{ or } x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2$$

Example 1 If a line makes angle 90°, 60° and 30° with the positive direction of x, y and z-axis respectively, find its direction cosines.

Solution Let the $d.c.$'s of the lines be l, m, n. Then $l = \cos 90^0 = 0$, $m = \cos 60^0 = \frac{1}{2}$,

$n = \cos 30^0 = \frac{\sqrt{3}}{2}$.

Example 2 If a line has direction ratios 2, – 1, – 2, determine its direction cosines.

Solution Direction cosines are

$$\frac{2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-1}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

or $$\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$$

Example 3 Find the direction cosines of the line passing through the two points (– 2, 4, – 5) and (1, 2, 3).

Solution We know the direction cosines of the line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Here P is (– 2, 4, – 5) and Q is (1, 2, 3).

So $$PQ = \sqrt{(1-(-2))^2 + (2-4)^2 + (3-(-5))^2} = \sqrt{77}$$

Thus, the direction cosines of the line joining two points is

$$\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$$

Example 4 Find the direction cosines of *x*, *y* and *z*-axis.

Solution The *x*-axis makes angles 0°, 90° and 90° respectively with *x*, *y* and *z*-axis. Therefore, the direction cosines of *x*-axis are cos 0°, cos 90°, cos 90° i.e., 1,0,0. Similarly, direction cosines of *y*-axis and *z*-axis are 0, 1, 0 and 0, 0, 1 respectively.

Example 5 Show that the points A (2, 3, – 4), B (1, – 2, 3) and C (3, 8, – 11) are collinear.

Solution Direction ratios of line joining A and B are

1 – 2, – 2 – 3, 3 + 4 i.e., – 1, – 5, 7.

The direction ratios of line joining B and C are

3 –1, 8 + 2, – 11 – 3, i.e., 2, 10, – 14.

It is clear that direction ratios of AB and BC are proportional, hence, AB is parallel to BC. But point B is common to both AB and BC. Therefore, A, B, C are collinear points.

EXERCISE 11.1

1. If a line makes angles 90°, 135°, 45° with the *x*, *y* and *z*-axes respectively, find its direction cosines.
2. Find the direction cosines of a line which makes equal angles with the coordinate axes.
3. If a line has the direction ratios –18, 12, – 4, then what are its direction cosines ?
4. Show that the points (2, 3, 4), (– 1, – 2, 1), (5, 8, 7) are collinear.
5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2).

11.3 Equation of a Line in Space

We have studied equation of lines in two dimensions in Class XI, we shall now study the vector and cartesian equations of a line in space.

A line is uniquely determined if

(i) it passes through a given point and has given direction, or

(ii) it passes through two given points.

11.3.1 *Equation of a line through a given point and parallel to a given vector* $\vec{b}$

Let $\vec{a}$ be the position vector of the given point A with respect to the origin O of the rectangular coordinate system. Let l be the line which passes through the point A and is parallel to a given vector $\vec{b}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the line (Fig 11.4).

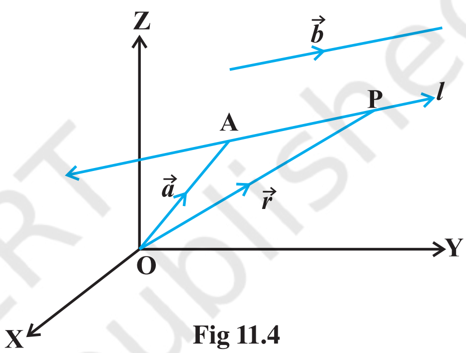


Fig 11.4

Then $\overrightarrow{AP}$ is parallel to the vector $\vec{b}$, i.e., $\overrightarrow{AP} = \lambda\vec{b}$, where λ is some real number.

But $$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$$

i.e. $$\lambda\vec{b} = \vec{r} - \vec{a}$$

Conversely, for each value of the parameter λ, this equation gives the position vector of a point P on the line. Hence, the vector equation of the line is given by

$$\vec{r} = \vec{a} + \lambda\,\vec{b} \qquad \text{... (1)}$$

Remark If $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$, then a, b, c are direction ratios of the line and conversely, if a, b, c are direction ratios of a line, then $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ will be the parallel to the line. Here, b should not be confused with $|\vec{b}|$.

Derivation of cartesian form from vector form

Let the coordinates of the given point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b, c. Consider the coordinates of any point P be (x, y, z). Then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\,;\; \vec{a} = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$$

and $$\vec{b} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$$

Substituting these values in (1) and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a;\;\; y = y_1 + \lambda\, b;\;\; z = z_1 + \lambda\, c \qquad \text{... (2)}$$

These are parametric equations of the line. Eliminating the parameter λ from (2), we get

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \qquad \ldots (3)$$

This is the Cartesian equation of the line.

Note If l, m, n are the direction cosines of the line, the equation of the line is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Example 6 Find the vector and the Cartesian equations of the line through the point (5, 2, – 4) and which is parallel to the vector $3\hat{i}+2\hat{j}-8\hat{k}$.

Solution We have

$$\vec{a} = 5\hat{i}+2\hat{j}-4\hat{k} \text{ and } \vec{b}=3\hat{i}+2\hat{j}-8\hat{k}$$

Therefore, the vector equation of the line is

$$\vec{r} = 5\hat{i}+2\hat{j}-4\hat{k}+\lambda(3\hat{i}+2\hat{j}-8\hat{k})$$

Now, $\vec{r}$ is the position vector of any point P(x, y, z) on the line.

Therefore, $\quad x\hat{i}+y\hat{j}+z\hat{k} = 5\hat{i}+2\hat{j}-4\hat{k}+\lambda(3\hat{i}+2\hat{j}-8\hat{k})$

$$= (5+3\lambda)\hat{i}+(2+2\lambda)\hat{j}+(-4-8\lambda)\hat{k}$$

Eliminating λ, we get

$$\frac{x-5}{3}=\frac{y-2}{2}=\frac{z+4}{-8}$$

which is the equation of the line in Cartesian form.

11.3.2 *Equation of a line passing through two given points*

Let $\vec{a}$ and $\vec{b}$ be the position vectors of two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2), respectively that are lying on a line (Fig 11.5).

Let $\vec{r}$ be the position vector of an arbitrary point P(x, y, z), then P is a point on the line if and only if $\overrightarrow{AP}=\vec{r}-\vec{a}$ and $\overrightarrow{AB}=\vec{b}-\vec{a}$ are collinear vectors. Therefore, P is on the line if and only if

$$\vec{r}-\vec{a}=\lambda(\vec{b}-\vec{a})$$

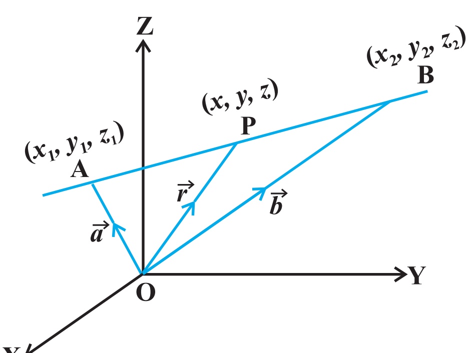


Fig 11.5

or $\quad \boldsymbol{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})}$, $\boldsymbol{\lambda \in}$ **R**. ... (1)

This is the vector equation of the line.

Derivation of cartesian form from vector form

We have

$$\vec{r} = x\hat{i} + y\,\hat{j} + z\hat{k},\ \vec{a} = x_1\hat{i} + y_1\,\hat{j} + z_1\,\hat{k} \text{ and } \vec{b} = x_2\hat{i} + y_2\,\hat{j} + z_2\,\hat{k},$$

Substituting these values in (1), we get

$$x\hat{i} + y\,\hat{j} + z\hat{k} = x_1\,\hat{i} + y_1\,\hat{j} + z_1\hat{k} + \lambda\,[(x_2 - x_1)\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}]$$

Equating the like coefficients of $\hat{i},\ \hat{j},\ \hat{k}$, we get

$$x = x_1 + \lambda\,(x_2 - x_1);\ y = y_1 + \lambda\,(y_2 - y_1);\ z = z_1 + \lambda\,(z_2 - z_1)$$

On eliminating λ, we obtain

$$\boldsymbol{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

which is the equation of the line in Cartesian form.

Example 7 Find the vector equation for the line passing through the points (–1, 0, 2) and (3, 4, 6).

Solution Let $\vec{a}$ and $\vec{b}$ be the position vectors of the point A (– 1, 0, 2) and B (3, 4, 6).

Then $\quad \vec{a} = -\hat{i} + 2\hat{k}$

and $\quad \vec{b} = 3\hat{i} + 4\,\hat{j} + 6\hat{k}$

Therefore $\quad \vec{b} - \vec{a} = 4\hat{i} + 4\,\hat{j} + 4\hat{k}$

Let $\vec{r}$ be the position vector of any point on the line. Then the vector equation of the line is

$$\vec{r} = -\,\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\,\hat{j} + 4\hat{k})$$

Example 8 The Cartesian equation of a line is

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$$

Find the vector equation for the line.

Solution Comparing the given equation with the standard form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

We observe that $\quad x_1 = -3,\ y_1 = 5,\ z_1 = -6;\ a = 2,\ b = 4,\ c = 2.$

Thus, the required line passes through the point (– 3, 5, – 6) and is parallel to the vector $2\hat{i}+4\hat{j}+2\hat{k}$. Let $\vec{r}$ be the position vector of any point on the line, then the vector equation of the line is given by

$$\vec{r}=(-3\hat{i}+5\hat{j}-6\hat{k})+\lambda\,(2\hat{i}+4\hat{j}+2\hat{k})$$

11.4 Angle between Two Lines

Let L_1 and L_2 be two lines passing through the origin and with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2, respectively. Let P be a point on L_1 and Q be a point on L_2. Consider the directed lines OP and OQ as given in Fig 11.6. Let θ be the acute angle between OP and OQ. Now recall that the directed line segments OP and OQ are vectors with components a_1, b_1, c_1 and a_2, b_2, c_2, respectively. Therefore, the angle θ between them is given by

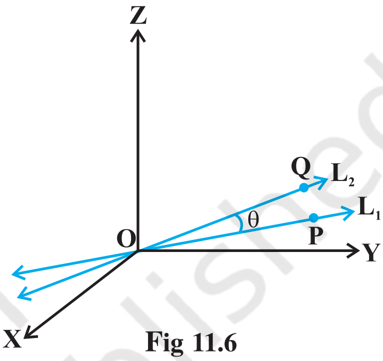


Fig 11.6

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right| \quad \text{... (1)}$$

The angle between the lines in terms of sin θ is given by

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{(a_1a_2+b_1b_2+c_1c_2)^2}{\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)}}$$

$$= \frac{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)-(a_1a_2+b_1b_2+c_1c_2)^2}}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}$$

$$= \frac{\sqrt{(a_1\,b_2-a_2\,b_1)^2+(b_1\,c_2-b_2\,c_1)^2+(c_1\,a_2-c_2\,a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \quad \text{... (2)}$$

Note In case the lines L_1 and L_2 do not pass through the origin, we may take lines L'_1 and L'_2 which are parallel to L_1 and L_2 respectively and pass through the origin.

If instead of direction ratios for the lines L_1 and L_2, direction cosines, namely, l_1, m_1, n_1 for L_1 and l_2, m_2, n_2 for L_2 are given, then (1) and (2) takes the following form:

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2| \quad (\text{as } l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2) \qquad \ldots (3)$$

and $$\sin\theta = \sqrt{(l_1 m_2 - l_2 m_1)^2 - (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2} \qquad \ldots (4)$$

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are

(i) perpendicular i.e. if $\theta = 90°$ by (1)

$$\boldsymbol{a_1 a_2 + b_1 b_2 + c_1 c_2 = 0}$$

(ii) parallel i.e. if $\theta = 0$ by (2)

$$\boldsymbol{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$$

Now, we find the angle between two lines when their equations are given. If θ is acute the angle between the lines

$$\vec{r} = \bar{a}_1 + \lambda \vec{b}_1 \text{ and } \vec{r} = \vec{a}_2 + \mu \vec{b}_2$$

then $$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

In Cartesian form, if θ is the angle between the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \qquad \ldots (1)$$

and $$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \qquad \ldots (2)$$

where, a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of the lines (1) and (2), respectively, then

$$\cos\theta = \left|\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

Example 9 Find the angle between the pair of lines given by

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

and $$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

Solution Here $\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

The angle θ between the two lines is given by

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right| = \left|\frac{(\hat{i}+2\hat{j}+2\hat{k})\cdot(3\hat{i}+2\hat{j}+6\hat{k})}{\sqrt{1+4+4}\,\sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3\times 7}\right| = \frac{19}{21}$$

Hence $\theta = \cos^{-1}\left(\frac{19}{21}\right)$

Example 10 Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$$

and
$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$$

Solution The direction ratios of the first line are 3, 5, 4 and the direction ratios of the second line are 1, 1, 2. If θ is the angle between them, then

$$\cos\theta = \left|\frac{3.1+5.1+4.2}{\sqrt{3^2+5^2+4^2}\,\sqrt{1^2+1^2+2^2}}\right| = \frac{16}{\sqrt{50}\,\sqrt{6}} = \frac{16}{5\sqrt{2}\,\sqrt{6}} = \frac{8\sqrt{3}}{15}$$

Hence, the required angle is $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$.

11.5 Shortest Distance between Two Lines

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from a point on one line onto the other line.

Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are non coplanar and are called *skew lines*. For example, let us consider a room of size 1, 3, 2 units along *x*, *y* and *z*-axes respectively Fig 11.7.

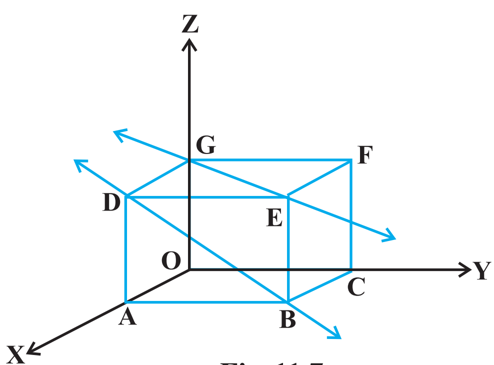


Fig 11.7

The line GE that goes diagonally across the ceiling and the line DB passes through one corner of the ceiling directly above A and goes diagonally down the wall. These lines are skew because they are not parallel and also never meet.

By the shortest distance between two lines we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

11.5.1 *Distance between two skew lines*

We now determine the shortest distance between two skew lines in the following way: Let l_1 and l_2 be two skew lines with equations (Fig. 11.8)

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \quad \text{... (1)}$$

and

$$\vec{r} = \vec{a}_2 + \mu \vec{b}_2 \quad \text{... (2)}$$

Take any point S on l_1 with position vector $\vec{a}_1$ and T on l_2, with position vector $\vec{a}_2$. Then the magnitude of the shortest distance vector will be equal to that of the projection of ST along the direction of the line of shortest distance (See 10.6.2).

If $\overrightarrow{PQ}$ is the shortest distance vector between l_1 and l_2, then it being perpendicular to both $\vec{b}_1$ and $\vec{b}_2$, the unit vector $\hat{n}$ along $\overrightarrow{PQ}$ would therefore be

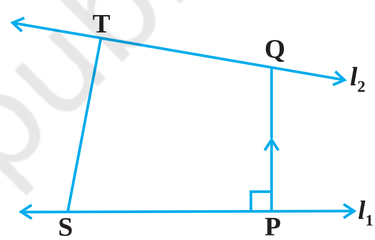


Fig 11.8

$$\hat{n} = \frac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|} \quad \text{... (3)}$$

Then $$\overrightarrow{PQ} = d\,\hat{n}$$

where, d is the magnitude of the shortest distance vector. Let θ be the angle between $\overrightarrow{ST}$ and $\overrightarrow{PQ}$. Then

$$PQ = ST\,|\cos\theta|$$

But

$$\cos\theta = \left| \frac{\overrightarrow{PQ} \cdot \overrightarrow{ST}}{|\overrightarrow{PQ}||\overrightarrow{ST}|} \right|$$

$$= \left| \frac{d\,\hat{n} \cdot (\vec{a}_2 - \vec{a}_1)}{d\ ST} \right| \quad (\text{since } \overrightarrow{ST} = \vec{a}_2 - \vec{a}_1)$$

$$= \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{ST\,|\vec{b}_1 \times \vec{b}_2|} \right| \quad \text{[From (3)]}$$

Hence, the required shortest distance is

$$d = PQ = ST\ |\cos\theta|$$

or
$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 \times \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Cartesian form

The shortest distance between the lines

$$l_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and
$$l_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

is
$$\left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right|$$

11.5.2 *Distance between parallel lines*

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be given by

$$\vec{r} = \vec{a}_1 + \lambda \vec{b} \qquad \dots (1)$$

and
$$\vec{r} = \vec{a}_2 + \mu \vec{b} \qquad \dots (2)$$

where, $\vec{a}_1$ is the position vector of a point S on l_1 and $\vec{a}_2$ is the position vector of a point T on l_2 Fig 11.9.

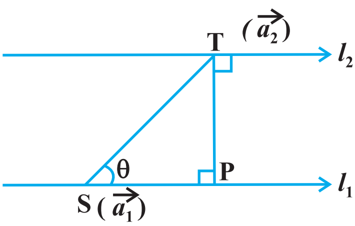


Fig 11.9

As l_1, l_2 are coplanar, if the foot of the perpendicular from T on the line l_1 is P, then the distance between the lines l_1 and l_2 = |TP|.

Let θ be the angle between the vectors $\overrightarrow{ST}$ and $\vec{b}$. Then

$$\vec{b} \times \overrightarrow{ST} = (|\vec{b}|\,|\overrightarrow{ST}| \sin\theta)\,\breve{n} \ \dots \qquad (3)$$

where $\hat{n}$ is the unit vector perpendicular to the plane of the lines l_1 and l_2.

But
$$\overrightarrow{ST} = \vec{a}_2 - \vec{a}_1$$

Therefore, from (3), we get

$$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|\, \text{PT}\, \hat{n} \qquad \text{(since PT = ST sin } \theta)$$

i.e.,
$$|\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|\, \text{PT} \cdot 1 \qquad (\text{as } |\hat{n}| = 1)$$

Hence, the distance between the given parallel lines is

$$d = |\overrightarrow{\mathbf{PT}}| = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

Example 11 Find the shortest distance between the lines l_1 and l_2 whose vector equations are

$$\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} - \hat{j} + \hat{k}) \qquad \text{... (1)}$$

and
$$\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu\,(3\hat{i} - 5\hat{j} + 2\hat{k}) \qquad \text{... (2)}$$

Solution Comparing (1) and (2) with $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ respectively, we get

$$\vec{a}_1 = \hat{i} + \hat{j},\ \vec{b}_1 = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k} \text{ and } \vec{b}_2 = 3\hat{i} - 5\hat{j} + 2\hat{k}$$

Therefore
$$\vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$$

and
$$\vec{b}_1 \times \vec{b}_2 = (2\hat{i} - \hat{j} + \hat{k}) \times (3\hat{i} - 5\hat{j} + 2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

So
$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{9 + 1 + 49} = \sqrt{59}$$

Hence, the shortest distance between the given lines is given by

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| = \frac{|3 - 0 + 7|}{\sqrt{59}} = \frac{10}{\sqrt{59}}$$

Example 12 Find the distance between the lines l_1 and l_2 given by

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda\,(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and
$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu\,(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Solution The two lines are parallel (Why?) We have

$$\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}, \ \vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k} \text{ and } \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Therefore, the distance between the lines is given by

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right| = \left| \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}}{\sqrt{4+9+36}} \right|$$

or
$$= \frac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{\sqrt{49}} = \frac{\sqrt{293}}{\sqrt{49}} = \frac{\sqrt{293}}{7}$$

EXERCISE 11.2

1. Show that the three lines with direction cosines

   $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \quad \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \quad \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

2. Show that the line through the points (1, – 1, 2), (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

3. Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1), (1, 2, 5).

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

5. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i} - j + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

6. Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

7. The cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.

8. Find the vector and the cartesian equations of the lines that passes through the origin and (5, – 2, 3).

9. Find the vector and the cartesian equations of the line that passes through the points (3, – 2, – 5), (3, – 2, 6).

10. Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

11. Find the angle between the following pair of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

12. Find the values of p so that the lines $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$

and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

13. Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

14. Find the shortest distance between the lines

$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and

$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

15. Find the shortest distance between the lines

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

16. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$

and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

17. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

11.6 Plane

A plane is determined uniquely if any one of the following is known:

(i) the normal to the plane and its distance from the origin is given, i.e., equation of a plane in normal form.

(ii) it passes through a point and is perpendicular to a given direction.

(iii) it passes through three given non collinear points.

Now we shall find vector and Cartesian equations of the planes.

11.6.1 *Equation of a plane in normal form*

Consider a plane whose perpendicular distance from the origin is $d\,(d \neq 0)$. Fig 11.10.

If $\overrightarrow{ON}$ is the normal from the origin to the plane, and $\hat{n}$ is the unit normal vector along $\overrightarrow{ON}$. Then $\overrightarrow{ON} = d\,\hat{n}$. Let P be any point on the plane. Therefore, $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$.

Therefore, $\overrightarrow{NP} \cdot \overrightarrow{ON} = 0$ ... (1)

Let $\vec{r}$ be the position vector of the point P, then $\overrightarrow{NP} = \vec{r} - d\,\hat{n}$ (as $\overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OP}$)

Therefore, (1) becomes

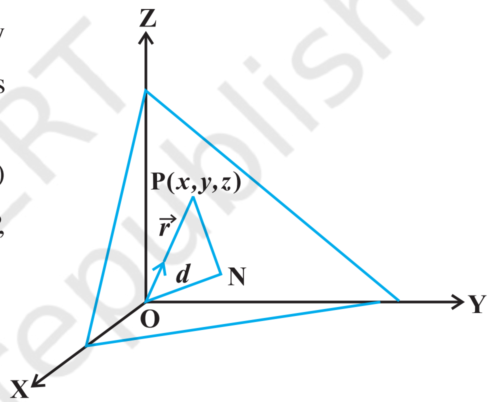


Fig 11.10

$$(\vec{r} - d\,\hat{n}) \cdot d\,\hat{n} = 0$$

or $$(\vec{r} - d\,\hat{n}) \cdot \hat{n} = 0 \qquad (d \neq 0)$$

or $$\vec{r} \cdot \hat{n} - d\,\hat{n} \cdot \hat{n} = 0$$

i.e., $$\boldsymbol{\vec{r} \cdot \hat{n} = d} \qquad (\text{as } \hat{n} \cdot \hat{n} = 1) \qquad \ldots (2)$$

This is the vector form of the equation of the plane.

Cartesian form

Equation (2) gives the vector equation of a plane, where $\hat{n}$ is the unit vector normal to the plane. Let P(x, y, z) be any point on the plane. Then

$$\overrightarrow{OP} = \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Let l, m, n be the direction cosines of $\hat{n}$. Then

$$\hat{n} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$$

Therefore, (2) gives

$$(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) \cdot (l\,\hat{i} + m\,\hat{j} + n\,\hat{k}) = d$$

i.e., $$\boldsymbol{lx + my + nz = d} \qquad \ldots (3)$$

This is the cartesian equation of the plane in the normal form.

