NCERT Solutions Class 11 Maths Chapter 4 Principle of Mathematical Induction

Prove the following by using the principle of mathematical induction for all

Question:1

$1+3+3^2+\ldots+3^{n-1}=\frac{(3^n-1)}{2}.$

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n):1+3+3^2+\ldots+3^{n-1}=\frac{(3^n-1)}{2}$

For $n=1$,

$P(1):1=\frac{(3^1-1)}{2}=\frac{2}{2}=1$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $1+3+3^2+\ldots+3^{k-1}=\frac{(3^k-1)}{2}$ ...(1)

We will now prove that $P(k+1)$ is also true.
Now, we have

$$1+3+3^2+\ldots+3^{(k+1)-1}$$

$$\Rightarrow(1+3+3^2+\ldots+3^{k-1})+3^k$$

$$\Rightarrow\frac{(3^k-1)}{2}+3^k \quad \ldots[\text{from }(1)]$$

$$\Rightarrow\frac{3^k-1+2\times3^k}{2}$$

$$\Rightarrow\frac{(1+2)3^k-1}{2}$$

$$\Rightarrow\frac{3\times3^k-1}{2}$$

$$\Rightarrow\frac{3^{k+1}-1}{2}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:2

$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n): 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

For $n = 1$,

$P(1): 1^3 = 1 = \left[\frac{1(1+1)}{2}\right]^2 = \left[\frac{1 \times 2}{2}\right]^2 = [1]^2 = 1$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $1^3 + 2^3 + 3^3 + \ldots + k^3 = \left[\frac{k(k+1)}{2}\right]^2 \qquad \ldots(1)$

We will now prove that $P(k+1)$ is also true.
Now, we have

$$1^3+2^3+3^3+\ldots+(k+1)^3$$

$$\Rightarrow (1^3+2^3+3^3+\ldots+k^3)+(k+1)^3$$

$$\Rightarrow \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3 \qquad \ldots[\text{from }(1)]$$

$$\Rightarrow \frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$\Rightarrow \frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$\Rightarrow \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}$$

$$\Rightarrow \frac{(k+1)^2\left[k^2+4k+4\right]}{4}$$

$$\Rightarrow \frac{(k+1)^2(k+2)^2}{4}$$

$$\Rightarrow \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$$\Rightarrow \left[\frac{(k+1)(k+1+1)}{2}\right]^2$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:3

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots n)}=\frac{2n}{(n+