Note Equation (3) shows that if $\vec{r} \cdot (a\,\hat{i} + b\,\hat{j} + c\,\hat{k}) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the Cartesian equation of the plane, where a, b and c are the direction ratios of the normal to the plane.

Example 13 Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$. Also find its cartesian form.

Solution Let $\vec{n} = 2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}$. Then

$$\hat{n} = \frac{\vec{n}}{|\,\vec{n}\,|} = \frac{2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}}{\sqrt{4 + 9 + 16}} = \frac{2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}}{\sqrt{29}}$$

Hence, the required equation of the plane is

$$\vec{r} \cdot \left(\frac{2}{\sqrt{29}}\,\hat{i} + \frac{-3}{\sqrt{29}}\,\hat{j} + \frac{4}{\sqrt{29}}\,\hat{k} \right) = \frac{6}{\sqrt{29}}$$

Example 14 Find the direction cosines of the unit vector perpendicular to the plane $\vec{r} \cdot (6\,\hat{i} - 3\,\hat{j} - 2\,\hat{k}) + 1 = 0$ passing through the origin.

Solution The given equation can be written as

$$\vec{r} \cdot (-6\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}) = 1 \qquad \ldots (1)$$

Now $$|-6\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}| = \sqrt{36 + 9 + 4} = 7$$

Therefore, dividing both sides of (1) by 7, we get

$$\vec{r} \cdot \left(-\frac{6}{7}\,\hat{i} + \frac{3}{7}\,\hat{j} + \frac{2}{7}\,\hat{k} \right) = \frac{1}{7}$$

which is the equation of the plane in the form $\vec{r} \cdot \hat{n} = d$.

This shows that $\hat{n} = -\frac{6}{7}\,\hat{i} + \frac{3}{7}\,\hat{j} + \frac{2}{7}\,\hat{k}$ is a unit vector perpendicular to the plane through the origin. Hence, the direction cosines of $\hat{n}$ are $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$.

Example 15 Find the distance of the plane $2x - 3y + 4z - 6 = 0$ from the origin.

Solution Since the direction ratios of the normal to the plane are 2, –3, 4; the direction cosines of it are

$$\frac{2}{\sqrt{2^2+(-3)^2+4^2}}, \frac{-3}{\sqrt{2^2+(-3)^2+4^2}}, \frac{4}{\sqrt{2^2+(-3)^2+4^2}}, \text{ i.e., } \frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

Hence, dividing the equation $2x-3y+4z-6=0$ i.e., $2x-3y+4z=6$ throughout by $\sqrt{29}$, we get

$$\frac{2}{\sqrt{29}}x + \frac{-3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

This is of the form $lx + my + nz = d$, where d is the distance of the plane from the origin. So, the distance of the plane from the origin is $\frac{6}{\sqrt{29}}$.

Example 16 Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x - 3y + 4z - 6 = 0$.

Solution Let the coordinates of the foot of the perpendicular P from the origin to the plane is (x_1, y_1, z_1) (Fig 11.11).

Then, the direction ratios of the line OP are x_1, y_1, z_1.

Writing the equation of the plane in the normal form, we have

$$\frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

where, $\frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$ are the direction cosines of the OP.

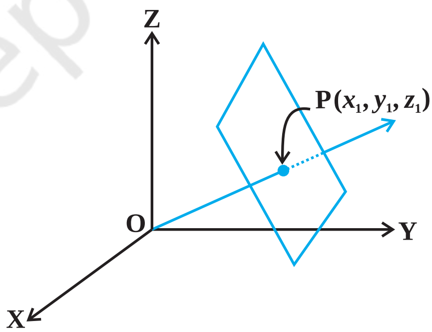


Fig 11.11

Since *d.c.'s* and direction ratios of a line are proportional, we have

$$\frac{x_1}{\frac{2}{\sqrt{29}}} = \frac{y_1}{\frac{-3}{\sqrt{29}}} = \frac{z_1}{\frac{4}{\sqrt{29}}} = k$$

i.e., $$x_1 = \frac{2k}{\sqrt{29}}, y_1 = \frac{-3k}{\sqrt{29}}, z_1 = \frac{4k}{\sqrt{29}}$$

Substituting these in the equation of the plane, we get $k = \frac{6}{\sqrt{29}}$.

Hence, the foot of the perpendicular is $\left(\frac{12}{29}, \frac{-18}{29}, \frac{24}{29}\right)$.

Note If d is the distance from the origin and l, m, n are the direction cosines of the normal to the plane through the origin, then the foot of the perpendicular is (ld, md, nd).

11.6.2 *Equation of a plane perpendicular to a given vector and passing through a given point*

In the space, there can be many planes that are perpendicular to the given vector, but through a given point $P(x_1, y_1, z_1)$, only one such plane exists (see Fig 11.12).

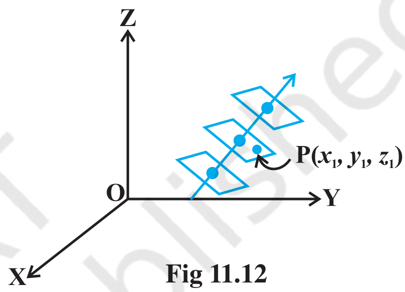


Fig 11.12

Let a plane pass through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$.

Let $\vec{r}$ be the position vector of any point P(x, y, z) in the plane. (Fig 11.13).

Then the point P lies in the plane if and only if $\overrightarrow{AP}$ is perpendicular to $\vec{N}$. i.e., $\overrightarrow{AP} \cdot \vec{N} = 0$. But $\overrightarrow{AP} = \vec{r} - \vec{a}$. Therefore, $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$ ... (1)

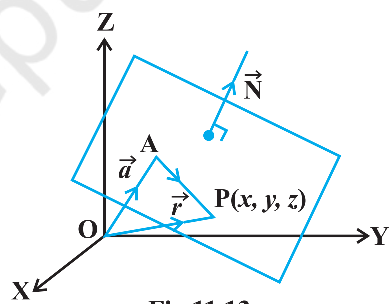


Fig 11.13

This is the vector equation of the plane.

Cartesian form

Let the given point A be (x_1, y_1, z_1), P be (x, y, z) and direction ratios of $\vec{N}$ are A, B and C. Then,

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ and } \vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

Now $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

So $\left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] \cdot (A\hat{i} + B\hat{j} + C\hat{k}) = 0$

i.e. $\mathbf{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}$

Example 17 Find the vector and cartesian equations of the plane which passes through the point (5, 2, – 4) and perpendicular to the line with direction ratios 2, 3, – 1.

Solution We have the position vector of point (5, 2, – 4) as $\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and the normal vector $\vec{N}$ perpendicular to the plane as $\vec{N} = 2\hat{i} + 3\hat{j} - \hat{k}$

Therefore, the vector equation of the plane is given by $(\vec{r} - \vec{a}).\vec{N} = 0$

or $$[\vec{r} - (5\hat{i} + 2\hat{j} - 4\hat{k})] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0 \qquad \ldots (1)$$

Transforming (1) into Cartesian form, we have

$$[(x-5)\hat{i} + (y-2)\hat{j} + (z+4)\hat{k}] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0$$

or $$2(x-5) + 3(y-2) - 1(z+4) = 0$$

i.e. $$2x + 3y - z = 20$$

which is the cartesian equation of the plane.

11.6.3 *Equation of a plane passing through three non collinear points*

Let R, S and T be three non collinear points on the plane with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively (Fig 11.14).

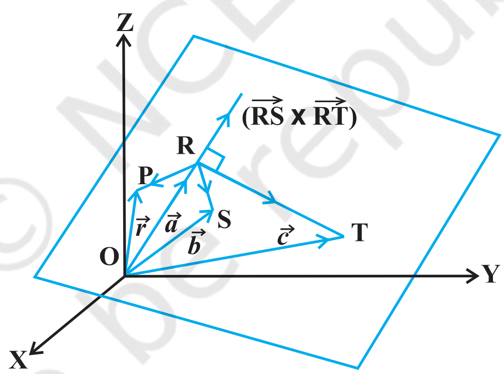


Fig 11.14

The vectors $\vec{RS}$ and $\vec{RT}$ are in the given plane. Therefore, the vector $\vec{RS} \times \vec{RT}$ is perpendicular to the plane containing points R, S and T. Let $\vec{r}$ be the position vector of any point P in the plane. Therefore, the equation of the plane passing through R and perpendicular to the vector $\vec{RS} \times \vec{RT}$ is

$$(\vec{r} - \vec{a}) \cdot (\vec{RS} \times \vec{RT}) = 0$$

or $$(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0 \qquad \ldots (1)$$

This is the equation of the plane in vector form passing through three noncollinear points.

Note Why was it necessary to say that the three points had to be non collinear? If the three points were on the same line, then there will be many planes that will contain them (Fig 11.15).

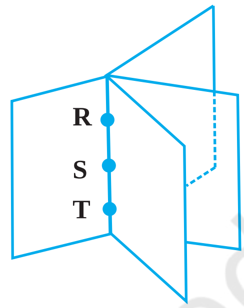


Fig 11.15

These planes will resemble the pages of a book where the line containing the points R, S and T are members in the binding of the book.

Cartesian form

Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of the points R, S and T respectively. Let (x, y, z) be the coordinates of any point P on the plane with position vector $\vec{r}$. Then

$$\overrightarrow{RP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$

$$\overrightarrow{RS} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\overrightarrow{RT} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$

Substituting these values in equation (1) of the vector form and expressing it in the form of a determinant, we have

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

which is the equation of the plane in Cartesian form passing through three non collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Example 18 Find the vector equations of the plane passing through the points R(2, 5, – 3), S(– 2, – 3, 5) and T(5, 3,– 3).

Solution Let $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$, $\vec{b} = -2\hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{c} = 5\hat{i} + 3\hat{j} - 3\hat{k}$

Then the vector equation of the plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is given by

$$(\vec{r} - \vec{a}) \cdot (\overrightarrow{RS} \times \overrightarrow{RT}) = 0 \qquad \text{(Why?)}$$

or $$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$

i.e. $$[\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})] \cdot [(-4\hat{i} - 8\hat{j} + 8\hat{k}) \times (3\hat{i} - 2\hat{j})] = 0$$

11.6.4 *Intercept form of the equation of a plane*

In this section, we shall deduce the equation of a plane in terms of the intercepts made by the plane on the coordinate axes. Let the equation of the plane be

$$\text{A}x + \text{B}y + \text{C}z + \text{D} = 0 \quad (\text{D} \neq 0) \qquad \ldots (1)$$

Let the plane make intercepts a, b, c on x, y and z axes, respectively (Fig 11.16).

Hence, the plane meets x, y and z-axes at $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, respectively.

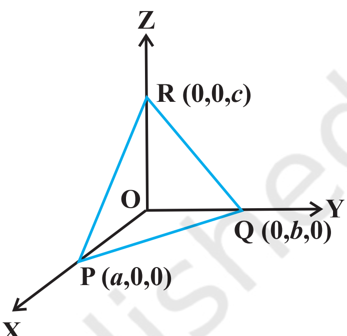


Fig 11.16

Therefore $\quad \text{A}a + \text{D} = 0 \text{ or } \text{A} = \dfrac{-\text{D}}{a}$

$$\text{B}b + \text{D} = 0 \text{ or } \text{B} = \frac{-\text{D}}{b}$$

$$\text{C}c + \text{D} = 0 \text{ or } \text{C} = \frac{-\text{D}}{c}$$

Substituting these values in the equation (1) of the plane and simplifying, we get

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \ldots (1)$$

which is the required equation of the plane in the intercept form.

Example 19 Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z-axis respectively.

Solution Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \ldots (1)$$

Here $\quad a = 2, b = 3, c = 4.$

Substituting the values of a, b and c in (1), we get the required equation of the plane as $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{4} = 1$ or $6x + 4y + 3z = 12$.

11.6.5 *Plane passing through the intersection of two given planes*

Let π_1 and π_2 be two planes with equations $\vec{r} \cdot \hat{n}_1 = d_1$ and $\vec{r} \cdot \hat{n}_2 = d_2$ respectively. The position vector of any point on the line of intersection must satisfy both the equations (Fig 11.17).

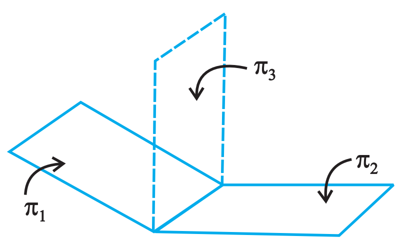


Fig 11.17

If $\vec{t}$ is the position vector of a point on the line, then

$$\vec{t}\cdot\hat{n}_1 = d_1 \text{ and } \vec{t}\cdot\hat{n}_2 = d_2$$

Therefore, for all real values of λ, we have

$$\vec{t}\cdot(\hat{n}_1+\lambda\hat{n}_2) = d_1+\lambda d_2$$

Since $\vec{t}$ is arbitrary, it satisfies for any point on the line.

Hence, the equation $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$ represents a plane π_3 which is such that if any vector $\vec{r}$ satisfies both the equations π_1 and π_2, it also satisfies the equation π_3 i.e., any plane passing through the intersection of the planes

$$\vec{r}\cdot\vec{n}_1 = d_1 \text{ and } \vec{r}\cdot\vec{n}_2 = d_2$$

has the equation $$\boldsymbol{\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2) = d_1 + \lambda d_2} \quad \ldots (1)$$

Cartesian form

In Cartesian system, let

$$\vec{n}_1 = A_1\hat{i}+B_2\hat{j}+C_1\hat{k}$$

$$\vec{n}_2 = A_2\hat{i}+B_2\hat{j}+C_2\hat{k}$$

and $$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

Then (1) becomes

$$x(A_1 + \lambda A_2) + y(B_1 + \lambda B_2) + z(C_1 + \lambda C_2) = d_1 + \lambda d_2$$

or $$\mathbf{(A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0} \quad \ldots (2)$$

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

Example 20 Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=6$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}+4\hat{k})=-5$, and the point (1, 1, 1).

Solution Here, $\vec{n}_1=\hat{i}+\hat{j}+\hat{k}$ and $\vec{n}_2 = 2\hat{i}+3\hat{j}+4\hat{k}$;

and $d_1 = 6$ and $d_2 = -5$

Hence, using the relation $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$, we get

$$\vec{r}\cdot[\hat{i}+\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}+4\hat{k})] = 6-5\lambda$$

or $$\vec{r}\cdot[(1+2\lambda)\hat{i}+(1+3\lambda)\hat{j}+(1+4\lambda)\hat{k}] = 6-5\lambda \quad \ldots (1)$$

where, λ is some real number.

Taking $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot [(1+2\lambda)\hat{i} + (1+3\lambda)\hat{j} + (1+4\lambda)\hat{k}] = 6 - 5\lambda$$

or $(1 + 2\lambda)\, x + (1 + 3\lambda)\, y + (1 + 4\lambda)\, z = 6 - 5\lambda$

or $(x + y + z - 6) + \lambda\ (2x + 3y + 4z + 5) = 0$ ... (2)

Given that the plane passes through the point (1,1,1), it must satisfy (2), i.e.

$$(1 + 1 + 1 - 6) + \lambda\,(2 + 3 + 4 + 5) = 0$$

or $\lambda = \dfrac{3}{14}$

Putting the values of λ in (1), we get

$$\vec{r}\left[\left(1+\frac{3}{7}\right)\hat{i} + \left(1+\frac{9}{14}\right)\hat{j} + \left(1+\frac{6}{7}\right)\hat{k}\right] = 6 - \frac{15}{14}$$

or $$\vec{r}\left(\frac{10}{7}\hat{i} + \frac{23}{14}\hat{j} + \frac{13}{7}\hat{k}\right) = \frac{69}{14}$$

or $$\vec{r} \cdot (20\hat{i} + 23\hat{j} + 26\hat{k}) = 69$$

which is the required vector equation of the plane.

11.7 Coplanarity of Two Lines

Let the given lines be

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \quad \text{... (1)}$$

and $$\vec{r} = \vec{a}_2 + \mu \vec{b}_2 \quad \text{... (2)}$$

The line (1) passes through the point, say A, with position vector $\vec{a}_1$ and is parallel to $\vec{b}_1$. The line (2) passes through the point, say B with position vector $\vec{a}_2$ and is parallel to $\vec{b}_2$.

Thus, $\overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$

The given lines are coplanar if and only if $\overrightarrow{AB}$ is perpendicular to $\vec{b}_1 \times \vec{b}_2$.

i.e. $\overrightarrow{AB} . (\vec{b}_1 \times \vec{b}_2) = 0$ or $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$

Cartesian form

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the points A and B respectively.

Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of $\vec{b}_1$ and $\vec{b}_2$, respectively. Then

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \text{ and } \vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

The given lines are coplanar if and only if $\overrightarrow{AB} \cdot (\vec{b}_1 \times \vec{b}_2) = 0$. In the cartesian form, it can be expressed as

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \qquad \ldots (4)$$

Example 21 Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and } \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \text{ are coplanar.}$$

Solution Here, $x_1 = -3$, $y_1 = 1$, $z_1 = 5$, $a_1 = -3$, $b_1 = 1$, $c_1 = 5$
$x_2 = -1$, $y_2 = 2$, $z_2 = 5$, $a_2 = -1$, $b_2 = 2$, $c_2 = 5$

Now, consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

Therefore, lines are coplanar.

11.8 Angle between Two Planes

Definition 2 The angle between two planes is defined as the angle between their normals (Fig 11.18 (a)). Observe that if θ is an angle between the two planes, then so is 180 – θ (Fig 11.18 (b)). We shall take the acute angle as the angles between two planes.

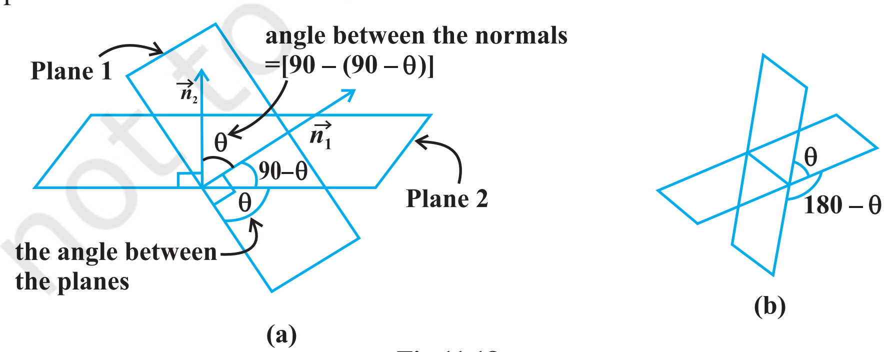


Fig 11.18

If $\vec{n}_1$ and $\vec{n}_2$ are normals to the planes and θ be the angle between the planes

$$\vec{r} \cdot \vec{n}_1 = d_1 \text{ and } \vec{r} \,.\, \vec{n}_2 = d_2 .$$

Then θ is the angle between the normals to the planes drawn from some common point.

We have, $$\cos\theta = \left|\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right|$$

Note The planes are perpendicular to each other if $\vec{n}_1 . \vec{n}_2 = 0$ and parallel if $\vec{n}_1$ is parallel to $\vec{n}_2$.

Cartesian form Let θ be the angle between the planes,

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \text{ and } A_2 x + B_2 y + C_2 z + D_2 = 0$$

The direction ratios of the normal to the planes are A_1, B_1, C_1 and A_2, B_2, C_2 respectively.

Therefore, $$\cos\theta = \left|\frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\;\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

Note

1. If the planes are at right angles, then θ = 90° and so cos θ = 0. Hence, cos θ = $A_1A_2 + B_1B_2 + C_1C_2 = 0$.
2. If the planes are parallel, then $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.

Example 22 Find the angle between the two planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$ using vector method.

Solution The angle between two planes is the angle between their normals. From the equation of the planes, the normal vectors are

$$\vec{N}_1 = 2\hat{i} + \hat{j} - 2\hat{k} \text{ and } \vec{N}_2 = 3\hat{i} - 6\hat{j} - 2\hat{k}$$

Therefore $$\cos\theta = \left|\frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|}\right| = \left|\frac{(2\hat{i} + \hat{j} - 2\hat{k})\cdot(3\hat{i} - 6\hat{j} - 2\hat{k})}{\sqrt{4+1+4}\;\sqrt{9+36+4}}\right| = \left(\frac{4}{21}\right)$$

Hence $$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

Example 23 Find the angle between the two planes $3x - 6y + 2z = 7$ and $2x + 2y - 2z = 5$.

Solution Comparing the given equations of the planes with the equations

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \text{ and } A_2 x + B_2 y + C_2 z + D_2 = 0$$

We get
$$A_1 = 3, B_1 = -6, C_1 = 2$$
$$A_2 = 2, B_2 = 2, C_2 = -2$$

$$\cos\theta = \left| \frac{3 \times 2 + (-6)(2) + (2)(-2)}{\sqrt{(3^2 + (-6)^2 + (-2)^2)}\sqrt{(2^2 + 2^2 + (-2)^2)}} \right|$$

$$= \left| \frac{-10}{7 \times 2\sqrt{3}} \right| = \frac{5}{7\sqrt{3}} = \frac{5\sqrt{3}}{21}$$

Therefore, $\theta = \cos^{-1}\left(\frac{5\sqrt{3}}{21}\right)$

11.9 Distance of a Point from a Plane

Vector form

Consider a point P with position vector $\vec{a}$ and a plane π_1 whose equation is $\vec{r} \cdot \hat{n} = d$ (Fig 11.19).

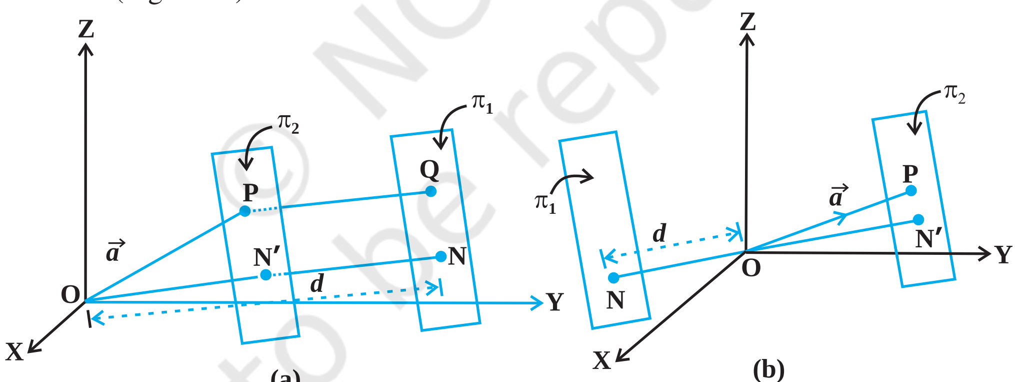


Fig 11.19

Consider a plane π_2 through P parallel to the plane π_1. The unit vector normal to π_2 is $\hat{n}$. Hence, its equation is $(\vec{r} - \vec{a}) \cdot \hat{n} = 0$

i.e., $$\vec{r} \cdot \hat{n} = \vec{a} \cdot \hat{n}$$

Thus, the distance ON′ of this plane from the origin is $|\vec{a} \cdot \hat{n}|$. Therefore, the distance PQ from the plane π_1 is (Fig. 11.21 (a))

i.e., $$\text{ON} - \text{ON}' = |d - \vec{a} \cdot \hat{n}|$$

which is the length of the perpendicular from a point to the given plane.

We may establish the similar results for (Fig 11.19 (b)).

Note

1. If the equation of the plane π_2 is in the form $\vec{r} \cdot \vec{N} = d$, where $\vec{N}$ is normal to the plane, then the perpendicular distance is $\dfrac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$.

2. The length of the perpendicular from origin O to the plane $\vec{r} \cdot \vec{N} = d$ is $\dfrac{|d|}{|\vec{N}|}$ (since $\vec{a} = 0$).

Cartesian form

Let $P(x_1, y_1, z_1)$ be the given point with position vector $\vec{a}$ and

$$Ax + By + Cz = D$$

be the Cartesian equation of the given plane. Then

$$\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$

$$\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

Hence, from Note 1, the perpendicular from P to the plane is

$$\left| \frac{(x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}) \cdot (A\hat{i} + B\hat{j} + C\hat{k}) - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

$$= \left| \frac{A x_1 + B y_1 + C z_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

Example 24 Find the distance of a point (2, 5, – 3) from the plane

$$\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$$

Solution Here, $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$, $\vec{N} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ and $d = 4$.

Therefore, the distance of the point (2, 5, – 3) from the given plane is

$$\frac{|(2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) - 4|}{|6\hat{i} - 3\hat{j} + 2\hat{k}|} = \frac{|12 - 15 - 6 - 4|}{\sqrt{36 + 9 + 4}} = \frac{13}{7}$$

11.10 Angle between a Line and a Plane

Definition 3 The angle between a line and a plane is the complement of the angle between the line and normal to the plane (Fig 11.20).

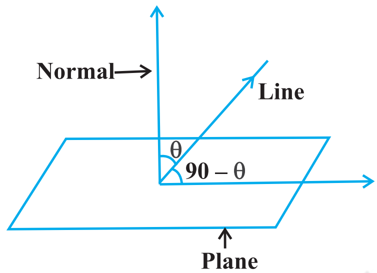


Fig 11.20

Vector form If the equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$ and the equation of the plane is $\vec{r} \cdot \vec{n} = d$. Then the angle θ between the line and the normal to the plane is

$$\cos\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}\right|$$

and so the angle φ between the line and the plane is given by 90 – θ, i.e.,

$$\sin(90-\theta) = \cos\theta$$

i.e.
$$\sin\phi = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\ |\vec{n}|}\right| \quad \text{or } \phi = \sin^{-1}\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

Example 25 Find the angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$$

and the plane $10x + 2y - 11z = 3$.

Solution Let θ be the angle between the line and the normal to the plane. Converting the given equations into vector form, we have

$$\vec{r} = (-\hat{i} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and
$$\vec{r}\cdot(10\hat{i} + 2\hat{j} - 11\hat{k}) = 3$$

Here
$$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k} \quad \text{and} \quad \vec{n} = 10\hat{i} + 2\hat{j} - 11\hat{k}$$

$$\sin\phi = \left|\frac{(2\hat{i} + 3\hat{j} + 6\hat{k})\cdot(10\hat{i} + 2\hat{j} - 11\hat{k})}{\sqrt{2^2 + 3^2 + 6^2}\ \sqrt{10^2 + 2^2 + 11^2}}\right|$$

$$= \left|\frac{-40}{7\times 15}\right| = \left|\frac{-8}{21}\right| = \frac{8}{21} \quad \text{or } \phi = \sin^{-1}\left(\frac{8}{21}\right)$$

EXERCISE 11.3

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.
   (a) $z = 2$ (b) $x + y + z = 1$
   (c) $2x + 3y - z = 5$ (d) $5y + 8 = 0$
2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.
3. Find the Cartesian equation of the following planes:
   (a) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$ (b) $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$
   (c) $\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$
4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.
   (a) $2x + 3y + 4z - 12 = 0$ (b) $3y + 4z - 6 = 0$
   (c) $x + y + z = 1$ (d) $5y + 8 = 0$
5. Find the vector and cartesian equations of the planes
   (a) that passes through the point $(1, 0, -2)$ and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.
   (b) that passes through the point $(1,4, 6)$ and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.
6. Find the equations of the planes that passes through three points.
   (a) $(1, 1, -1)$, $(6, 4, -5)$, $(-4, -2, 3)$
   (b) $(1, 1, 0)$, $(1, 2, 1)$, $(-2, 2, -1)$
7. Find the intercepts cut off by the plane $2x + y - z = 5$.
8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.
9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point $(2, 2, 1)$.
10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$, $\vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point $(2, 1, 3)$.
11. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

12. Find the angle between the planes whose vector equations are

$$\vec{r}\cdot(2\,\hat{i}+2\,\hat{j}-3\,\hat{k})=5 \text{ and } \vec{r}\cdot(3\,\hat{i}-3\,\hat{j}+5\,\hat{k})=3.$$

13. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

	Point	Plane
(a)	(0, 0, 0)	$3x - 4y + 12z = 3$
(b)	(3, – 2, 1)	$2x - y + 2z + 3 = 0$
(c)	(2, 3, – 5)	$x + 2y - 2z = 9$
(d)	(– 6, 0, 0)	$2x - 3y + 6z - 2 = 0$

Miscellaneous Examples

Example 26 A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Solution A cube is a rectangular parallelopiped having equal length, breadth and height. Let OADBFEGC be the cube with each side of length a units. (Fig 11.21)

The four diagonals are OE, AF, BG and CD.

The direction cosines of the diagonal OE which is the line joining two points O and E are

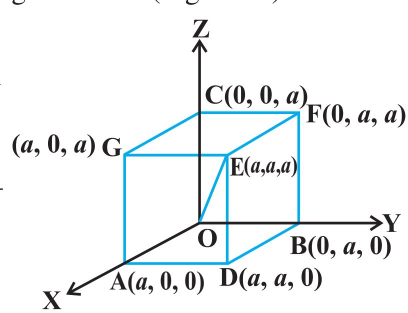


Fig 11.21

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

i.e., $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

Similarly, the direction cosines of AF, BG and CD are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$, respectively.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OE, AF, BG, CD, respectively. Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l + m + n);\ \cos\beta = \frac{1}{\sqrt{3}}(-l + m + n);$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l - m + n);\ \cos\delta = \frac{1}{\sqrt{3}}(l + m - n) \qquad \text{(Why?)}$$

Squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\left[(l + m + n)^2 + (-l + m + n)^2\right] + (l - m + n)^2 + (l + m - n)^2]$$

$$= \frac{1}{3}\left[4(l^2 + m^2 + n^2)\right] = \frac{4}{3} \qquad (\text{as } l^2 + m^2 + n^2 = 1)$$

Example 27 Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.

Solution The equation of the plane containing the given point is

$$A(x - 1) + B(y + 1) + C(z - 2) = 0 \qquad \ldots (1)$$

Applying the condition of perpendicularly to the plane given in (1) with the planes

$$2x + 3y - 2z = 5 \text{ and } x + 2y - 3z = 8, \text{ we have}$$

$$2A + 3B - 2C = 0 \text{ and } A + 2B - 3C = 0$$

Solving these equations, we find A = – 5C and B = 4C. Hence, the required equation is

$$-5C(x - 1) + 4C(y + 1) + C(z - 2) = 0$$

i.e. $\quad 5x - 4y - z = 7$

Example 28 Find the distance between the point P(6, 5, 9) and the plane determined by the points A (3, – 1, 2), B (5, 2, 4) and C(– 1, – 1, 6).

Solution Let A, B, C be the three points in the plane. D is the foot of the perpendicular drawn from a point P to the plane. PD is the required distance to be determined, which is the projection of $\overrightarrow{AP}$ on $\overrightarrow{AB} \times \overrightarrow{AC}$.

Hence, PD = the dot product of $\overrightarrow{AP}$ with the unit vector along $\overrightarrow{AB} \times \overrightarrow{AC}$.

So $$\overrightarrow{AP} = 3\hat{i} + 6\hat{j} + 7\hat{k}$$

and $$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 12\hat{i} - 16\hat{j} + 12\hat{k}$$

Unit vector along $$\overrightarrow{AB} \times \overrightarrow{AC} = \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

Hence $$PD = (3\hat{i} + 6\hat{j} + 7\hat{k}) \cdot \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

$$= \frac{3\sqrt{34}}{17}$$

Alternatively, find the equation of the plane passing through A, B and C and then compute the distance of the point P from the plane.

Example 29 Show that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$$

and $$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$$ are coplanar.

Solution

Here
$$x_1 = a - d \qquad x_2 = b - c$$
$$y_1 = a \qquad y_2 = b$$
$$z_1 = a + d \qquad z_2 = b + c$$
$$a_1 = \alpha - \delta \qquad a_2 = \beta - \gamma$$
$$b_1 = \alpha \qquad b_2 = \beta$$
$$c_1 = \alpha + \delta \qquad c_2 = \beta + \gamma$$

Now consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b-c-a+d & b-a & b+c-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}$$

Adding third column to the first column, we get

$$2\begin{vmatrix} b-a & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

Since the first and second columns are identical. Hence, the given two lines are coplanar.

Example 30 Find the coordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) crosses the XY-plane.

Solution The vector equation of the line through the points A and B is

$$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda[(5-3)\hat{i} + (1-4)\hat{j} + (6-1)\hat{k}]$$

i.e. $$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda(2\hat{i} - 3\hat{j} + 5\hat{k}) \qquad \dots (1)$$

Let P be the point where the line AB crosses the XY-plane. Then the position vector of the point P is of the form $x\hat{i} + y\hat{j}$.

This point must satisfy the equation (1). (Why ?)

i.e. $$x\hat{i} + y\hat{j} = (3+2\lambda)\hat{i} + (4-3\lambda)\hat{j} + (1+5\lambda)\hat{k}$$

Equating the like coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$x = 3 + 2\lambda$$
$$y = 4 - 3\lambda$$
$$0 = 1 + 5\lambda$$

Solving the above equations, we get

$$x = \frac{13}{5} \text{ and } y = \frac{23}{5}$$

Hence, the coordinates of the required point are $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

Miscellaneous Exercise on Chapter 11

1. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, – 1), (4, 3, – 1).

2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $m_1 n_2 - m_2 n_1,\ n_1 l_2 - n_2 l_1,\ l_1 m_2 - l_2 m_1$

3. Find the angle between the lines whose direction ratios are a, b, c and $b - c, c - a, a - b$.

4. Find the equation of a line parallel to x-axis and passing through the origin.

5. If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

6. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

7. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane $\vec{r}\,.\,(\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.

8. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 2$.

9. Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$.

10. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4,1) crosses the YZ-plane.

11. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX-plane.

12. Find the coordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane $2x + y + z = 7$.

13. Find the equation of the plane passing through the point (– 1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

14. If the points (1, 1, p) and (– 3, 0, 1) be equidistant from the plane $\vec{r}\cdot(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, then find the value of p.

15. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r}\cdot(2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis.

16. If O be the origin and the coordinates of P be (1, 2, – 3), then find the equation of the plane passing through P and perpendicular to OP.

17. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}\cdot(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$, $\vec{r}\cdot(2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\vec{r}\cdot(5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$.

18. Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$.

19. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}\cdot(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}\cdot(3\hat{i} + \hat{j} + \hat{k}) = 6$.

20. Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

21. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Choose the correct answer in Exercises 22 and 23.

22. Distance between the two planes: $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is

(A) 2 units (B) 4 units (C) 8 units (D) $\frac{2}{\sqrt{29}}$ units

23. The planes: $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

(A) Perpendicular (B) Parallel

(C) intersect y-axis (D) passes through $\left(0, 0, \frac{5}{4}\right)$

Summary

- **Direction cosines of a line** are the cosines of the angles made by the line with the positive directions of the coordinate axes.
- If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.
- Direction cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

  $$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

  where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- **Direction ratios of a line** are the numbers which are proportional to the direction cosines of a line.
- If l, m, n are the direction cosines and a, b, c are the direction ratios of a line

then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} \; ; m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \; ; n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- **Skew lines** are lines in space which are neither parallel nor intersecting. They lie in different planes.
- **Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the acute angle between the two lines; then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$.
- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- The vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$.
- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.
- If θ is the acute angle between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$, then

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1|\,|\vec{b}_2|} \right|$$

- If $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$

  are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$.

- Shortest distance between two skew lines is the line segment perpendicular to both the lines.
- Shortest distance between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is

$$\left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

- Shortest distance between the lines: $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and

$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

- Distance between parallel lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is

$$\left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

- In the vector form, equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is $\vec{r} \cdot \hat{n} = d$.
- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}) . \vec{N} = 0$.
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

- Equation of a plane passing through three non collinear points (x_1, y_1, z_1),

(x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Vector equation of a plane that contains three non collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$
- Equation of a plane that cuts the coordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- Vector equation of a plane that passes through the intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$, where λ is any nonzero constant.
- Cartesian equation of a plane that passes through the intersection of two given planes $A_1 x + B_1 y + C_1 z + D_1 = 0$ and $A_2 x + B_2 y + C_2 z + D_2 = 0$ is $(A_1 x + B_1 y + C_1 z + D_1) + \lambda(A_2 x + B_2 y + C_2 z + D_2) = 0$.
- Two lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ are coplanar if

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$$

- In the cartesian form two lines $= \dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ *and* $\dfrac{x - x_2}{a_2}$

$$= \frac{y - y_2}{b_2} = \frac{z - z_2}{C_2} \text{ are coplanar if } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

- In the vector form, if θ is the angle between the two planes, $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then $\theta = \cos^{-1} \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$.
- The angle ϕ between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \hat{n} = d$ is

$$\sin \phi = \left| \frac{\vec{b} \cdot \hat{n}}{|\vec{b}||\hat{n}|} \right|$$

- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2 x + B_2 y + C_2 z + D_2 = 0$ is given by

$$\cos \theta = \left| \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

- The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r} \cdot \hat{n} = d$ is $| d - \vec{a} \cdot \hat{n}|$
- The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is

$$\left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$

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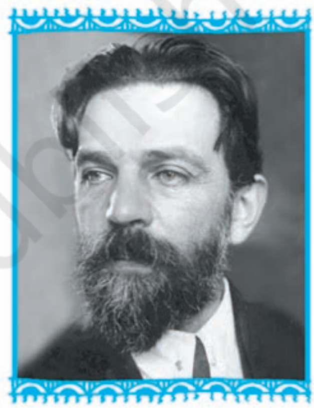
Chapter 12

LINEAR PROGRAMMING

❖ *The mathematical experience of the student is incomplete if he never had the opportunity to solve a problem invented by himself. – G. POLYA* ❖

12.1 Introduction

In earlier classes, we have discussed systems of linear equations and their applications in day to day problems. In Class XI, we have studied linear inequalities and systems of linear inequalities in two variables and their solutions by graphical method. Many applications in mathematics involve systems of inequalities/equations. In this chapter, we shall apply the systems of linear inequalities/equations to solve some real life problems of the type as given below:



L. Kantorovich

A furniture dealer deals in only two items–tables and chairs. He has Rs 50,000 to invest and has storage space of at most 60 pieces. A table costs Rs 2500 and a chair Rs 500. He estimates that from the sale of one table, he can make a profit of Rs 250 and that from the sale of one chair a profit of Rs 75. He wants to know how many tables and chairs he should buy from the available money so as to maximise his total profit, assuming that he can sell all the items which he buys.

Such type of problems which seek to maximise (or, minimise) profit (or, cost) form a general class of problems called **optimisation problems**. Thus, an optimisation problem may involve finding maximum profit, minimum cost, or minimum use of resources etc.

A special but a very important class of optimisation problems is **linear programming problem.** The above stated optimisation problem is an example of linear programming problem. Linear programming problems are of much interest because of their wide applicability in industry, commerce, management science etc.

In this chapter, we shall study some linear programming problems and their solutions by graphical method only, though there are many other methods also to solve such problems.

12.2 Linear Programming Problem and its Mathematical Formulation

We begin our discussion with the above example of furniture dealer which will further lead to a mathematical formulation of the problem in two variables. In this example, we observe

(i) The dealer can invest his money in buying tables or chairs or combination thereof. Further he would earn different profits by following different investment strategies.

(ii) There are certain **overriding conditions** or **constraints** viz., his investment is limited to a **maximum** of Rs 50,000 and so is his storage space which is for a maximum of 60 pieces.

Suppose he decides to buy tables only and no chairs, so he can buy 50000 ÷ 2500, i.e., 20 tables. His profit in this case will be Rs (250 × 20), i.e., **Rs 5000.**

Suppose he chooses to buy chairs only and no tables. With his capital of Rs 50,000, he can buy 50000 ÷ 500, i.e. 100 chairs. But he can store only 60 pieces. Therefore, he is forced to buy only 60 chairs which will give him a total profit of Rs (60 × 75), i.e., **Rs 4500**.

There are many other possibilities, for instance, he may choose to buy 10 tables and 50 chairs, as he can store only 60 pieces. Total profit in this case would be Rs (10 × 250 + 50 × 75), i.e., **Rs 6250** and so on.

We, thus, find that the dealer can invest his money in different ways and he would earn different profits by following different investment strategies.

Now the problem is : How should he invest his money in order to get maximum profit? To answer this question, let us try to formulate the problem mathematically.

12.2.1 *Mathematical formulation of the problem*

Let x be the number of tables and y be the number of chairs that the dealer buys. Obviously, x and y must be non-negative, i.e.,

$$\left.\begin{array}{l} x \geq 0 \quad \text{... (1)} \\ y \geq 0 \quad \text{... (2)} \end{array}\right\} \text{(Non-negative constraints)}$$

The dealer is constrained by the maximum amount he can invest (Here it is Rs 50,000) and by the maximum number of items he can store (Here it is 60).

Stated mathematically,

$$2500x + 500y \leq 50000 \text{ (investment constraint)}$$

or
$$5x + y \leq 100 \qquad \text{... (3)}$$

and
$$x + y \leq 60 \text{ (storage constraint)} \qquad \text{... (4)}$$

The dealer wants to invest in such a way so as to maximise his profit, say, Z which stated as a function of x and y is given by

Z = 250x + 75y (called *objective function*) ... (5)

Mathematically, the given problems now reduces to:

Maximise Z = 250x + 75y

subject to the constraints:

$$5x + y \leq 100$$
$$x + y \leq 60$$
$$x \geq 0, \; y \geq 0$$

So, we have to maximise the linear function Z subject to certain conditions determined by a set of linear inequalities with variables as non-negative. There are also some other problems where we have to minimise a linear function subject to certain conditions determined by a set of linear inequalities with variables as non-negative. Such problems are called **Linear Programming Problems.**

Thus, a Linear Programming Problem is one that is concerned with finding the **optimal value** (maximum or minimum value) of a linear function (called **objective function**) of several variables (say x and y), subject to the conditions that the variables are **non-negative** and satisfy a set of linear inequalities (called **linear constraints).** The term **linear** implies that all the mathematical relations used in the problem are **linear relations** while the term programming refers to the method of determining a particular **programme** or plan of action.

Before we proceed further, we now formally define some terms (which have been used above) which we shall be using in the linear programming problems:

Objective function Linear function Z = $ax + by$, where a, b are constants, which has to be maximised or minimized is called a linear **objective function.**

In the above example, Z = 250x + 75y is a linear objective function. Variables x and y are called **decision variables**.

Constraints The linear inequalities or equations or restrictions on the variables of a linear programming problem are called **constraints**. The conditions $x \geq 0$, $y \geq 0$ are called non-negative restrictions. In the above example, the set of inequalities (1) to (4) are **constraints**.

Optimisation problem A problem which seeks to maximise or minimise a linear function (say of two variables x and y) subject to certain constraints as determined by a set of linear inequalities is called an **optimisation problem**. Linear programming problems are special type of optimisation problems. The above problem of investing a given sum by the dealer in purchasing chairs and tables is an example of an optimisation problem as well as of a linear programming problem.

We will now discuss how to find solutions to a linear programming problem. In this chapter, we will be concerned only with the graphical method.

12.2.2 *Graphical method of solving linear programming problems*

In Class XI, we have learnt how to graph a system of linear inequalities involving two variables x and y and to find its solutions graphically. Let us refer to the problem of investment in tables and chairs discussed in Section 12.2. We will now solve this problem graphically. Let us graph the constraints stated as linear inequalities:

$$5x + y \le 100 \quad \text{... (1)}$$

$$x + y \le 60 \quad \text{... (2)}$$

$$x \ge 0 \quad \text{... (3)}$$

$$y \ge 0 \quad \text{... (4)}$$

The graph of this system (shaded region) consists of the points common to all half planes determined by the inequalities (1) to (4) (Fig 12.1). Each point in this region represents a **feasible choice** open to the dealer for investing in tables and chairs. The region, therefore, is called the **feasible region** for the problem. Every point of this region is called a **feasible solution** to the problem. Thus, we have,

Feasible region The common region determined by all the constraints including non-negative constraints $x, y \ge 0$ of a linear programming problem is called the **feasible region** (or solution region) for the problem. In Fig 12.1, the region OABC (shaded) is the feasible region for the problem. The region other than feasible region is called an **infeasible region**.

Feasible solutions Points within and on the boundary of the feasible region represent feasible solutions of the constraints. In Fig 12.1, every point within and on the boundary of the feasible region OABC represents feasible solution to the problem. For example, the point (10, 50) is a feasible solution of the problem and so are the points (0, 60), (20, 0) etc.

Any point outside the feasible region is called an **infeasible solution.** For example, the point (25, 40) is an infeasible solution of the problem.

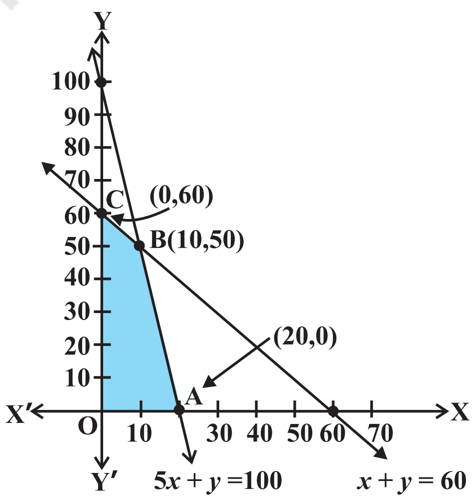


Fig 12.1

Optimal (feasible) solution: Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**

Now, we see that every point in the feasible region OABC satisfies all the constraints as given in (1) to (4), and since there are **infinitely many points**, it is not evident how we should go about finding a point that gives a maximum value of the objective function Z = 250x + 75y. To handle this situation, we use the following theorems which are fundamental in solving linear programming problems. The proofs of these theorems are beyond the scope of the book.

Theorem 1 Let R be the feasible region (convex polygon) for a linear programming problem and let Z = $ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point* (vertex) of the feasible region.

Theorem 2 Let R be the feasible region for a linear programming problem, and let Z = $ax + by$ be the objective function. If R is **bounded****, then the objective function Z has both a **maximum** and a **minimum** value on R and each of these occurs at a corner point (vertex) of R.

Remark If R is **unbounded**, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R. (By Theorem 1).

In the above example, the corner points (vertices) of the bounded (feasible) region are: O, A, B and C and it is easy to find their coordinates as (0, 0), (20, 0), (10, 50) and (0, 60) respectively. Let us now compute the values of Z at these points.

We have

Vertex of the Feasible Region	Corresponding value of Z (in Rs)	
O (0,0)	0	
C (0,60)	4500	
B (10,50)	**6250** ←	Maximum
A (20,0)	5000	

* A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.

** A feasible region of a system of linear inequalities is said to be bounded if it can be enclosed within a circle. Otherwise, it is called unbounded. Unbounded means that the feasible region does extend indefinitely in any direction.

We observe that the maximum profit to the dealer results from the investment strategy (10, 50), i.e. buying 10 tables and 50 chairs.

This method of solving linear programming problem is referred as **Corner Point Method**. The method comprises of the following steps:

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
2. Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m, respectively denote the largest and smallest values of these points.
3. (i) When the feasible region is **bounded**, M and m are the maximum and minimum values of Z.

   (ii) In case, the feasible region is **unbounded**, we have:
4. (a) M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

   (b) Similarly, m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

We will now illustrate these steps of Corner Point Method by considering some examples:

Example 1 Solve the following linear programming problem graphically:

Maximise $Z = 4x + y$ ... (1)

subject to the constraints:

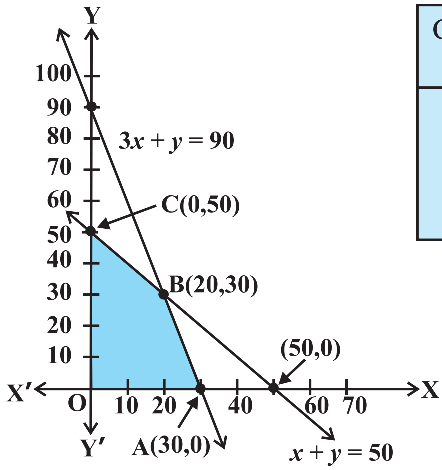
$$x + y \leq 50 \quad \text{... (2)}$$

$$3x + y \leq 90 \quad \text{... (3)}$$

$$x \geq 0, y \geq 0 \quad \text{... (4)}$$

Solution The shaded region in Fig 12.2 is the feasible region determined by the system of constraints (2) to (4). We observe that the feasible region OABC is **bounded.** So, we now use Corner Point Method to determine the maximum value of Z.

The coordinates of the corner points O, A, B and C are (0, 0), (30, 0), (20, 30) and (0, 50) respectively. Now we evaluate Z at each corner point.



Corner Point	Corresponding value of Z	
(0, 0)	0	
(30, 0)	**120** ←	Maximum
(20, 30)	110	
(0, 50)	50	

Fig 12.2

Hence, maximum value of Z is 120 at the point (30, 0).

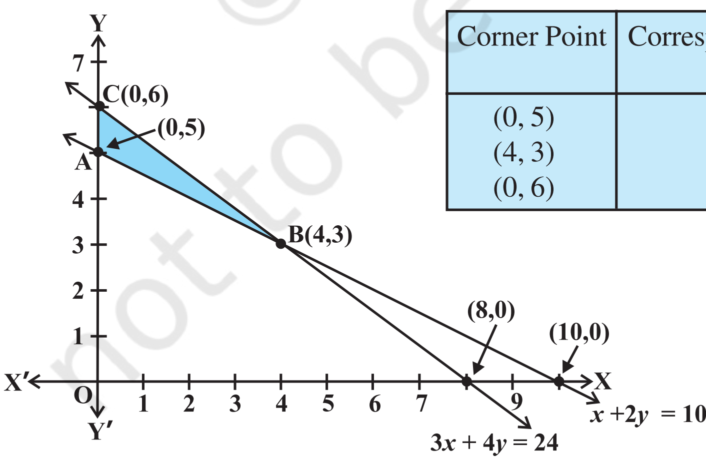
Example 2 Solve the following linear programming problem graphically:

Minimise $Z = 200\,x + 500\,y$ ... (1)

subject to the constraints:

$$x + 2y \geq 10 \quad \text{... (2)}$$
$$3x + 4y \leq 24 \quad \text{... (3)}$$
$$x \geq 0,\ y \geq 0 \quad \text{... (4)}$$

Solution The shaded region in Fig 12.3 is the feasible region ABC determined by the system of constraints (2) to (4), which is **bounded**. The coordinates of corner points



Corner Point	Corresponding value of Z	
(0, 5)	2500	
(4, 3)	**2300** ←	Minimum
(0, 6)	3000	

Fig 12.3

A, B and C are (0,5), (4,3) and (0,6) respectively. Now we evaluate Z $= 200x + 500y$ at these points.

Hence, minimum value of Z is 2300 attained at the point (4, 3)

Example 3 Solve the following problem graphically:

Minimise and Maximise $Z = 3x + 9y$ ... (1)

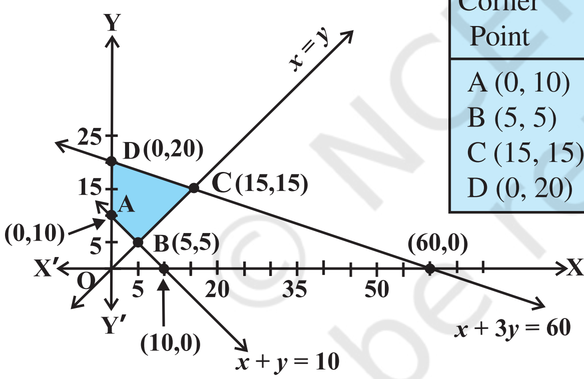
subject to the constraints: $x + 3y \leq 60$ ... (2)

$x + y \geq 10$ ... (3)

$x \leq y$ ... (4)

$x \geq 0, y \geq 0$ ... (5)

Solution First of all, let us graph the feasible region of the system of linear inequalities (2) to (5). The feasible region ABCD is shown in the Fig 12.4. Note that the region is bounded. The coordinates of the corner points A, B, C and D are (0, 10), (5, 5), (15,15) and (0, 20) respectively.



Corner Point	Corresponding value of $Z = 3x + 9y$	
A (0, 10)	90	
B (5, 5)	**60** ←	Minimum
C (15, 15)	**180**	Maximum (Multiple optimal solutions)
D (0, 20)	**180**	

Fig 12.4

We now find the minimum and maximum value of Z. From the table, we find that the minimum value of Z is 60 at the point B (5, 5) of the feasible region.

The maximum value of Z on the feasible region occurs at the two corner points C (15, 15) and D (0, 20) and it is 180 in each case.

Remark Observe that in the above example, the problem has multiple optimal solutions at the corner points C and D, i.e. the both points produce same maximum value 180. In such cases, you can see that every point on the line segment CD joining the two corner points C and D also give the same maximum value. Same is also true in the case if the two points produce same minimum value.

Example 4 Determine graphically the minimum value of the objective function

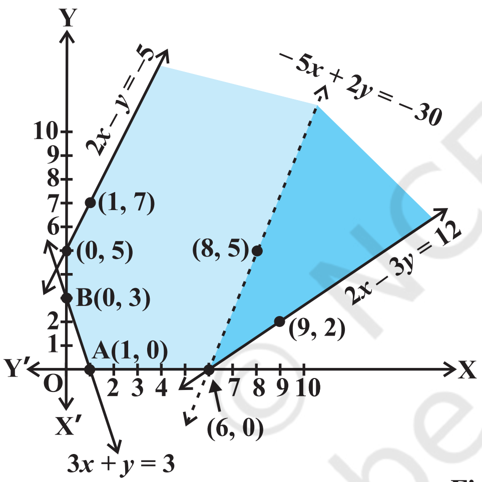
$$Z = -50x + 20y \qquad \text{... (1)}$$

subject to the constraints:

$$2x - y \geq -5 \qquad \text{... (2)}$$
$$3x + y \geq 3 \qquad \text{... (3)}$$
$$2x - 3y \leq 12 \qquad \text{... (4)}$$
$$x \geq 0, y \geq 0 \qquad \text{... (5)}$$

Solution First of all, let us graph the feasible region of the system of inequalities (2) to (5). The feasible region (shaded) is shown in the Fig 12.5. Observe that the feasible region is **unbounded**.

We now evaluate Z at the corner points.



Corner Point	$Z = -50x + 20y$	
(0, 5)	100	
(0, 3)	**60**	
(1, 0)	**–50**	
(6, 0)	**–300** ←	smallest

Fig 12.5

From this table, we find that – 300 is the smallest value of Z at the corner point (6, 0). Can we say that minimum value of Z is – 300? Note that if the region would have been bounded, this smallest value of Z is the minimum value of Z (Theorem 2). But here we see that the feasible region is unbounded. Therefore, – 300 may or may not be the minimum value of Z. To decide this issue, we graph the inequality

$$-50x + 20y < -300 \text{ (see Step 3(ii) of corner Point Method.)}$$

i.e., $$-5x + 2y < -30$$

and check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then –300 will not be the minimum value of Z. Otherwise, –300 will be the minimum value of Z.

As shown in the Fig 12.5, it has common points. Therefore, Z = –50 x + 20 y has no minimum value subject to the given constraints.

In the above example, can you say whether $z = -50x + 20y$ has the maximum value 100 at (0,5)? For this, check whether the graph of $-50x + 20y > 100$ has points in common with the feasible region. (Why?)

Example 5 Minimise $Z = 3x + 2y$

subject to the constraints:

$$x + y \geq 8 \quad \ldots (1)$$

$$3x + 5y \leq 15 \quad \ldots (2)$$

$$x \geq 0, y \geq 0 \quad \ldots (3)$$

Solution Let us graph the inequalities (1) to (3) (Fig 12.6). Is there any feasible region? Why is so?

From Fig 12.6, you can see that there is no point satisfying all the constraints simultaneously. Thus, the problem is having no feasible region and hence no feasible solution.

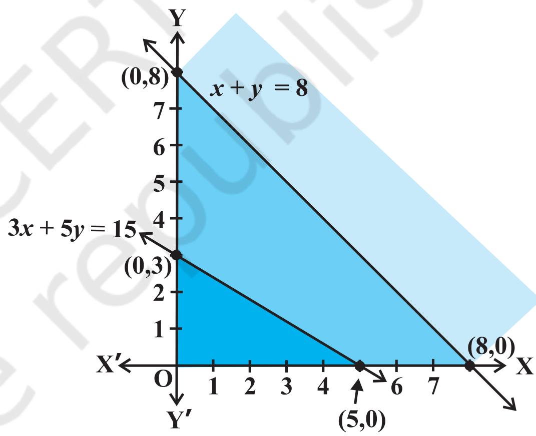


Fig 12.6

Remarks From the examples which we have discussed so far, we notice some general features of linear programming problems:

(i) The feasible region is always a convex region.

(ii) The maximum (or minimum) solution of the objective function occurs at the vertex (corner) of the feasible region. If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.

EXERCISE 12.1

Solve the following Linear Programming Problems graphically:

1. Maximise $Z = 3x + 4y$

subject to the constraints : $x + y \leq 4$, $x \geq 0$, $y \geq 0$.

2. Minimise $Z = -3x + 4y$

   subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$.

3. Maximise $Z = 5x + 3y$

   subject to $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x \geq 0$, $y \geq 0$.

4. Minimise $Z = 3x + 5y$

   such that $x + 3y \geq 3$, $x + y \geq 2$, $x, y \geq 0$.

5. Maximise $Z = 3x + 2y$

   subject to $x + 2y \leq 10$, $3x + y \leq 15$, $x, y \geq 0$.

6. Minimise $Z = x + 2y$

   subject to $2x + y \geq 3$, $x + 2y \geq 6$, $x, y \geq 0$.

Show that the minimum of Z occurs at more than two points.

7. Minimise and Maximise $Z = 5x + 10y$

   subject to $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x, y \geq 0$.

8. Minimise and Maximise $Z = x + 2y$

   subject to $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$; $x, y \geq 0$.

9. Maximise $Z = -x + 2y$, subject to the constraints:

   $x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$.

10. Maximise $Z = x + y$, subject to $x - y \leq -1$, $-x + y \leq 0$, $x, y \geq 0$.

12.3 Different Types of Linear Programming Problems

A few important linear programming problems are listed below:

1. **Manufacturing problems** In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hour per unit of product, warehouse space per unit of the output etc., in order to make maximum profit.
2. **Diet problems** In these problems, we determine the amount of different kinds of constituents/nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each constituent/nutrients.
3. **Transportation problems** In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

Let us now solve some of these types of linear programming problems:

Example 6 (Diet problem): A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. Food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food 'I' and Rs 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimise the cost of such a mixture.

Solution Let the mixture contain x kg of Food 'I' and y kg of Food 'II'. Clearly, $x \geq 0$, $y \geq 0$. We make the following table from the given data:

Resources	Food		Requirement
	I (x)	II (y)	
Vitamin A (units/kg)	2	1	8
Vitamin C (units/kg)	1	2	10
Cost (Rs/kg)	50	70	

Since the mixture must contain at least 8 units of vitamin A and 10 units of vitamin C, we have the constraints:

$$2x + y \geq 8$$

$$x + 2y \geq 10$$

Total cost Z of purchasing x kg of food 'I' and y kg of Food 'II' is

$$Z = 50x + 70y$$

Hence, the mathematical formulation of the problem is:

Minimise $\quad Z = 50x + 70y \qquad \ldots (1)$

subject to the constraints:

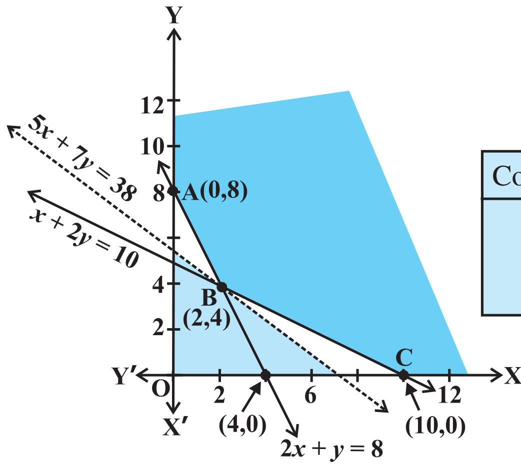
$$2x + y \geq 8 \qquad \ldots (2)$$

$$x + 2y \geq 10 \qquad \ldots (3)$$

$$x, y \geq 0 \qquad \ldots (4)$$

Let us graph the inequalities (2) to (4). The feasible region determined by the system is shown in the Fig 12.7. Here again, observe that the feasible region is **unbounded**.

Let us evaluate Z at the corner points A(0,8), B(2,4) and C(10,0).



Corner Point	$Z = 50x + 70y$	
(0,8)	560	
(2,4)	**380** ←	Minimum
(10,0)	500	

Fig 12.7

In the table, we find that smallest value of Z is 380 at the point (2,4). Can we say that the minimum value of Z is 380? Remember that the feasible region is unbounded. Therefore, we have to draw the graph of the inequality

$$50x + 70y < 380 \text{ i.e., } \quad 5x + 7y < 38$$

to check whether the resulting open half plane has any point common with the feasible region. From the Fig 12.7, we see that it has no points in common.

Thus, the minimum value of Z is 380 attained at the point (2, 4). Hence, the optimal mixing strategy for the dietician would be to mix 2 kg of Food 'I' and 4 kg of Food 'II', and with this strategy, the minimum cost of the mixture will be Rs 380.

Example 7 (Allocation problem) A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society?

Solution Let x hectare of land be allocated to crop X and y hectare to crop Y. Obviously, $x \geq 0$, $y \geq 0$.

Profit per hectare on crop X = Rs 10500

Profit per hectare on crop Y = Rs 9000

Therefore, total profit = Rs $(10500x + 9000y)$

The mathematical formulation of the problem is as follows:

Maximise $Z = 10500\,x + 9000\,y$

subject to the constraints:

$$x + y \le 50 \quad \text{(constraint related to land)} \qquad \dots (1)$$

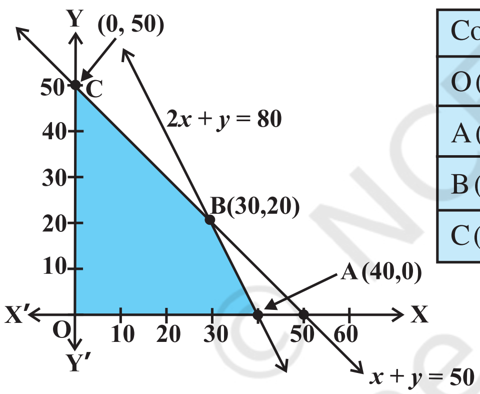
$$20x + 10y \le 800 \text{ (constraint related to use of herbicide)}$$

i.e. $$2x + y \le 80 \qquad \dots (2)$$

$$x \ge 0,\, y \ge 0 \qquad \text{(non negative constraint)} \qquad \dots (3)$$

Let us draw the graph of the system of inequalities (1) to (3). The feasible region OABC is shown (shaded) in the Fig 12.8. Observe that the feasible region is **bounded**.

The coordinates of the corner points O, A, B and C are (0, 0), (40, 0), (30, 20) and (0, 50) respectively. Let us evaluate the objective function $Z = 10500\,x + 9000y$ at these vertices to find which one gives the maximum profit.



Corner Point	$Z = 10500x + 9000y$	
O (0, 0)	0	
A (40, 0)	420000	
B (30, 20)	**495000** ←	Maximum
C (0,50)	450000	

Fig 12.8

Hence, the society will get the maximum profit of Rs 4,95,000 by allocating 30 hectares for crop X and 20 hectares for crop Y.

Example 8 (Manufacturing problem) A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?

Solution Suppose x is the number of pieces of Model A and y is the number of pieces of Model B. Then

$$\text{Total profit (in Rs)} = 8000\,x + 12000\,y$$

Let $$Z = 8000\,x + 12000\,y$$

We now have the following mathematical model for the given problem.

Maximise $Z = 8000\,x + 12000\,y$ ... (1)

subject to the constraints:

$9x + 12y \leq 180$ (Fabricating constraint)

i.e. $3x + 4y \leq 60$ ... (2)

$x + 3y \leq 30$ (Finishing constraint) ... (3)

$x \geq 0,\ y \geq 0$ (non-negative constraint) ... (4)

The feasible region (shaded) OABC determined by the linear inequalities (2) to (4) is shown in the Fig 12.9. Note that the feasible region is bounded.

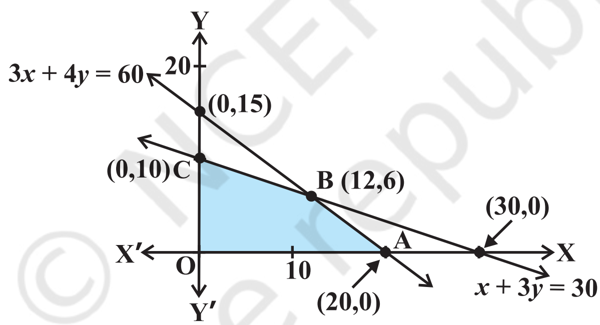


Fig 12.9

Let us evaluate the objective function Z at each corner point as shown below:

Corner Point	$Z = 8000\,x + 12000\,y$	
0 (0, 0)	0	
A (20, 0)	160000	
B (12, 6)	**168000** ←	Maximum
C (0, 10)	120000	

We find that maximum value of Z is 1,68,000 at B (12, 6). Hence, the company should produce 12 pieces of Model A and 6 pieces of Model B to realise maximum profit and maximum profit then will be Rs 1,68,000.

EXERCISE 12.2

1. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units / kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.
2. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.
3. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.
   (i) What number of rackets and bats must be made if the factory is to work at full capacity?
   (ii) If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.
4. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?
5. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.

6. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?

7. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?

8. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.

9. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

10. There are two types of fertilisers F_1 and F_2. F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs 6/kg and F_2 costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

11. The corner points of the feasible region determined by the following system of linear inequalities:

$2x + y \le 10$, $x + 3y \le 15$, $x, y \ge 0$ are (0, 0), (5, 0), (3, 4) and (0, 5). Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both (3, 4) and (0, 5) is

(A) $p = q$ (B) $p = 2q$ (C) $p = 3q$ (D) $q = 3p$

Miscellaneous Examples

Example 9 **(Diet problem)** A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

Solution Let x and y be the number of packets of food P and Q respectively. Obviously $x \geq 0, y \geq 0$. Mathematical formulation of the given problem is as follows:
Minimise $Z = 6x + 3y$ (vitamin A)

subject to the constraints

$12x + 3y \geq 240$ (constraint on calcium), i.e. $\quad 4x + y \geq 80$ ... (1)

$4x + 20y \geq 460$ (constraint on iron), i.e. $\quad x + 5y \geq 115$ ... (2)

$6x + 4y \leq 300$ (constraint on cholesterol), i.e. $3x + 2y \leq 150$ ... (3)

$x \geq 0, y \geq 0$ ... (4)

Let us graph the inequalities (1) to (4).

The feasible region (shaded) determined by the constraints (1) to (4) is shown in Fig 12.10 and note that it is bounded.

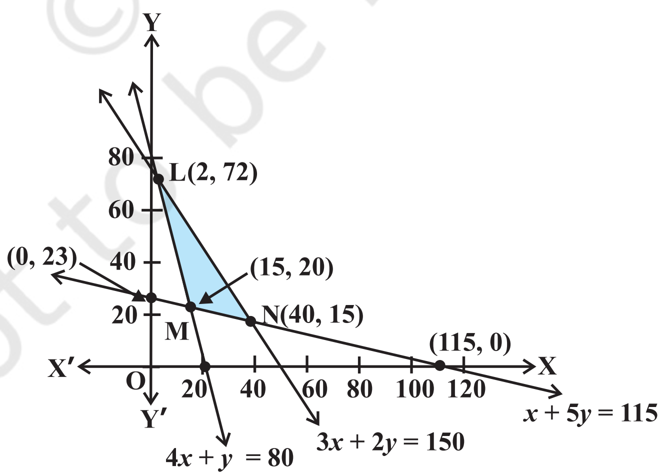


Fig 12.10

The coordinates of the corner points L, M and N are (2, 72), (15, 20) and (40, 15) respectively. Let us evaluate Z at these points:

Corner Point	$Z = 6x + 3y$	
(2, 72)	228	
(15, 20)	**150** $\leftarrow$	Minimum
(40, 15)	285	

From the table, we find that Z is minimum at the point (15, 20). Hence, the amount of vitamin A under the constraints given in the problem will be minimum, if 15 packets of food P and 20 packets of food Q are used in the special diet. The minimum amount of vitamin A will be 150 units.

Example 10 **(Manufacturing problem)** A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines.

The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Items	**Number of hours required on machines**		
	I	II	III
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

Solution Let x and y be the number of items M and N respectively.
Total profit on the production = Rs $(600x + 400y)$
Mathematical formulation of the given problem is as follows:
Maximise $Z = 600x + 400y$
subject to the constraints:

$$x + 2y \le 12 \text{ (constraint on Machine I)} \quad \text{... (1)}$$
$$2x + y \le 12 \text{ (constraint on Machine II)} \quad \text{... (2)}$$
$$x + \frac{5}{4}y \ge 5 \text{ (constraint on Machine III)} \quad \text{... (3)}$$
$$x \ge 0,\ y \ge 0 \quad \text{... (4)}$$

Let us draw the graph of constraints (1) to (4). ABCDE is the feasible region (shaded) as shown in Fig 12.11 determined by the constraints (1) to (4). Observe that the feasible region is bounded, coordinates of the corner points A, B, C, D and E are (5, 0) (6, 0), (4, 4), (0, 6) and (0, 4) respectively.

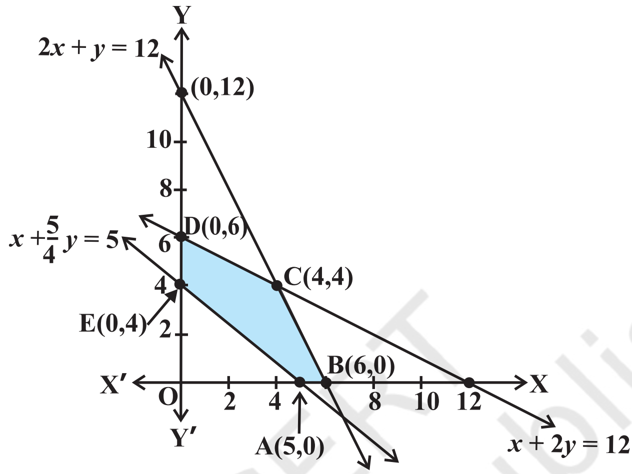


Fig 12.11

Let us evaluate Z = 600 x + 400 y at these corner points.

Corner point	Z = 600 x + 400 y
(5, 0)	3000
(6, 0)	3600
(4, 4)	**4000** ← Maximum
(0, 6)	2400
(0, 4)	1600

We see that the point (4, 4) is giving the maximum value of Z. Hence, the manufacturer has to produce 4 units of each item to get the maximum profit of Rs 4000.

Example 11 (Transportation problem) There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

From/To	Cost (in Rs)		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Solution The problem can be explained diagrammatically as follows (Fig 12.12):

Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then $(8 - x - y)$ units will be transported to depot at C (Why?)

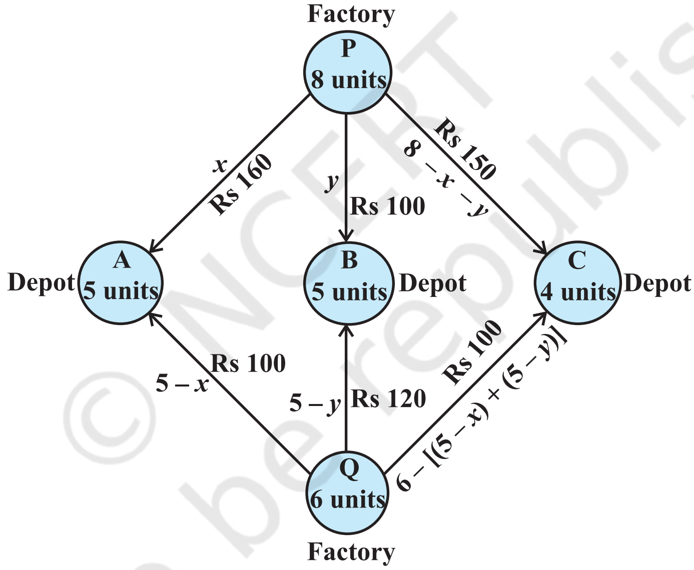


Fig 12.12

Hence, we have $x \geq 0, y \geq 0$ and $8 - x - y \geq 0$

i.e. $x \geq 0, y \geq 0$ and $x + y \leq 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5 - x)$ units need to be transported from the factory at Q. Obviously, $5 - x \geq 0$, i.e. $x \leq 5$.

Similarly, $(5 - y)$ and $6 - (5 - x + 5 - y) = x + y - 4$ units are to be transported from the factory at Q to the depots at B and C respectively.

Thus, $5 - y \geq 0$, $x + y - 4 \geq 0$

i.e. $y \leq 5$, $x + y \geq 4$

Total transportation cost Z is given by

$$Z = 160\,x + 100\,y + 150\,(8 - x - y) + 100\,(5 - x) + 120\,(5 - y) + 100\,(x + y - 4)$$
$$= 10\,(x - 7\,y + 190)$$

Therefore, the problem reduces to

Minimise $Z = 10\,(x - 7y + 190)$

subject to the constraints:

$x \geq 0, y \geq 0$ ... (1)

$x + y \leq 8$ ... (2)

$x \leq 5$ ... (3)

$y \leq 5$ ... (4)

and $x + y \geq 4$ ... (5)

The shaded region ABCDEF represented by the constraints (1) to (5) is the feasible region (Fig 12.13).

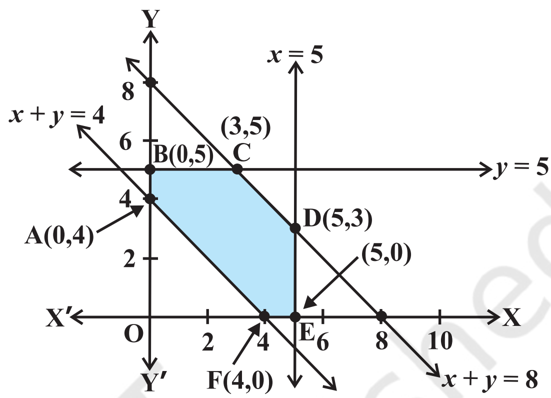


Fig 12.13

Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0). Let us evaluate Z at these points.

Corner Point	$Z = 10\,(x - 7\,y + 190)$	
(0, 4)	1620	
(0, 5)	1550 ←	Minimum
(3, 5)	1580	
(5, 3)	1740	
(5, 0)	1950	
(4, 0)	1940	

From the table, we see that the minimum value of Z is 1550 at the point (0, 5).

Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e., Rs 1550.

Miscellaneous Exercise on Chapter 12

1. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

2. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

3. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	**Vitamin A**	**Vitamin B**	**Vitamin C**
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

4. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Types of Toys	**Machines**		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

5. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

6. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per quintal (in Rs)		
From/To	**A**	**B**
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

7. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From / To	**A**	**B**
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

8. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine.

If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	**Brand Q**
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

9. Refer to Question 8. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

10. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?

Summary

- A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum) of a linear function of several variables (called **objective function)** subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear **constraints**). Variables are sometimes called **decision variables** and are **non-negative.**
- A few important linear programming problems are:
  - (i) Diet problems
  - (ii) Manufacturing problems
  - (iii) Transportation problems
- The common region determined by all the constraints including the non-negative constraints $x \geq 0, y \geq 0$ of a linear programming problem is called the **feasible region** (or **solution region**) for the problem.
- Points within and on the boundary of the feasible region represent **feasible solutions** of the constraints.

  Any point outside the feasible region is an **infeasible solution.**

- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**
- The following Theorems are fundamental in solving linear programming problems:

  Theorem 1 Let R be the feasible region (convex polygon) for a linear programming problem and let Z = $ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

  Theorem 2 Let R be the feasible region for a linear programming problem, and let Z = $ax + by$ be the objective function. If R is **bounded**, then the objective function Z has both a **maximum** and a **minimum** value on R and each of these occurs at a corner point (vertex) of R.
- If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R.
- **Corner point method** for solving a linear programming problem. The method comprises of the following steps:
  - (i) Find the feasible region of the linear programming problem and determine its corner points (vertices).
  - (ii) Evaluate the objective function Z = $ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.
  - (iii) If the feasible region is bounded, M and m respectively are the maximum and minimum values of the objective function.

  If the feasible region is unbounded, then
  - (i) M is the maximum value of the objective function, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.
  - (ii) m is the minimum value of the objective function, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.
- If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

Historical Note

In the World War II, when the war operations had to be planned to economise expenditure, maximise damage to the enemy, linear programming problems came to the forefront.

The first problem in linear programming was formulated in 1941 by the Russian mathematician, L. Kantorovich and the American economist, F. L. Hitchcock, both of whom worked at it independently of each other. This was the well known *transportation problem*. In 1945, an English economist, G.Stigler, described yet another linear programming problem – that of determining an *optimal diet*.

In 1947, the American economist, G. B. Dantzig suggested an efficient method known as the simplex method which is an iterative procedure to solve any linear programming problem in a finite number of steps.

L. Katorovich and American mathematical economist, T. C. Koopmans were awarded the nobel prize in the year 1975 in economics for their pioneering work in linear programming. With the advent of computers and the necessary softwares, it has become possible to apply linear programming model to increasingly complex problems in many areas.

Chapter 13

PROBABILITY

❖ *The theory of probabilities is simply the Science of logic quantitatively treated. – C.S. PEIRCE* ❖

13.1 Introduction

Pierre de Fermat (1601-1665)

In earlier Classes, we have studied the probability as a measure of uncertainty of events in a random experiment. We discussed the axiomatic approach formulated by Russian Mathematician, A.N. Kolmogorov (1903-1987) and treated probability as a function of outcomes of the experiment. We have also established equivalence between the axiomatic theory and the classical theory of probability in case of equally likely outcomes. On the basis of this relationship, we obtained probabilities of events associated with discrete sample spaces. We have also studied the addition rule of probability. In this chapter, we shall discuss the important concept of conditional probability of an event given that another event has occurred, which will be helpful in understanding the Bayes' theorem, multiplication rule of probability and independence of events. We shall also learn an important concept of random variable and its probability distribution and also the mean and variance of a probability distribution. In the last section of the chapter, we shall study an important discrete probability distribution called Binomial distribution. Throughout this chapter, we shall take up the experiments having equally likely outcomes, unless stated otherwise.

13.2 Conditional Probability

Uptill now in probability, we have discussed the methods of finding the probability of events. If we have two events from the same sample space, does the information about the occurrence of one of the events affect the probability of the other event? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur.

Consider the experiment of tossing three fair coins. The sample space of the experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Since the coins are fair, we can assign the probability $\frac{1}{8}$ to each sample point. Let E be the event 'at least two heads appear' and F be the event 'first coin shows tail'. Then

$$E = \{HHH, HHT, HTH, THH\}$$

and $$F = \{THH, THT, TTH, TTT\}$$

Therefore $$P(E) = P(\{HHH\}) + P(\{HHT\}) + P(\{HTH\}) + P(\{THH\})$$

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2} \text{ (Why ?)}$$

and $$P(F) = P(\{THH\}) + P(\{THT\}) + P(\{TTH\}) + P(\{TTT\})$$

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

Also $$E \cap F = \{THH\}$$

with $$P(E \cap F) = P(\{THH\}) = \frac{1}{8}$$

Now, suppose we are given that the first coin shows tail, i.e. F occurs, then what is the probability of occurrence of E? With the information of occurrence of F, we are sure that the cases in which first coin does not result into a tail should not be considered while finding the probability of E. This information reduces our sample space from the set S to its subset F for the event E. In other words, the additional information really amounts to telling us that the situation may be considered as being that of a new random experiment for which the sample space consists of all those outcomes only which are favourable to the occurrence of the event F.

Now, the sample point of F which is favourable to event E is THH.

Thus, Probability of E considering F as the sample space = $\frac{1}{4}$,

or Probability of E given that the event F has occurred = $\frac{1}{4}$

This probability of the event E is called the *conditional probability of E given that F has already occurred*, and is denoted by P (E|F).

Thus $$P(E|F) = \frac{1}{4}$$

Note that the elements of F which favour the event E are the common elements of E and F, i.e. the sample points of $E \cap F$.

Thus, we can also write the conditional probability of E given that F has occurred as

$$P(E|F) = \frac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F}$$

$$= \frac{n(E \cap F)}{n(F)}$$

Dividing the numerator and the denominator by total number of elementary events of the sample space, we see that P(E|F) can also be written as

$$P(E|F) = \frac{\frac{n(E \cap F)}{n(S)}}{\frac{n(F)}{n(S)}} = \frac{P(E \cap F)}{P(F)} \qquad \text{... (1)}$$

Note that (1) is valid only when $P(F) \neq 0$ i.e., $F \neq \phi$ (Why?)

Thus, we can define the conditional probability as follows :

Definition 1 If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, i.e. P (E|F) is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \text{ provided } P(F) \neq 0$$

13.2.1 *Properties of conditional probability*

Let E and F be events of a sample space S of an experiment, then we have

Property 1 P(S|F) = P(F|F) = 1

We know that

$$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Also $$P(F|F) = \frac{P(F \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Thus $$P(S|F) = P(F|F) = 1$$

Property 2 *If* A *and* B *are any two events of a sample space* S *and* F *is an event of* S *such that* $P(F) \neq 0$, *then*

$$P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$$

In particular, if A *and* B *are disjoint events, then*

$$P((A \cup B)|F) = P(A|F) + P(B|F)$$

We have

$$P((A \cup B)|F) = \frac{P[(A \cup B) \cap F]}{P(F)}$$

$$= \frac{P[(A \cap F) \cup (B \cap F)]}{P(F)}$$

(by distributive law of union of sets over intersection)

$$= \frac{P(A \cap F) + P(B \cap F) - P(A \cap B \cap F)}{P(F)}$$

$$= \frac{P(A \cap F)}{P(F)} + \frac{P(B \cap F)}{P(F)} - \frac{P[(A \cap B) \cap F]}{P(F)}$$

$$= P(A|F) + P(B|F) - P((A \cap B)|F)$$

When A and B are disjoint events, then

$$P((A \cap B)|F) = 0$$

$$\Rightarrow \quad P((A \cup B)|F) = P(A|F) + P(B|F)$$

Property 3 $P(E'|F) = 1 - P(E|F)$

From Property 1, we know that $P(S|F) = 1$

$\Rightarrow \quad P(E \cup E'|F) = 1 \qquad$ since $S = E \cup E'$

$\Rightarrow \quad P(E|F) + P(E'|F) = 1 \qquad$ since E and E′ are disjoint events

Thus, $\quad P(E'|F) = 1 - P(E|F)$

Let us now take up some examples.

Example 1 If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, evaluate $P(A|B)$.

Solution We have $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{4}{13}}{\frac{9}{13}} = \dfrac{4}{9}$

Example 2 A family has two children. What is the probability that both the children are boys given that at least one of them is a boy ?

Solution Let b stand for boy and g for girl. The sample space of the experiment is

$$S = \{(b, b), (g, b), (b, g), (g, g)\}$$

Let E and F denote the following events :

E : 'both the children are boys'

F : 'at least one of the child is a boy'

Then $$E = \{(b,b)\} \text{ and } F = \{(b,b), (g,b), (b,g)\}$$

Now $$E \cap F = \{(b,b)\}$$

Thus $$P(F) = \frac{3}{4} \text{ and } P(E \cap F) = \frac{1}{4}$$

Therefore $$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example 3 Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?

Solution Let A be the event 'the number on the card drawn is even' and B be the event 'the number on the card drawn is greater than 3'. We have to find P(A|B).

Now, the sample space of the experiment is S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Then $$A = \{2, 4, 6, 8, 10\}, \ B = \{4, 5, 6, 7, 8, 9, 10\}$$

and $$A \cap B = \{4, 6, 8, 10\}$$

Also $$P(A) = \frac{5}{10}, \ P(B) = \frac{7}{10} \text{ and } P(A \cap B) = \frac{4}{10}$$

Then $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{10}}{\frac{7}{10}} = \frac{4}{7}$$

Example 4 In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in Class XII given that the chosen student is a girl?

Solution Let E denote the event that a student chosen randomly studies in Class XII and F be the event that the randomly chosen student is a girl. We have to find P (E|F).

Now $\quad P(F) = \dfrac{430}{1000} = 0.43$ and $P(E \cap F) = \dfrac{43}{1000} = 0.043$ (Why?)

Then $\quad P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{0.043}{0.43} = 0.1$

Example 5 A die is thrown three times. Events A and B are defined as below:

A : 4 on the third throw

B : 6 on the first and 5 on the second throw

Find the probability of A given that B has already occurred.

Solution The sample space has 216 outcomes.

Now
$$A = \begin{Bmatrix} (1,1,4) & (1,2,4) & ... & (1,6,4) & (2,1,4) & (2,2,4) & ... & (2,6,4) \\ (3,1,4) & (3,2,4) & ... & (3,6,4) & (4,1,4) & (4,2,4) & ... & (4,6,4) \\ (5,1,4) & (5,2,4) & ... & (5,6,4) & (6,1,4) & (6,2,4) & ... & (6,6,4) \end{Bmatrix}$$

$$B = \{(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)\}$$

and $\quad A \cap B = \{(6,5,4)\}$.

Now $\quad P(B) = \dfrac{6}{216}$ and $P(A \cap B) = \dfrac{1}{216}$

Then $\quad P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{216}}{\frac{6}{216}} = \dfrac{1}{6}$

Example 6 A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

Solution Let E be the event that 'number 4 appears at least once' and F be the event that 'the sum of the numbers appearing is 6'.

Then, $\quad E = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (1,4), (2,4), (3,4), (5,4), (6,4)\}$

and $\quad F = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

We have $\quad P(E) = \dfrac{11}{36}$ and $P(F) = \dfrac{5}{36}$

Also $\quad E \cap F = \{(2,4), (4,2)\}$

Therefore $\quad P(E \cap F) = \frac{2}{36}$

Hence, the required probability

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

For the conditional probability discussed above, we have considered the elementary events of the experiment to be equally likely and the corresponding definition of the probability of an event was used. However, the same definition can also be used in the general case where the elementary events of the sample space are not equally likely, the probabilities $P(E \cap F)$ and $P(F)$ being calculated accordingly. Let us take up the following example.

Example 7 Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail, then throw a die. Find the conditional probability of the event that 'the die shows a number greater than 4' given that 'there is at least one tail'.

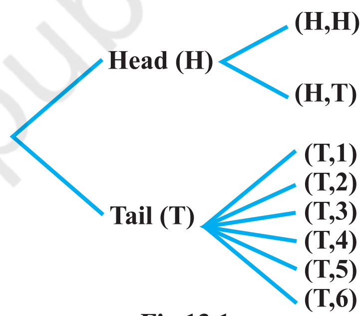


Fig 13.1

Solution The outcomes of the experiment can be represented in following diagrammatic manner called the 'tree diagram'.

The sample space of the experiment may be described as

$$S = \{(H,H), (H,T), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

where (H, H) denotes that both the tosses result into head and (T, i) denote the first toss result into a tail and the number i appeared on the die for i = 1,2,3,4,5,6.

Thus, the probabilities assigned to the 8 elementary events

(H, H), (H, T), (T, 1), (T, 2), (T, 3) (T, 4), (T, 5), (T, 6)

are $\frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$ respectively which is clear from the Fig 13.2.

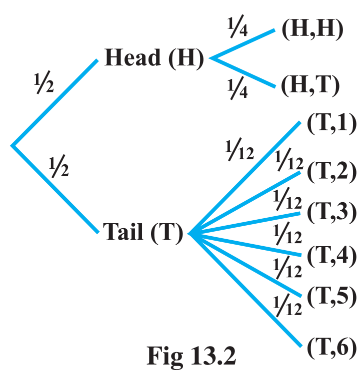


Fig 13.2

Let F be the event that 'there is at least one tail' and E be the event 'the die shows a number greater than 4'. Then

$$F = \{(H,T), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

$$E = \{(T,5), (T,6)\} \text{ and } E \cap F = \{(T,5), (T,6)\}$$

Now $$P(F) = P(\{(H,T)\}) + P(\{(T,1)\}) + P(\{(T,2)\}) + P(\{(T,3)\}) + P(\{(T,4)\}) + P(\{(T,5)\}) + P(\{(T,6)\})$$

$$= \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4}$$

and $$P(E \cap F) = P(\{(T,5)\}) + P(\{(T,6)\}) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

Hence $$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{6}}{\frac{3}{4}} = \frac{2}{9}$$

EXERCISE 13.1

1. Given that E and F are events such that P(E) = 0.6, P(F) = 0.3 and $P(E \cap F) = 0.2$, find P(E|F) and P(F|E)
2. Compute P(A|B), if P(B) = 0.5 and $P(A \cap B) = 0.32$
3. If P(A) = 0.8, P(B) = 0.5 and P(B|A) = 0.4, find
   (i) $P(A \cap B)$ (ii) P(A|B) (iii) $P(A \cup B)$
4. Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$
5. If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find
   (i) $P(A \cap B)$ (ii) P(A|B) (iii) P(B|A)

Determine P(E|F) in Exercises 6 to 9.

6. A coin is tossed three times, where
   (i) E : head on third toss , F : heads on first two tosses
   (ii) E : at least two heads , F : at most two heads
   (iii) E : at most two tails , F : at least one tail

7. Two coins are tossed once, where

(i) E : tail appears on one coin, F : one coin shows head

(ii) E : no tail appears, F : no head appears

8. A die is thrown three times,

E : 4 appears on the third toss, F : 6 and 5 appears respectively on first two tosses

9. Mother, father and son line up at random for a family picture

E : son on one end, F : father in middle

10. A black and a red dice are rolled.

(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

11. A fair die is rolled. Consider events $E = \{1,3,5\}$, $F = \{2,3\}$ and $G = \{2,3,4,5\}$ Find

(i) $P(E|F)$ and $P(F|E)$ (ii) $P(E|G)$ and $P(G|E)$

(iii) $P((E \cup F)|G)$ and $P((E \cap F)|G)$

12. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?

13. An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

14. Given that the two numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

15. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

In each of the Exercises 16 and 17 choose the correct answer:

16. If $P(A) = \frac{1}{2}$, $P(B) = 0$, then $P(A|B)$ is

(A) 0 (B) $\frac{1}{2}$

(C) not defined (D) 1

17. If A and B are events such that P(A|B) = P(B|A), then

(A) $A \subset B$ but $A \neq B$ (B) A = B

(C) $A \cap B = \phi$ (D) P(A) = P(B)

13.3 Multiplication Theorem on Probability

Let E and F be two events associated with a sample space S. Clearly, the set $E \cap F$ denotes the event that both E and F have occurred. In other words, $E \cap F$ denotes the simultaneous occurrence of the events E and F. The event $E \cap F$ is also written as EF.

Very often we need to find the probability of the event EF. For example, in the experiment of drawing two cards one after the other, we may be interested in finding the probability of the event 'a king and a queen'. The probability of event EF is obtained by using the conditional probability as obtained below :

We know that the conditional probability of event E given that F has occurred is denoted by P(E|F) and is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$$

From this result, we can write

$$P(E \cap F) = P(F) \, . \, P(E|F) \qquad \text{... (1)}$$

Also, we know that

$$P(F|E) = \frac{P(F \cap E)}{P(E)}, P(E) \neq 0$$

or $$P(F|E) = \frac{P(E \cap F)}{P(E)} \quad (\text{since } E \cap F = F \cap E)$$

Thus, $$P(E \cap F) = P(E). \; P(F|E) \qquad \text{.... (2)}$$

Combining (1) and (2), we find that

$$P(E \cap F) = P(E) \; P(F|E)$$

$$= P(F) \; P(E|F) \text{ provided } P(E) \neq 0 \text{ and } P(F) \neq 0.$$

The above result is known as the *multiplication rule of probability*.

Let us now take up an example.

Example 8 An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

Solution Let E and F denote respectively the events that first and second ball drawn are black. We have to find $P(E \cap F)$ or P (EF).

Now $\qquad P(E) = P\text{ (black ball in first draw)} = \frac{10}{15}$

Also given that the first ball drawn is black, i.e., event E has occurred, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e. $$P(F|E) = \frac{9}{14}$$

By multiplication rule of probability, we have

$$P(E \cap F) = P(E)\ P(F|E)$$

$$= \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$$

Multiplication rule of probability for more than two events If E, F and G are three events of sample space, we have

$$P(E \cap F \cap G) = P(E)\ P(F|E)\ P(G|(E \cap F)) = P(E)\ P(F|E)\ P(G|EF)$$

Similarly, the multiplication rule of probability can be extended for four or more events.

The following example illustrates the extension of multiplication rule of probability for three events.

Example 9 Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace?

Solution Let K denote the event that the card drawn is king and A be the event that the card drawn is an ace. Clearly, we have to find P (KKA)

Now $$P(K) = \frac{4}{52}$$

Also, P (K|K) is the probability of second king with the condition that one king has already been drawn. Now there are three kings in $(52 - 1) = 51$ cards.

Therefore $$P(K|K) = \frac{3}{51}$$

Lastly, P(A|KK) is the probability of third drawn card to be an ace, with the condition that two kings have already been drawn. Now there are four aces in left 50 cards.

Therefore $$\mathrm{P(A|KK)} = \frac{4}{50}$$

By multiplication law of probability, we have

$$\begin{aligned}\mathrm{P(KKA)} &= \mathrm{P(K)} \quad \mathrm{P(K|K)} \quad \mathrm{P(A|KK)}\\ &= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}\end{aligned}$$

13.4 Independent Events

Consider the experiment of drawing a card from a deck of 52 playing cards, in which the elementary events are assumed to be equally likely. If E and F denote the events 'the card drawn is a spade' and 'the card drawn is an ace' respectively, then

$$\mathrm{P(E)} = \frac{13}{52} = \frac{1}{4} \text{ and } \mathrm{P(F)} = \frac{4}{52} = \frac{1}{13}$$

Also E and F is the event ' the card drawn is the ace of spades' so that

$$\mathrm{P(E \cap F)} = \frac{1}{52}$$

Hence $$\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}} = \frac{\frac{1}{52}}{\frac{1}{13}} = \frac{1}{4}$$

Since $\mathrm{P(E)} = \frac{1}{4} = \mathrm{P(E|F)}$, we can say that the occurrence of event F has not affected the probability of occurrence of the event E.

We also have

$$\mathrm{P(F|E)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(E)}} = \frac{\frac{1}{52}}{\frac{1}{4}} = \frac{1}{13} = \mathrm{P(F)}$$

Again, $\mathrm{P(F)} = \frac{1}{13} = \mathrm{P(F|E)}$ shows that occurrence of event E has not affected the probability of occurrence of the event F.

Thus, E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other.

Such events are called *independent events*.

Definition 2 Two events E and F are said to be independent, if

$$P(F|E) = P(F) \text{ provided } P(E) \neq 0$$

and
$$P(E|F) = P(E) \text{ provided } P(F) \neq 0$$

Thus, in this definition we need to have $P(E) \neq 0$ and $P(F) \neq 0$

Now, by the multiplication rule of probability, we have

$$P(E \cap F) = P(E) \,.\, P(F|E) \qquad \dots (1)$$

If E and F are independent, then (1) becomes

$$P(E \cap F) = P(E) \,.\, P(F) \qquad \dots (2)$$

Thus, using (2), the independence of two events is also defined as follows:

Definition 3 Let E and F be two events associated with the same random experiment, then E and F are said to be independent if

$$P(E \cap F) = P(E) \,.\, P(F)$$

Remarks

(i) Two events E and F are said to be dependent if they are not independent, i.e. if

$$P(E \cap F) \neq P(E) \,.\, P(F)$$

(ii) Sometimes there is a confusion between independent events and mutually exclusive events. Term 'independent' is defined in terms of 'probability of events' whereas mutually exclusive is defined in term of events (subset of sample space). Moreover, mutually exclusive events never have an outcome common, but independent events, may have common outcome. Clearly, 'independent' and 'mutually exclusive' do not have the same meaning.

In other words, two independent events having nonzero probabilities of occurrence can not be mutually exclusive, and conversely, i.e. two mutually exclusive events having nonzero probabilities of occurrence can not be independent.

(iii) Two experiments are said to be independent if for every pair of events E and F, where E is associated with the first experiment and F with the second experiment, the probability of the simultaneous occurrence of the events E and F when the two experiments are performed is the product of P(E) and P(F) calculated separately on the basis of two experiments, i.e., $P(E \cap F) = P(E) \,.\, P(F)$

(iv) Three events A, B and C are said to be mutually independent, if

$$P(A \cap B) = P(A)\, P(B)$$
$$P(A \cap C) = P(A)\, P(C)$$
$$P(B \cap C) = P(B)\, P(C)$$

and
$$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

If at least one of the above is not true for three given events, we say that the events are not independent.

Example 10 A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent ?

Solution We know that the sample space is S = {1, 2, 3, 4, 5, 6}

Now $\quad E = \{3, 6\}, F = \{2, 4, 6\}$ and $E \cap F = \{6\}$

Then $\quad P(E) = \frac{2}{6} = \frac{1}{3},\ P(F) = \frac{3}{6} = \frac{1}{2}$ and $P(E \cap F) = \frac{1}{6}$

Clearly $\quad P(E \cap F) = P(E).\ P(F)$

Hence $\quad$ E and F are independent events.

Example 11 An unbiased die is thrown twice. Let the event A be 'odd number on the first throw' and B the event 'odd number on the second throw'. Check the independence of the events A and B.

Solution If all the 36 elementary events of the experiment are considered to be equally likely, we have

$$P(A) = \frac{18}{36} = \frac{1}{2} \text{ and } P(B) = \frac{18}{36} = \frac{1}{2}$$

Also $\quad P(A \cap B) = P$ (odd number on both throws)

$$= \frac{9}{36} = \frac{1}{4}$$

Now $\quad P(A)\ P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Clearly $\quad P(A \cap B) = P(A) \times P(B)$

Thus, $\quad$ A and B are independent events

Example 12 Three coins are tossed simultaneously. Consider the event E 'three heads or three tails', F 'at least two heads' and G 'at most two heads'. Of the pairs (E,F), (E,G) and (F,G), which are independent? which are dependent?

Solution The sample space of the experiment is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Clearly $\quad E = \{HHH, TTT\}, F = \{HHH, HHT, HTH, THH\}$

and $\quad G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Also $\quad E \cap F = \{HHH\}, E \cap G = \{TTT\}, F \cap G = \{HHT, HTH, THH\}$

Therefore $\quad P(E) = \frac{2}{8} = \frac{1}{4}, P(F) = \frac{4}{8} = \frac{1}{2}, P(G) = \frac{7}{8}$

and $\quad P(E \cap F) = \frac{1}{8}, P(E \cap G) = \frac{1}{8}, P(F \cap G) = \frac{3}{8}$

Also $\quad P(E) \, . \, P(F) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}, P(E) \cdot P(G) = \frac{1}{4} \times \frac{7}{8} = \frac{7}{32}$

and $\quad P(F) \, . \, P(G) = \frac{1}{2} \times \frac{7}{8} = \frac{7}{16}$

Thus $\quad P(E \cap F) = P(E) \, . \, P(F)$

$\quad P(E \cap G) \neq P(E) \, . \, P(G)$

and $\quad P(F \cap G) \neq P(F) \, . \, P(G)$

Hence, the events (E and F) are independent, and the events (E and G) and (F and G) are dependent.

Example 13 Prove that if E and F are independent events, then so are the events E and F′.

Solution Since E and F are independent, we have

$$P(E \cap F) = P(E) \, . \, P(F) \qquad \text{....(1)}$$

From the venn diagram in Fig 13.3, it is clear that $E \cap F$ and $E \cap F'$ are mutually exclusive events and also $E = (E \cap F) \cup (E \cap F')$.

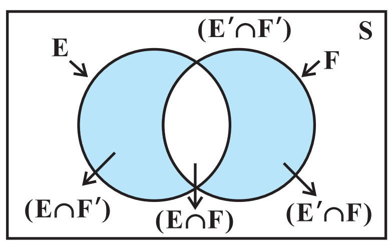


Fig 13.3

Therefore $\quad P(E) = P(E \cap F) + P(E \cap F')$

or $\quad P(E \cap F') = P(E) - P(E \cap F)$

$\quad = P(E) - P(E) \, . \, P(F)$

(by (1))

$\quad = P(E) \, (1 - P(F))$

$\quad = P(E) . \; P(F')$

Hence, E and F′ are independent

Note In a similar manner, it can be shown that if the events E and F are independent, then

(a) E' and F are independent,

(b) E' and F' are independent

Example 14 If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by $1 - P(A')\, P(B')$

Solution We have

$$
\begin{aligned}
P(\text{at least one of A and B}) &= P(A \cup B)\\
&= P(A) + P(B) - P(A \cap B)\\
&= P(A) + P(B) - P(A)\, P(B)\\
&= P(A) + P(B)\, [1 - P(A)]\\
&= P(A) \;+\; P(B).\, P(A')\\
&= 1 - P(A') + P(B)\, P(A')\\
&= 1 - P(A')\;\; [1 - P(B)]\\
&= 1 - P(A')\; P\,(B')
\end{aligned}
$$

EXERCISE 13.2

1. If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

2. Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

3. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

4. A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

5. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even,' and B be the event, 'the number is red'. Are A and B independent?

6. Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

7. Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are (i) mutually exclusive (ii) independent.

8. Let A and B be independent events with P(A) = 0.3 and P(B) = 0.4. Find

(i) $P(A \cap B)$ (ii) $P(A \cup B)$

(iii) $P(A|B)$ (iv) $P(B|A)$

9. If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find P (not A and not B).

10. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not A or not B}) = \frac{1}{4}$. State whether A and B are independent ?

11. Given two independent events A and B such that P(A) = 0.3, P(B) = 0.6. Find

(i) P(A and B) (ii) P(A and not B)

(iii) P(A or B) (iv) P(neither A nor B)

12. A die is tossed thrice. Find the probability of getting an odd number at least once.

13. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that

(i) both balls are red.

(ii) first ball is black and second is red.

(iii) one of them is black and other is red.

14. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(i) the problem is solved (ii) exactly one of them solves the problem.

15. One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?

(i) E : 'the card drawn is a spade'
F : 'the card drawn is an ace'

(ii) E : 'the card drawn is black'
F : 'the card drawn is a king'

(iii) E : 'the card drawn is a king or queen'
F : 'the card drawn is a queen or jack'.

16. In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

Choose the correct answer in Exercises 17 and 18.

17. The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{1}{12}$ (D) $\frac{1}{36}$

18. Two events A and B will be independent, if

(A) A and B are mutually exclusive

(B) $P(A'B') = [1 - P(A)]\,[1 - P(B)]$

(C) $P(A) = P(B)$

(D) $P(A) + P(B) = 1$

13.5 Bayes' Theorem

Consider that there are two bags I and II. Bag I contains 2 white and 3 red balls and Bag II contains 4 white and 5 red balls. One ball is drawn at random from one of the bags. We can find the probability of selecting any of the bags (i.e. $\frac{1}{2}$) or probability of drawing a ball of a particular colour (say white) from a particular bag (say Bag I). In other words, we can find the probability that the ball drawn is of a particular colour, if we are given the bag from which the ball is drawn. But, can we find the probability that the ball drawn is from a particular bag (say Bag II), if the colour of the ball drawn is given? Here, we have to find the reverse probability of Bag II to be selected when an event occurred after it is known. Famous mathematician, John Bayes' solved the problem of finding reverse probability by using conditional probability. The formula developed by him is known as *'Bayes theorem'* which was published posthumously in 1763. Before stating and proving the Bayes' theorem, let us first take up a definition and some preliminary results.

13.5.1 *Partition of a sample space*

A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of the sample space S if

(a) $E_i \cap E_j = \phi,\ i \neq j,\ i, j = 1, 2, 3, ..., n$

(b) $E_1 \cup E_2 \cup ... \cup E_n = S$ and

(c) $P(E_i) > 0$ for all $i = 1, 2, ..., n$.

In other words, the events $E_1, E_2, ..., E_n$ represent a partition of the sample space S if they are pairwise disjoint, exhaustive and have nonzero probabilities.

As an example, we see that any nonempty event E and its complement E′ form a partition of the sample space S since they satisfy $E \cap E' = \phi$ and $E \cup E' = S$.

From the Venn diagram in Fig 13.3, one can easily observe that if E and F are any two events associated with a sample space S, then the set $\{E \cap F', E \cap F, E' \cap F, E' \cap F'\}$ is a partition of the sample space S. It may be mentioned that the partition of a sample space is not unique. There can be several partitions of the same sample space.

We shall now prove a theorem known as *Theorem of total probability*.

13.5.2 *Theorem of total probability*

Let $\{E_1, E_2, ..., E_n\}$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2, ..., E_n$ has nonzero probability of occurrence. Let A be any event associated with S, then

$$P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + ... + P(E_n)\,P(A|E_n)$$

$$= \sum_{j=1}^{n} P(E_j)\,P(A|E_j)$$

Proof Given that $E_1, E_2, ..., E_n$ is a partition of the sample space S (Fig 13.4). Therefore,

$$S = E_1 \cup E_2 \cup ... \cup E_n \quad ... (1)$$

and $\quad E_i \cap E_j = \phi,\ i \neq j,\ i, j = 1, 2, ..., n$

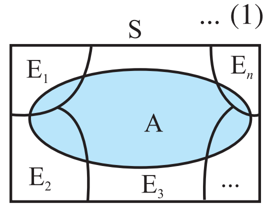


Fig 13.4

Now, we know that for any event A,

$$A = A \cap S$$
$$= A \cap (E_1 \cup E_2 \cup ... \cup E_n)$$
$$= (A \cap E_1) \cup (A \cap E_2) \cup ... \cup (A \cap E_n)$$

Also $A \cap E_i$ and $A \cap E_j$ are respectively the subsets of E_i and E_j. We know that E_i and E_j are disjoint, for $i \neq j$, therefore, $A \cap E_i$ and $A \cap E_j$ are also disjoint for all $i \neq j,\ i, j = 1, 2, ..., n$.

Thus, $\quad P(A) = P\,[(A \cap E_1) \cup (A \cap E_2) \cup ..... \cup (A \cap E_n)]$

$$= P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_n)$$

Now, by multiplication rule of probability, we have

$$P(A \cap E_i) = P(E_i)\,P(A|E_i) \text{ as } P(E_i) \neq 0 \ \forall i = 1, 2, ..., n$$

Therefore, $\quad \mathrm{P}(\mathrm{A}) = \mathrm{P}(\mathrm{E}_1)\,\mathrm{P}(\mathrm{A}|\mathrm{E}_1) + \mathrm{P}(\mathrm{E}_2)\,\mathrm{P}(\mathrm{A}|\mathrm{E}_2) + \ldots + \mathrm{P}(\mathrm{E}_n)\mathrm{P}(\mathrm{A}|\mathrm{E}_n)$

or $$\mathrm{P}(\mathrm{A}) = \sum_{j=1}^{n} \mathrm{P}(\mathrm{E}_j)\,\mathrm{P}(\mathrm{A}|\mathrm{E}_j)$$

Example 15 A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

Solution Let A be the event that the construction job will be completed on time, and B be the event that there will be a strike. We have to find P(A).
We have

$$\mathrm{P}(\mathrm{B}) = 0.65,\ \mathrm{P}(\text{no strike}) = \mathrm{P}(\mathrm{B}') = 1 - \mathrm{P}(\mathrm{B}) = 1 - 0.65 = 0.35$$
$$\mathrm{P}(\mathrm{A}|\mathrm{B}) = 0.32,\ \mathrm{P}(\mathrm{A}|\mathrm{B}') = 0.80$$

Since events B and B′ form a partition of the sample space S, therefore, by theorem on total probability, we have

$$\begin{aligned}\mathrm{P}(\mathrm{A}) &= \mathrm{P}(\mathrm{B})\ \mathrm{P}(\mathrm{A}|\mathrm{B}) + \mathrm{P}(\mathrm{B}')\ \mathrm{P}(\mathrm{A}|\mathrm{B}')\\ &= 0.65 \times 0.32 + 0.35 \times 0.8\\ &= 0.208 + 0.28 = 0.488\end{aligned}$$

Thus, the probability that the construction job will be completed in time is 0.488.

We shall now state and prove the Bayes' theorem.

Bayes' Theorem If $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_n$ are n non empty events which constitute a partition of sample space S, i.e. $\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_n$ are pairwise disjoint and $\mathrm{E}_1 \cup \mathrm{E}_2 \cup \ldots \cup \mathrm{E}_n = \mathrm{S}$ and A is any event of nonzero probability, then

$$\mathrm{P}(\mathrm{E}_i|\mathrm{A}) = \frac{\mathrm{P}(\mathrm{E}_i)\,\mathrm{P}(\mathrm{A}|\mathrm{E}_i)}{\sum_{j=1}^{n} \mathrm{P}(\mathrm{E}_j)\,\mathrm{P}(\mathrm{A}|\mathrm{E}_j)} \quad \text{for any } i = 1, 2, 3, \ldots, n$$

Proof By formula of conditional probability, we know that

$$\begin{aligned}\mathrm{P}(\mathrm{E}_i|\mathrm{A}) &= \frac{\mathrm{P}(\mathrm{A} \cap \mathrm{E}_i)}{\mathrm{P}(\mathrm{A})}\\ &= \frac{\mathrm{P}(\mathrm{E}_i)\,\mathrm{P}(\mathrm{A}|\mathrm{E}_i)}{\mathrm{P}(\mathrm{A})} \quad \text{(by multiplication rule of probability)}\\ &= \frac{\mathrm{P}(\mathrm{E}_i)\,\mathrm{P}(\mathrm{A}|\mathrm{E}_i)}{\sum_{j=1}^{n} \mathrm{P}(\mathrm{E}_j)\,\mathrm{P}(\mathrm{A}|\mathrm{E}_j)} \quad \text{(by the result of theorem of total probability)}\end{aligned}$$

Remark The following terminology is generally used when Bayes' theorem is applied.

The events $E_1, E_2, ..., E_n$ are called *hypotheses.*

The probability $P(E_i)$ is called the *priori probability* of the hypothesis E_i

The conditional probability $P(E_i|A)$ is called *a posteriori probability* of the hypothesis E_i.

Bayes' theorem is also called the formula for the probability of "causes". Since the E_i's are a partition of the sample space S, one and only one of the events E_i occurs (i.e. one of the events E_i must occur and only one can occur). Hence, the above formula gives us the probability of a particular E_i (i.e. a "Cause"), given that the event A has occurred.

The Bayes' theorem has its applications in variety of situations, few of which are illustrated in following examples.

Example 16 Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.

Solution Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II and A be the event of drawing a red ball.

Then $\quad P(E_1) = P(E_2) = \frac{1}{2}$

Also $\quad P(A|E_1) = P(\text{drawing a red ball from Bag I}) = \frac{3}{7}$

and $\quad P(A|E_2) = P(\text{drawing a red ball from Bag II}) = \frac{5}{11}$

Now, the probability of drawing a ball from Bag II, being given that it is red, is $P(E_2|A)$

By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)} = \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \frac{35}{68}$$

Example 17 Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

Solution Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $\qquad P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, let A be the event that 'the coin drawn is of gold'

Then $\qquad P(A|E_1) = P(\text{a gold coin from bag I}) = \frac{2}{2} = 1$

$P(A|E_2) = P(\text{a gold coin from bag II}) = 0$

$P(A|E_3) = P(\text{a gold coin from bag III}) = \frac{1}{2}$

Now, the probability that the other coin in the box is of gold

= the probability that gold coin is drawn from the box I.

$= P(E_1|A)$

By Bayes' theorem, we know that

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)+P(E_3)P(A|E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times 0+\frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

Example 18 Suppose that the reliability of a HIV test is specified as follows:

Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV–ive but 1% are diagnosed as showing HIV+ive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV+ive. What is the probability that the person actually has HIV?

Solution Let E denote the event that the person selected is actually having HIV and A the event that the person's HIV test is diagnosed as +ive. We need to find P(E|A).

Also E′ denotes the event that the person selected is actually not having HIV.

Clearly, {E, E′} is a partition of the sample space of all people in the population. We are given that

$$P(E) = 0.1\% = \frac{0.1}{100} = 0.001$$

$$P(E') = 1 - P(E) = 0.999$$

$P(A|E)$ = P(Person tested as HIV+ive given that he/she is actually having HIV)

$$= 90\% = \frac{90}{100} = 0.9$$

and $P(A|E')$ = P(Person tested as HIV +ive given that he/she is actually not having HIV)

$$= 1\% = \frac{1}{100} = 0.01$$

Now, by Bayes' theorem

$$P(E|A) = \frac{P(E)\,P(A|E)}{P(E)\,P(A|E) + P(E')\,P(A|E')}$$

$$= \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{90}{1089}$$

$$= 0.083 \text{ approx.}$$

Thus, the probability that a person selected at random is actually having HIV given that he/she is tested HIV+ive is 0.083.

Example 19 In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

Solution Let events B_1, B_2, B_3 be the following :

B_1 : the bolt is manufactured by machine A

B_2 : the bolt is manufactured by machine B

B_3 : the bolt is manufactured by machine C

Clearly, B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space.

Let the event E be 'the bolt is defective'.

The event E occurs with B_1 or with B_2 or with B_3. Given that,

$$P(B_1) = 25\% = 0.25, \;\; P(B_2) = 0.35 \text{ and } P(B_3) = 0.40$$

Again $P(E|B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A = 5% = 0.05

Similarly, $P(E|B_2) = 0.04$, $P(E|B_3) = 0.02$.

Hence, by Bayes' Theorem, we have

$$P(B_2|E) = \frac{P(B_2)P(E|B_2)}{P(B_1)P(E|B_1)+P(B_2)P(E|B_2)+P(B_3)P(E|B_3)}$$

$$= \frac{0.35\times 0.04}{0.25\times 0.05+0.35\times 0.04+0.40\times 0.02}$$

$$= \frac{0.0140}{0.0345} = \frac{28}{69}$$

Example 20 A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$, and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Solution Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus, scooter, and other means of transport respectively.

Then $\quad P(T_1) = \frac{3}{10}, P(T_2) = \frac{1}{5}, P(T_3) = \frac{1}{10}$ and $P(T_4) = \frac{2}{5}$ (given)

$P(E|T_1)$ = Probability that the doctor arriving late comes by train = $\frac{1}{4}$

Similarly, $P(E|T_2) = \frac{1}{3}$, $P(E|T_3) = \frac{1}{12}$ and $P(E|T_4) = 0$, since he is not late if he comes by other means of transport.

Therefore, by Bayes' Theorem, we have

$P(T_1|E)$ = Probability that the doctor arriving late comes by train

$$= \frac{P(T_1)P(E|T_1)}{P(T_1)P(E|T_1)+P(T_2)P(E|T_2)+P(T_3)P(E|T_3)+P(T_4)P(E|T_4)}$$

$$= \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0} = \frac{3}{40}\times\frac{120}{18} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

Example 21 A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Solution Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur.

Then $\quad P(S_1)$ = Probability that *six* occurs = $\frac{1}{6}$

$P(S_2)$ = Probability that *six* does not occur = $\frac{5}{6}$

$P(E|S_1)$ = Probability that the man reports that *six* occurs when *six* has actually occurred on the die

= Probability that the man speaks the truth = $\frac{3}{4}$

$P(E|S_2)$ = Probability that the man reports that *six* occurs when *six* has not actually occurred on the die

= Probability that the man does not speak the truth $= 1 - \frac{3}{4} = \frac{1}{4}$

Thus, by Bayes' theorem, we get

$P(S_1|E)$ = Probability that the report of the man that *six* has occurred is actually a *six*

$$= \frac{P(S_1)P(E|S_1)}{P(S_1)P(E|S_1) + P(S_2)P(E|S_2)}$$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{1}{8} \times \frac{24}{8} = \frac{3}{8}$$

Hence, the required probability is $\frac{3}{8}$.

EXERCISE 13.3

1. An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

2. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

3. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

4. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

5. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

6. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

7. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

8. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

9. Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are

0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

10. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

11. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

12. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

13. Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(A) $\frac{4}{5}$ (B) $\frac{1}{2}$ (C) $\frac{1}{5}$ (D) $\frac{2}{5}$

14. If A and B are two events such that $A \subset B$ and $P(B) \neq 0$, then which of the following is correct?

(A) $P(A|B) = \frac{P(B)}{P(A)}$ (B) $P(A|B) < P(A)$

(C) $P(A|B) \geq P(A)$ (D) None of these

13.6 Random Variables and its Probability Distributions

We have already learnt about random experiments and formation of sample spaces. In most of these experiments, we were not only interested in the particular outcome that occurs but rather in some number associated with that outcomes as shown in following examples/experiments.

(i) In tossing two dice, we may be interested in the sum of the numbers on the two dice.

(ii) In tossing a coin 50 times, we may want the number of heads obtained.

(iii) In the experiment of taking out four articles (one after the other) at random from a lot of 20 articles in which 6 are defective, we want to know the number of defectives in the sample of four and not in the particular sequence of defective and nondefective articles.

In all of the above experiments, we have a rule which assigns to each outcome of the experiment a single real number. This single real number may vary with different outcomes of the experiment. Hence, it is a variable. Also its value depends upon the outcome of a random experiment and, hence, is called random variable. A random variable is usually denoted by X.

If you recall the definition of a function, you will realise that the random variable X is really speaking a function whose domain is the set of outcomes (or sample space) of a random experiment. A random variable can take any real value, therefore, its co-domain is the set of real numbers. Hence, a random variable can be defined as follows :

Definition 4 A random variable is a real valued function whose domain is the sample space of a random experiment.

For example, let us consider the experiment of tossing a coin two times in succession.

The sample space of the experiment is $S = \{HH, HT, TH, TT\}$.

If X denotes the number of heads obtained, then X is a random variable and for each outcome, its value is as given below :

$$X(HH) = 2,\ X(HT) = 1,\ X(TH) = 1,\ X(TT) = 0.$$

More than one random variables can be defined on the same sample space. For example, let Y denote the number of heads minus the number of tails for each outcome of the above sample space S.

Then $\qquad Y(HH) = 2,\ Y(HT) = 0,\ Y(TH) = 0,\ Y(TT) = -2.$

Thus, X and Y are two different random variables defined on the same sample space S.

Example 22 A person plays a game of tossing a coin thrice. For each head, he is given Rs 2 by the organiser of the game and for each tail, he has to give Rs 1.50 to the organiser. Let X denote the amount gained or lost by the person. Show that X is a random variable and exhibit it as a function on the sample space of the experiment.

Solution X is a number whose values are defined on the outcomes of a random experiment. Therefore, X is a random variable.

Now, sample space of the experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Then $\quad X(HHH) = \text{Rs } (2 \times 3) = \text{Rs } 6$

$$X(HHT) = X(HTH) = X(THH) = \text{Rs } (2 \times 2 - 1 \times 1.50) = \text{Rs } 2.50$$

$$X(HTT) = X(THT) = (TTH) = \text{Rs } (1 \times 2) - (2 \times 1.50) = -\text{ Re } 1$$

and $\quad X(TTT) = -\text{ Rs } (3 \times 1.50) = -\text{ Rs } 4.50$

where, minus sign shows the loss to the player. Thus, for each element of the sample space, X takes a unique value, hence, X is a function on the sample space whose range is

$$\{-1,\ 2.50,\ -4.50,\ 6\}$$

Example 23 A bag contains 2 white and 1 red balls. One ball is drawn at random and then put back in the box after noting its colour. The process is repeated again. If X denotes the number of red balls recorded in the two draws, describe X.

Solution Let the balls in the bag be denoted by w_1, w_2, r. Then the sample space is

$$S = \{w_1\, w_1,\ w_1\, w_2,\ w_2\, w_2,\ w_2\, w_1,\ w_1\, r,\ w_2\, r,\ r\, w_1,\ r\, w_2,\ r\, r\}$$

Now, for $\quad \omega \in S$

$$X(\omega) = \text{number of red balls}$$

Therefore

$$X(\{w_1\, w_1\}) = X(\{w_1\, w_2\}) = X(\{w_2\, w_2\}) = X(\{w_2\, w_1\}) = 0$$

$$X(\{w_1\, r\}) = X(\{w_2\, r\}) = X(\{r\, w_1\}) = X(\{r\, w_2\}) = 1 \text{ and } X(\{r\, r\}) = 2$$

Thus, X is a random variable which can take values 0, 1 or 2.

13.6.1 *Probability distribution of a random variable*

Let us look at the experiment of selecting one family out of ten families $f_1, f_2, ..., f_{10}$ in such a manner that each family is equally likely to be selected. Let the families $f_1, f_2, ..., f_{10}$ have 3, 4, 3, 2, 5, 4, 3, 6, 4, 5 members, respectively.

Let us select a family and note down the number of members in the family denoting X. Clearly, X is a random variable defined as below :

$$X(f_1) = 3,\ X(f_2) = 4,\ X(f_3) = 3,\ X(f_4) = 2,\ X(f_5) = 5,$$

$$X(f_6) = 4,\ X(f_7) = 3,\ X(f_8) = 6,\ X(f_9) = 4,\ X(f_{10}) = 5$$

Thus, X can take any value 2,3,4,5 or 6 depending upon which family is selected.

Now, X will take the value 2 when the family f_4 is selected. X can take the value 3 when any one of the families f_1, f_3, f_7 is selected.

Similarly, $\quad X = 4$, when family f_2, f_6 or f_9 is selected,

$\quad X = 5$, when family f_5 or f_{10} is selected

and $\quad X = 6$, when family f_8 is selected.

Since we had assumed that each family is equally likely to be selected, the probability that family f_4 is selected is $\frac{1}{10}$.

Thus, the probability that X can take the value 2 is $\frac{1}{10}$. We write $P(X = 2) = \frac{1}{10}$

Also, the probability that any one of the families f_1, f_3 or f_7 is selected is

$$P(\{f_1, f_3, f_7\}) = \frac{3}{10}$$

Thus, the probability that X can take the value 3 = $\frac{3}{10}$

We write $$P(X = 3) = \frac{3}{10}$$

Similarly, we obtain

$$P(X = 4) = P(\{f_2, f_6, f_9\}) = \frac{3}{10}$$

$$P(X = 5) = P(\{f_5, f_{10}\}) = \frac{2}{10}$$

and $$P(X = 6) = P(\{f_8\}) = \frac{1}{10}$$

Such a description giving the values of the random variable along with the corresponding probabilities is called the *probability distribution of the random variable X.*

In general, the probability distribution of a random variable X is defined as follows:

Definition 5 The probability distribution of a random variable X is the system of numbers

X	:	x_1	x_2	...	x_n
P(X)	:	p_1	p_2	...	p_n

where, $$p_i > 0, \sum_{i=1}^{n} p_i = 1, i = 1, 2,..., n$$

The real numbers $x_1, x_2,..., x_n$ are the possible values of the random variable X and p_i ($i = 1,2,..., n$) is the probability of the random variable X taking the value x_i i.e., $P(X = x_i) = p_i$

Note If x_i is one of the possible values of a random variable X, the statement X = x_i is true only at some point (s) of the sample space. Hence, the probability that X takes value x_i is always nonzero, i.e. $P(X = x_i) \neq 0$.

Also for all possible values of the random variable X, all elements of the sample space are covered. Hence, the sum of all the probabilities in a probability distribution must be one.

Example 24 Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

Solution The number of aces is a random variable. Let it be denoted by X. Clearly, X can take the values 0, 1, or 2.

Now, since the draws are done with replacement, therefore, the two draws form independent experiments.

Therefore,

$$P(X = 0) = P(\text{non-ace and non-ace})$$
$$= P(\text{non-ace}) \times P(\text{non-ace})$$
$$= \frac{48}{52} \times \frac{48}{52} = \frac{144}{169}$$
$$P(X = 1) = P(\text{ace and non-ace or non-ace and ace})$$
$$= P(\text{ace and non-ace}) + P(\text{non-ace and ace})$$
$$= P(\text{ace}).\ P(\text{non-ace}) + P\ (\text{non-ace})\ .\ P(\text{ace})$$
$$= \frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52} = \frac{24}{169}$$

and

$$P(X = 2) = P\ (\text{ace and ace})$$
$$= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$$

Thus, the required probability distribution is

X	0	1	2
P(X)	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Example 25 Find the probability distribution of number of doublets in three throws of a pair of dice.

Solution Let X denote the number of doublets. Possible doublets are

$$(1,1)\,,\,(2,2),\ (3,3),\ (4,4),\ (5,5),\,(6,6)$$

Clearly, X can take the value 0, 1, 2, or 3.

Probability of getting a doublet $= \frac{6}{36} = \frac{1}{6}$

Probability of not getting a doublet $= 1 - \frac{1}{6} = \frac{5}{6}$

Now $\quad$ $P(X = 0) = P$ (no doublet) $= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$

$P(X = 1) = P$ (one doublet and two non-doublets)

$$= \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$= 3\ \frac{1}{6} \times \frac{5^2}{6^2}\ = \frac{75}{216}$$

$P(X = 2) = P$ (two doublets and one non-doublet)

$$= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} = 3\ \frac{1}{6^2} \times \frac{5}{6}\ = \frac{15}{216}$$

and $\quad$ $P(X = 3) = P$ (three doublets)

$$= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$$

Thus, the required probability distribution is

X	0	1	2	3
P(X)	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Verification Sum of the probabilities

$$\sum_{i=1}^{n} p_i = \frac{125}{216} + \frac{75}{216} + \frac{15}{216} + \frac{1}{216}$$

$$= \frac{125 + 75 + 15 + 1}{216} = \frac{216}{216} = 1$$

Example 26 Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the values x, has the following form, where k is some unknown constant.

$$P(X = x) = \begin{cases} 0.1, \text{ if } x = 0 \\ kx, \text{ if } x = 1 \text{ or } 2 \\ k(5 - x), \text{ if } x = 3 \text{ or } 4 \\ 0, \text{ otherwise} \end{cases}$$

(a) Find the value of k.

(b) What is the probability that you study at least two hours ? Exactly two hours? At most two hours?

Solution The probability distribution of X is

X	0	1	2	3	4
P(X)	0.1	k	$2k$	$2k$	k

(a) We know that $\sum_{i=1}^{n} p_i = 1$

Therefore $0.1 + k + 2k + 2k + k = 1$

i.e. $k = 0.15$

(b) P(you study at least two hours) $= P(X \geq 2)$

$= P(X = 2) + P(X = 3) + P(X = 4)$

$= 2k + 2k + k = 5k = 5 \times 0.15 = 0.75$

P(you study exactly two hours) $= P(X = 2)$

$= 2k = 2 \times 0.15 = 0.3$

P(you study at most two hours) $= P(X \leq 2)$

$= P(X = 0) + P(X = 1) + P(X = 2)$

$= 0.1 + k + 2k = 0.1 + 3k = 0.1 + 3 \times 0.15$

$= 0.55$

13.6.2 *Mean of a random variable*

In many problems, it is desirable to describe some feature of the random variable by means of a single number that can be computed from its probability distribution. Few such numbers are mean, median and mode. In this section, we shall discuss mean only. Mean is a measure of location or central tendency in the sense that it roughly locates a *middle* or *average value* of the random variable.

Definition 6 Let X be a random variable whose possible values $x_1, x_2, x_3, ..., x_n$ occur with probabilities $p_1, p_2, p_3, ..., p_n$, respectively. The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i p_i$ i.e. the mean of X is the weighted average of the possible values of X, each value being weighted by its probability with which it occurs.

The mean of a random variable X is also called the expectation of X, denoted by E(X).

Thus,
$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + ... + x_n p_n.$$

In other words, the mean or expectation of a random variable X is the sum of the products of all possible values of X by their respective probabilities.

Example 27 Let a pair of dice be thrown and the random variable X be the sum of the numbers that appear on the two dice. Find the mean or expectation of X.

Solution The sample space of the experiment consists of 36 elementary events in the form of ordered pairs (x_i, y_i), where $x_i = 1, 2, 3, 4, 5, 6$ and $y_i = 1, 2, 3, 4, 5, 6$.

The random variable X i.e. the sum of the numbers on the two dice takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

Now
$$P(X = 2) = P(\{(1,1)\}) = \frac{1}{36}$$
$$P(X = 3) = P(\{(1,2), (2,1)\}) = \frac{2}{36}$$
$$P(X = 4) = P(\{(1,3), (2,2), (3,1)\}) = \frac{3}{36}$$
$$P(X = 5) = P(\{(1,4), (2,3), (3,2), (4,1)\}) = \frac{4}{36}$$
$$P(X = 6) = P(\{(1,5), (2,4), (3,3), (4,2), (5,1)\}) = \frac{5}{36}$$
$$P(X = 7) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}) = \frac{6}{36}$$
$$P(X = 8) = P(\{(2,6), (3,5), (4,4), (5,3), (6,2)\}) = \frac{5}{36}$$

$$P(X = 9) = P(\{(3,6), (4,5), (5,4), (6,3)\}) = \frac{4}{36}$$

$$P(X = 10) = P(\{(4,6), (5,5), (6,4)\}) = \frac{3}{36}$$

$$P(X = 11) = P(\{(5,6), (6,5)\}) = \frac{2}{36}$$

$$P(X = 12) = P(\{(6,6)\}) = \frac{1}{36}$$

The probability distribution of X is

X or x_i	2	3	4	5	6	7	8	9	10	11	12
P(X) or p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Therefore,

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i = 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36}$$

$$+ 6\times\frac{5}{36} + 7\times\frac{6}{36} + 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}$$

$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36} = 7$$

Thus, the mean of the sum of the numbers that appear on throwing two fair dice is 7.

13.6.3 *Variance of a random variable*

The mean of a random variable does not give us information about the variability in the values of the random variable. In fact, if the variance is small, then the values of the random variable are close to the mean. Also random variables with different probability distributions can have equal means, as shown in the following distributions of X and Y.

X	1	2	3	4
P(X)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{2}{8}$

Y	–1	0	4	5	6
P(Y)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Clearly $\quad E(X) = 1\times\frac{1}{8}+2\times\frac{2}{8}+3\times\frac{3}{8}+4\times\frac{2}{8}=\frac{22}{8}=2.75$

and $\quad E(Y) = -1\times\frac{1}{8}+0\times\frac{2}{8}+4\times\frac{3}{8}+5\times\frac{1}{8}= 6\times\frac{1}{8}=\frac{22}{8}=2.75$

The variables X and Y are different, however their means are same. It is also easily observable from the diagramatic representation of these distributions (Fig 13.5).

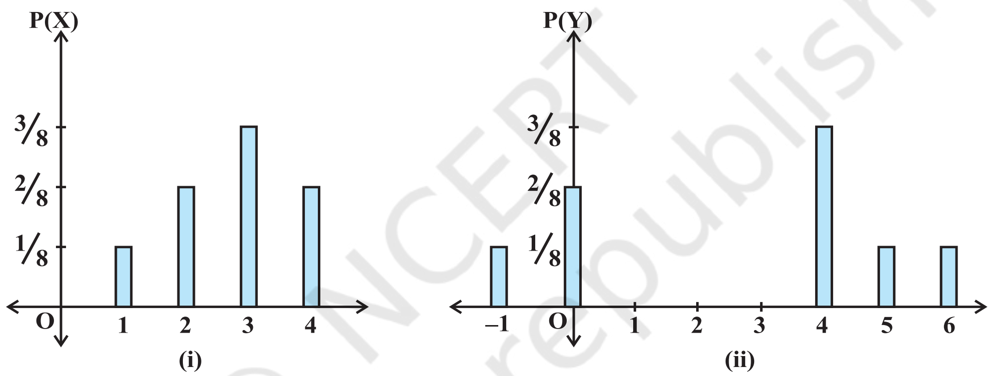


Fig 13.5

To distinguish X from Y, we require a measure of the extent to which the values of the random variables spread out. In Statistics, we have studied that the variance is a measure of the spread or scatter in data. Likewise, the variability or spread in the values of a random variable may be measured by variance.

Definition 7 Let X be a random variable whose possible values $x_1, x_2,...,x_n$ occur with probabilities $p(x_1), p(x_2),..., p(x_n)$ respectively.

Let $\mu = E(X)$ be the mean of X. The variance of X, denoted by Var (X) or σ_x^2 is defined as

$$\sigma_x^2 = \text{Var}(X) = \sum_{i=1}^{n}(x_i - \mu)^2\, p(x_i)$$

or equivalently $\quad \sigma_x^2 = E(X - \mu)^2$

The non-negative number

$$\sigma_x = \sqrt{\text{Var(X)}} = \sqrt{\sum_{i=1}^{n}(x_i - \mu)^2\, p(x_i)}$$

is called the *standard deviation* of the random variable X.

Another formula to find the variance of a random variable. We know that,

$$\text{Var (X)} = \sum_{i=1}^{n}(x_i - \mu)^2\, p(x_i)$$

$$= \sum_{i=1}^{n}(x_i^2 + \mu^2 - 2\mu\, x_i)\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2\, p(x_i) + \sum_{i=1}^{n}\mu^2\, p(x_i) - \sum_{i=1}^{n} 2\mu x_i\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2\, p(x_i) + \mu^2 \sum_{i=1}^{n} p(x_i) - 2\mu \sum_{i=1}^{n} x_i\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2\, p(x_i) + \mu^2 - 2\mu^2 \left[\text{since} \sum_{i=1}^{n} p\,(x_i) = 1 \text{ and } \mu = \sum_{i=1}^{n} x_i\, p(x_i)\right]$$

$$= \sum_{i=1}^{n} x_i^2\, p(x_i) - \mu^2$$

or $$\text{Var (X)} = \sum_{i=1}^{n} x_i^2\, p(x_i) - \left(\sum_{i=1}^{n} x_i\, p(x_i)\right)^2$$

or $$\text{Var (X)} = \text{E}(\text{X}^2) - [\text{E(X)}]^2, \text{ where } \text{E}(\text{X}^2) = \sum_{i=1}^{n} x_i^2\, p(x_i)$$

Example 28 Find the variance of the number obtained on a throw of an unbiased die.

Solution The sample space of the experiment is S = {1, 2, 3, 4, 5, 6}.

Let X denote the number obtained on the throw. Then X is a random variable which can take values 1, 2, 3, 4, 5, or 6.

Also $\qquad P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$

Therefore, the Probability distribution of X is

X	1	2	3	4	5	6
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Now $\qquad E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$

$$= 1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=\frac{21}{6}$$

Also $\qquad E(X^2) = 1^2\times\frac{1}{6}+2^2\times\frac{1}{6}+3^2\times\frac{1}{6}+4^2\times\frac{1}{6}+5^2\times\frac{1}{6}+6^2\times\frac{1}{6}=\frac{91}{6}$

Thus, $\qquad \text{Var }(X) = E(X^2) - (E(X))^2$

$$= \frac{91}{6}-\left(\frac{21}{6}\right)^2=\frac{91}{6}-\frac{441}{36} \quad =\frac{35}{12}$$

Example 29 Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

Solution Let X denote the number of kings in a draw of two cards. X is a random variable which can assume the values 0, 1 or 2.

Now $\qquad P(X = 0) = P\text{ (no king) } = \frac{^{48}C_2}{^{52}C_2} = \frac{\frac{48!}{2!(48-2)!}}{\frac{52!}{2!(52-2)!}} = \frac{48\times 47}{52\times 51} = \frac{188}{221}$

$P(X = 1) = P\text{ (one king and one non-king) } = \frac{^{4}C_1\ ^{48}C_1}{^{52}C_2}$

$$= \frac{4\times 48\times 2}{52\times 51} = \frac{32}{221}$$

and $\quad P(X = 2) = P \text{ (two kings)} = \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{4\times3}{52\times51} = \frac{1}{221}$

Thus, the probability distribution of X is

X	0	1	2
P(X)	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Now $\quad$ Mean of $\quad X = E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$

$$= 0\times\frac{188}{221}+1\times\frac{32}{221}+2\times\frac{1}{221}=\frac{34}{221}$$

Also $\quad E(X^2) = \sum_{i=1}^{n} x_i^2\, p(x_i)$

$$= 0^2\times\frac{188}{221}+1^2\times\frac{32}{221}+2^2\times\frac{1}{221}=\frac{36}{221}$$

Now $\quad Var(X) = E(X^2) - [E(X)]^2$

$$= \frac{36}{221}-\left(\frac{34}{221}\right)^2=\frac{6800}{(221)^2}$$

Therefore $\quad \sigma_x = \sqrt{Var(X)} = \frac{\sqrt{6800}}{221} = 0.37$

EXERCISE 13.4

1. State which of the following are not the probability distributions of a random variable. Give reasons for your answer.

(i)

X	0	1	2
P(X)	0.4	0.4	0.2

(ii)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	– 0.1	0.3

(iii)

Y	– 1	0	1
P(Y)	0.6	0.1	0.2

(iv)

Z	3	2	1	0	–1
P(Z)	0.3	0.2	0.4	0.1	0.05

2. An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represent the number of black balls. What are the possible values of X? Is X a random variable ?

3. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are possible values of X?

4. Find the probability distribution of

(i) number of heads in two tosses of a coin.

(ii) number of tails in the simultaneous tosses of three coins.

(iii) number of heads in four tosses of a coin.

5. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as

(i) number greater than 4

(ii) six appears on at least one die

6. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

7. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

8. A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
P(X)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Determine

(i) k (ii) $P(X < 3)$

(iii) $P(X > 6)$ (iv) $P(0 < X < 3)$

9. The random variable X has a probability distribution P(X) of the following form, where k is some number :

$$P(X) = \begin{cases} k, & if \quad x=0 \\ 2k, & if \quad x=1 \\ 3k, & if \quad x=2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the value of k.

(b) Find P $(X < 2)$, P $(X \leq 2)$, $P(X \geq 2)$.

10. Find the mean number of heads in three tosses of a fair coin.

11. Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

12. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E(X).

13. Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of X.

14. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.

15. In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random and we take X = 0 if he opposed, and X = 1 if he is in favour. Find E(X) and Var (X).

Choose the correct answer in each of the following:

16. The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

(A) 1 (B) 2 (C) 5 (D) $\frac{8}{3}$

17. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is

(A) $\frac{37}{221}$ (B) $\frac{5}{13}$ (C) $\frac{1}{13}$ (D) $\frac{2}{13}$

13.7 Bernoulli Trials and Binomial Distribution

13.7.1 *Bernoulli trials*

Many experiments are dichotomous in nature. For example, a tossed coin shows a 'head' or 'tail', a manufactured item can be 'defective' or 'non-defective', the response to a question might be 'yes' or 'no', an egg has 'hatched' or 'not hatched', the decision is 'yes' or 'no' etc. In such cases, it is customary to call one of the outcomes a 'success' and the other 'not success' or 'failure'. For example, in tossing a coin, if the occurrence of the head is considered a success, then occurrence of tail is a failure.

Each time we toss a coin or roll a die or perform any other experiment, we call it a trial. If a coin is tossed, say, 4 times, the number of trials is 4, each having exactly two outcomes, namely, success or failure. The outcome of any trial is independent of the outcome of any other trial. In each of such trials, the probability of success or failure remains constant. Such independent trials which have only two outcomes usually referred as 'success' or 'failure' are called *Bernoulli trials*.

Definition 8 Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(i) There should be a finite number of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes : success or failure.

(iv) The probability of success remains the same in each trial.

For example, throwing a die 50 times is a case of 50 Bernoulli trials, in which each trial results in success (say an even number) or failure (an odd number) and the probability of success (p) is same for all 50 throws. Obviously, the successive throws of the die are independent experiments. If the die is fair and have six numbers 1 to 6 written on six faces, then $p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$ = probability of failure.

Example 30 Six balls are drawn successively from an urn containing 7 red and 9 black balls. Tell whether or not the trials of drawing balls are Bernoulli trials when after each draw the ball drawn is

(i) replaced (ii) not replaced in the urn.

Solution

(i) The number of trials is finite. When the drawing is done with replacement, the probability of success (say, red ball) is $p = \frac{7}{16}$ which is same for all six trials (draws). Hence, the drawing of balls with replacements are Bernoulli trials.

(ii) When the drawing is done without replacement, the probability of success (i.e., red ball) in first trial is $\frac{7}{16}$, in 2nd trial is $\frac{6}{15}$ if the first ball drawn is red or $\frac{7}{15}$ if the first ball drawn is black and so on. Clearly, the probability of success is not same for all trials, hence the trials are not Bernoulli trials.

13.7.2 *Binomial distribution*

Consider the experiment of tossing a coin in which each trial results in success (say, heads) or failure (tails). Let S and F denote respectively success and failure in each trial. Suppose we are interested in finding the ways in which we have one success in six trials.

Clearly, six different cases are there as listed below:

SFFFFF, FSFFFF, FFSFFF, FFFSFF, FFFFSF, FFFFFS.

Similarly, two successes and four failures can have $\frac{6!}{4!\times 2!}$ combinations. It will be lengthy job to list all of these ways. Therefore, calculation of probabilities of 0, 1, 2,..., n number of successes may be lengthy and time consuming. To avoid the lengthy calculations and listing of all the possible cases, for the probabilities of number of successes in n-Bernoulli trials, a formula is derived. For this purpose, let us take the experiment made up of three Bernoulli trials with probabilities p and $q = 1 - p$ for success and failure respectively in each trial. The sample space of the experiment is the set

$$S = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

The number of successes is a random variable X and can take values 0, 1, 2, or 3. The probability distribution of the number of successes is as below :

$$\begin{aligned}
P(X = 0) &= P(\text{no success}) \\
&= P(\{FFF\}) = P(F)\ P(F)\ P(F) \\
&= q\ .\ q\ .\ q = q^3 \text{ since the trials are independent} \\
P(X = 1) &= P(\text{one successes}) \\
&= P(\{SFF, FSF, FFS\}) \\
&= P(\{SFF\}) + P(\{FSF\}) + P(\{FFS\}) \\
&= P(S)\ P(F)\ P(F) + P(F)\ P(S)\ P(F) + P(F)\ P(F)\ P(S) \\
&= p.q.q + q.p.q + q.q.p = 3pq^2 \\
P(X = 2) &= P\ (\text{two successes}) \\
&= P(\{SSF, SFS, FSS\}) \\
&= P(\{SSF\}) + P\ (\{SFS\}) + P(\{FSS\})
\end{aligned}$$

$$= P(S)\ P(S)\ P(F) + P(S)\ P(F)\ P(S) + P(F)\ P(S)\ P(S)$$
$$= p.p.q. + p.q.p + q.p.p = 3p^2q$$

and $\qquad P(X = 3) = P(\text{three success}) = P(\{SSS\})$
$$= P(S)\ .\ P(S)\ .\ P(S) = p^3$$

Thus, the probability distribution of X is

X	0	1	2	3
P(X)	q^3	$3q^2p$	$3qp^2$	p^3

Also, the binominal expansion of $(q + p)^3$ is

$$q^3 + 3q^2p + 3qp^2 + p^3$$

Note that the probabilities of 0, 1, 2 or 3 successes are respectively the 1st, 2nd, 3rd and 4th term in the expansion of $(q + p)^3$.

Also, since $q + p = 1$, it follows that the sum of these probabilities, as expected, is 1.

Thus, we may conclude that in an experiment of n-Bernoulli trials, the probabilities of 0, 1, 2,..., n successes can be obtained as 1st, 2nd,...,$(n + 1)^{\text{th}}$ terms in the expansion of $(q + p)^n$. To prove this assertion (result), let us find the probability of x-successes in an experiment of n-Bernoulli trials.

Clearly, in case of x successes (S), there will be $(n - x)$ failures (F).

Now, x successes (S) and $(n - x)$ failures (F) can be obtained in $\dfrac{n!}{x!(n-x)!}$ ways.

In each of these ways, the probability of x successes and $(n - x)$ failures is

$$= P(x \text{ successes})\ .\ P(n\text{–}x) \text{ failures is}$$

$$= \underbrace{P(S).P(S)...P(S)}_{x\,\text{times}} \cdot \underbrace{P(F).P(F)...P(F)}_{(n-x)\text{ times}} = p^x\ q^{n-x}$$

Thus, the probability of x successes in n-Bernoulli trials is $\dfrac{n!}{x!(n-x)!}p^x\ q^{n-x}$

or ${}^nC_x\, p^x\ q^{n-x}$

Thus $\qquad P(x \text{ successes}) = {}^nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, 2,...,n.\ (q = 1 - p)$

Clearly, P(x successes), i.e. ${}^nC_x\, p^x\, q^{n-x}$ is the $(x + 1)^{\text{th}}$ term in the binomial expansion of $(q + p)^n$.

Thus, the probability distribution of number of successes in an experiment consisting of n Bernoulli trials may be obtained by the binomial expansion of $(q + p)^n$. Hence, this distribution of number of successes X can be written as

X	0	1	2	...	x	...	n
P(X)	${}^nC_0\, q^n$	${}^nC_1\, q^{n-1}p^1$	${}^nC_2\, q^{n-2}p^2$		${}^nC_x\, q^{n-x}p^x$		${}^nC_n\, p^n$

The above probability distribution is known as *binomial distribution* with parameters n and p, because for given values of n and p, we can find the complete probability distribution.

The probability of x successes P(X = x) is also denoted by P(x) and is given by

$$P(x) = {}^nC_x q^{n-x}p^x, \quad x = 0, 1,..., n. \ (q = 1 - p)$$

This P(x) is called the *probability function* of the binomial distribution.

A binomial distribution with n-Bernoulli trials and probability of success in each trial as p, is denoted by B(n, p).

Let us now take up some examples.

Example 31 If a fair coin is tossed 10 times, find the probability of

(i) exactly six heads

(ii) at least six heads

(iii) at most six heads

Solution The repeated tosses of a coin are Bernoulli trials. Let X denote the number of heads in an experiment of 10 trials.

Clearly, X has the binomial distribution with $n = 10$ and $p = \frac{1}{2}$

Therefore $\quad P(X = x) = {}^nC_x q^{n-x}p^x, \ x = 0, 1, 2,...,n$

Here $\quad n = 10, \ p = \frac{1}{2}, \ q = 1 - p = \frac{1}{2}$

Therefore $\quad P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x} \left(\frac{1}{2}\right)^x = {}^{10}C_x \left(\frac{1}{2}\right)^{10}$

Now (i) $P(X = 6) = {}^{10}C_6 \left(\frac{1}{2}\right)^{10} = \frac{10!}{6! \times 4!}\frac{1}{2^{10}} = \frac{105}{512}$

(ii) P(at least six heads) = P(X ≥ 6)

$$= P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left[\left(\frac{10!}{6!\times 4!}\right)+\left(\frac{10!}{7!\times 3!}\right)+\left(\frac{10!}{8!\times 2!}\right)+\left(\frac{10!}{9!\times 1!}\right)+\left(\frac{10!}{10!}\right)\right]\frac{1}{2^{10}} = \frac{193}{512}$$

(iii) P(at most six heads) = P(X ≤ 6)

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$+ P(X=4) + P(X=5) + P(X=6)$$

$$= \left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10}$$

$$+ {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$$

$$= \frac{848}{1024} = \frac{53}{64}$$

Example 32 Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Solution Let X denote the number of defective eggs in the 10 eggs drawn. Since the drawing is done with replacement, the trials are Bernoulli trials. Clearly, X has the binomial distribution with $n = 10$ and $p = \frac{10}{100} = \frac{1}{10}$.

Therefore $$q = 1 - p = \frac{9}{10}$$

Now P(at least one defective egg) = P(X ≥ 1) = 1 – P (X = 0)

$$= 1 - {}^{10}C_0\left(\frac{9}{10}\right)^{10} = 1 - \frac{9^{10}}{10^{10}}$$

EXERCISE 13.5

1. A die is thrown 6 times. If ‘getting an odd number’ is a success, what is the probability of
   (i) 5 successes? (ii) at least 5 successes?
   (iii) at most 5 successes?

2. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

3. There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

4. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(i) all the five cards are spades?

(ii) only 3 cards are spades?

(iii) none is a spade?

5. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs

(i) none

(ii) not more than one

(iii) more than one

(iv) at least one

will fuse after 150 days of use.

6. A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

7. In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.

8. Suppose X has a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome.

(Hint : $P(X = 3)$ is the maximum among all $P(x_i)$, $x_i = 0,1,2,3,4,5,6$)

9. On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing ?

10. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize

(a) at least once (b) exactly once (c) at least twice?

11. Find the probability of getting 5 exactly twice in 7 throws of a die.

12. Find the probability of throwing at most 2 sixes in 6 throws of a single die.

13. It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

In each of the following, choose the correct answer:

14. In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is

(A) 10^{-1} (B) $\left(\frac{1}{2}\right)^5$ (C) $\left(\frac{9}{10}\right)^5$ (D) $\frac{9}{10}$

15. The probability that a student is not a swimmer is $\frac{1}{5}\cdot$ Then the probability that out of five students, four are swimmers is

(A) ${}^5C_4\left(\frac{4}{5}\right)^4\frac{1}{5}$ (B) $\left(\frac{4}{5}\right)^4\frac{1}{5}$

(C) ${}^5C_1\frac{1}{5}\left(\frac{4}{5}\right)^4$ (D) None of these

Miscellaneous Examples

Example 33 Coloured balls are distributed in four boxes as shown in the following table:

Box	Colour			
	Black	**White**	**Red**	**Blue**
I	3	4	5	6
II	2	2	2	2
III	1	2	3	1
IV	4	3	1	5

A box is selected at random and then a ball is randomly drawn from the selected box. The colour of the ball is black, what is the probability that ball drawn is from the box III?

Solution Let A, E_1, E_2, E_3 and E_4 be the events as defined below :

A : a black ball is selected $\qquad$ E_1 : box I is selected

E_2 : box II is selected $\qquad$ E_3 : box III is selected

E_4 : box IV is selected

Since the boxes are chosen at random,

Therefore $\qquad P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$

Also $\qquad P(A|E_1) = \frac{3}{18}$, $P(A|E_2) = \frac{2}{8}$, $P(A|E_3) = \frac{1}{7}$ and $P(A|E_4) = \frac{4}{13}$

P(box III is selected, given that the drawn ball is black) = $P(E_3|A)$. By Bayes' theorem,

$$P(E_3|A) = \frac{P(E_3)\cdot P(A|E_3)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)+P(E_3)P(A|E_3)+P(E_4)P(A|E_4)}$$

$$= \frac{\frac{1}{4}\times\frac{1}{7}}{\frac{1}{4}\times\frac{3}{18}+\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{7}+\frac{1}{4}\times\frac{4}{13}} = 0.165$$

Example 34 Find the mean of the Binomial distribution $B\left(4, \frac{1}{3}\right)$.

Solution Let X be the random variable whose probability distribution is $B\left(4, \frac{1}{3}\right)$.

Here $\qquad n = 4,\ p = \frac{1}{3}$ and $q = 1 - \frac{1}{3} = \frac{2}{3}$

We know that $\qquad P(X = x) = {}^4C_x \left(\frac{2}{3}\right)^{4-x}\left(\frac{1}{3}\right)^x,\ x = 0, 1, 2, 3, 4.$

i.e. the distribution of X is

x_i	$P(x_i)$	$x_i\, P(x_i)$
0	${}^4C_0\left(\frac{2}{3}\right)^4$	0
1	${}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$	${}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$

2	${}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$	$2\left({}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2\right)$
3	${}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3$	$3\left({}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3\right)$
4	${}^4C_4\left(\frac{1}{3}\right)^4$	$4\left({}^4C_4\left(\frac{1}{3}\right)^4\right)$

Now Mean (μ) $= \sum_{i=1}^{4} x_i\, p(x_i)$

$$= 0 + {}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 2\cdot{}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 3\cdot{}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + 4\cdot{}^4C_4\left(\frac{1}{3}\right)^4$$

$$= 4\times\frac{2^3}{3^4} + 2\times6\times\frac{2^2}{3^4} + 3\times4\times\frac{2}{3^4} + 4\times1\times\frac{1}{3^4}$$

$$= \frac{32+48+24+4}{3^4} = \frac{108}{81} = \frac{4}{3}$$

Example 35 The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99?

Solution Let the shooter fire n times. Obviously, n fires are n Bernoulli trials. In each trial, p = probability of hitting the target = $\frac{3}{4}$ and q = probability of not hitting the target = $\frac{1}{4}$. Then $P(X = x) = {}^nC_x\, q^{n-x}\, p^x = {}^nC_x\left(\frac{1}{4}\right)^{n-x}\left(\frac{3}{4}\right)^x = {}^nC_x\frac{3^x}{4^n}$.

Now, given that,

P(hitting the target at least once) > 0.99

i.e. $\qquad P(x \geq 1) > 0.99$

Therefore, $1 - P(x = 0) > 0.99$

or $$1 - {}^nC_0 \frac{1}{4^n} > 0.99$$

or $${}^nC_0 \frac{1}{4^n} < 0.01 \quad \text{i.e.} \quad \frac{1}{4^n} < 0.01$$

or $$4^n > \frac{1}{0.01} = 100 \qquad \ldots (1)$$

The minimum value of n to satisfy the inequality (1) is 4.

Thus, the shooter must fire 4 times.

Example 36 A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.

Solution Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

Thus, $$P(S) = \frac{1}{6}, \; P(F) = \frac{5}{6}$$

$$P(\text{A wins in the first throw}) = P(S) = \frac{1}{6}$$

A gets the third throw, when the first throw by A and second throw by B result into failures.

Therefore, $$P(\text{A wins in the 3rd throw}) = P(FFS) = P(F)P(F)P(S) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$= \left(\frac{5}{6}\right)^2 \times \frac{1}{6}$$

$$P(\text{A wins in the 5th throw}) = P\,(FFFFS) = \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) \text{ and so on.}$$

Hence, $$P(\text{A wins}) = \frac{1}{6} + \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) + \ldots$$

$$= \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$$

$$P(\text{B wins}) = 1 - P(\text{A wins}) = 1-\frac{6}{11}=\frac{5}{11}$$

Remark If $a + ar + ar^2 + ... + ar^{n-1} + ...$, where $|r| < 1$, then sum of this infinite G.P. is given by $\frac{a}{1-r}\cdot$ (Refer A.1.3 of Class XI Text book).

Example 37 If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the set ups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly setup.

Solution Let A be the event that the machine produces 2 acceptable items.

Also let B_1 represent the event of correct set up and B_2 represent the event of incorrect setup.

Now $\quad P(B_1) = 0.8, P(B_2) = 0.2$

$$P(A|B_1) = 0.9 \times 0.9 \text{ and } P(A|B_2) = 0.4 \times 0.4$$

Therefore
$$P(B_1|A) = \frac{P(B_1)\,P(A|B_1)}{P(B_1)\,P(A|B_1)+P(B_2)\,P(A|B_2)}$$

$$= \frac{0.8\times0.9\times0.9}{0.8\times0.9\times0.9+0.2\times0.4\times0.4} = \frac{648}{680} = 0.95$$

Miscellaneous Exercise on Chapter 13

1. A and B are two events such that P (A) ≠ 0. Find P(B|A), if
(i) A is a subset of B (ii) $A \cap B = \phi$

2. A couple has two children,
(i) Find the probability that both children are males, if it is known that at least one of the children is male.
(ii) Find the probability that both children are females, if it is known that the elder child is a female.

3. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

4. Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right-handed?

5. An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

(i) all will bear 'X' mark.

(ii) not more than 2 will bear 'Y' mark.

(iii) at least one ball will bear 'Y' mark.

(iv) the number of balls with 'X' mark and 'Y' mark will be equal.

6. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

7. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.

8. If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?

9. An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be atleast 4 successes.

10. How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

11. In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins / loses.

12. Suppose we have four boxes A,B,C and D containing coloured marbles as given below:

Box	**Marble colour**		
	Red	**White**	**Black**
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from box A?, box B?, box C?

13. Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

14. If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$).

15. An electronic assembly consists of two subsystems, say, A and B. From previous testing procedures, the following probabilities are assumed to be known:

$$P(\text{A fails}) = 0.2$$
$$P(\text{B fails alone}) = 0.15$$
$$P(\text{A and B fail}) = 0.15$$

Evaluate the following probabilities

(i) $P(\text{A fails}|\text{B has failed})$ (ii) $P(\text{A fails alone})$

16. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Choose the correct answer in each of the following:

17. If A and B are two events such that $P(A) \neq 0$ and $P(B \mid A) = 1$, then

(A) $A \subset B$ (B) $B \subset A$ (C) $B = \phi$ (D) $A = \phi$

18. If $P(A|B) > P(A)$, then which of the following is correct :

(A) $P(B|A) < P(B)$ (B) $P(A \cap B) < P(A) \, . \, P(B)$

(C) $P(B|A) > P(B)$ (D) $P(B|A) = P(B)$

19. If A and B are any two events such that $P(A) + P(B) - P(\text{A and B}) = P(A)$, then

(A) $P(B|A) = 1$ (B) $P(A|B) = 1$

(C) $P(B|A) = 0$ (D) $P(A|B) = 0$

Summary

The salient features of the chapter are –

- The conditional probability of an event E, given the occurrence of the event F is given by $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$, $P(F) \neq 0$
- $0 \leq P(E|F) \leq 1$, $P(E'|F) = 1 - P(E|F)$
  $P((E \cup F)|G) = P(E|G) + P(F|G) - P((E \cap F)|G)$
- $P(E \cap F) = P(E)\, P(F|E)$, $P(E) \neq 0$
  $P(E \cap F) = P(F)\, P(E|F)$, $P(F) \neq 0$
- If E and F are independent, then
  $P(E \cap F) = P(E)\, P(F)$
  $P(E|F) = P(E)$, $P(F) \neq 0$
  $P(F|E) = P(F)$, $P(E) \neq 0$
- **Theorem of total probability**
  Let $\{E_1, E_2, ..., E_n)$ be a partition of a sample space and suppose that each of $E_1, E_2, ..., E_n$ has nonzero probability. Let A be any event associated with S, then
  $P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + ... + P(E_n)\, P(A|E_n)$
- **Bayes' theorem** If $E_1, E_2, ..., E_n$ are events which constitute a partition of sample space S, i.e. $E_1, E_2, ..., E_n$ are pairwise disjoint and $E_1 \cup E_2 \cup ... \cup E_n = S$ and A be any event with nonzero probability, then

$$P(E_i \mid A) = \frac{P(E_i)\, P(A|E_i)}{\sum_{j=1}^{n} P(E_j)\, P(A|E_j)}$$

- A random variable is a real valued function whose domain is the sample space of a random experiment.
- The probability distribution of a random variable X is the system of numbers

X	:	x_1	x_2	...	x_n
P(X)	:	p_1	p_2	...	p_n

where, $p_i > 0, \sum_{i=1}^{n} p_i = 1, i = 1, 2, ..., n$

- Let X be a random variable whose possible values $x_1, x_2, x_3, ..., x_n$ occur with probabilities $p_1, p_2, p_3, ... p_n$ respectively. The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i p_i$.

  The mean of a random variable X is also called the expectation of X, denoted by E (X).
- Let X be a random variable whose possible values $x_1, x_2, ..., x_n$ occur with probabilities $p(x_1), p(x_2), ..., p(x_n)$ respectively.

  Let μ = E(X) be the mean of X. The variance of X, denoted by Var (X) or σ_x^2, is defined as $\sigma_x^2 = \text{Var}(X) = \sum_{i=1}^{n}(x_i - \mu)^2 p(x_i)$

  or equivalently $\sigma_x^2 = E(X - \mu)^2$

  The non-negative number

$$\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{\sum_{i=1}^{n}(x_i - \mu)^2 p(x_i)}$$

  is called the standard deviation of the random variable X.
- $\text{Var}(X) = E(X^2) - [E(X)]^2$
- Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :
  (i) There should be a finite number of trials.
  (ii) The trials should be independent.
  (iii) Each trial has exactly two outcomes : success or failure.
  (iv) The probability of success remains the same in each trial.

  For Binomial distribution B (n, p), $P(X = x) = {}^nC_x\, q^{n-x} p^x$, $x = 0, 1, ..., n$ $(q = 1 - p)$

Historical Note

The earliest indication on measurement of chances in game of dice appeared in 1477 in a commentary on Dante's Divine Comedy. A treatise on gambling named *liber de Ludo Alcae*, by Geronimo Carden (1501-1576) was published posthumously in 1663. In this treatise, he gives the number of favourable cases for each event when two dice are thrown.

Galileo (1564-1642) gave casual remarks concerning the correct evaluation of chance in a game of three dice. Galileo analysed that when three dice are thrown, the sum of the number that appear is more likely to be 10 than the sum 9, because the number of cases favourable to 10 are more than the number of cases for the appearance of number 9.

Apart from these early contributions, it is generally acknowledged that the true origin of the science of probability lies in the correspondence between two great men of the seventeenth century, Pascal (1623-1662) and Pierre de Fermat (1601-1665). A French gambler, Chevalier de Metre asked Pascal to explain some seeming contradiction between his theoretical reasoning and the observation gathered from gambling. In a series of letters written around 1654, Pascal and Fermat laid the first foundation of science of probability. Pascal solved the problem in algebraic manner while Fermat used the method of combinations.

Great Dutch Scientist, Huygens (1629-1695), became acquainted with the content of the correspondence between Pascal and Fermat and published a first book on probability, "*De Ratiociniis in Ludo Aleae*" containing solution of many interesting rather than difficult problems on probability in games of chances.

The next great work on probability theory is by Jacob Bernoulli (1654-1705), in the form of a great book, "*Ars Conjectendi*" published posthumously in 1713 by his nephew, Nicholes Bernoulli. To him is due the discovery of one of the most important probability distribution known as Binomial distribution. The next remarkable work on probability lies in 1993. A. N. Kolmogorov (1903-1987) is credited with the axiomatic theory of probability. His book, 'Foundations of probability' published in 1933, introduces probability as a set function and is considered a 'classic!'.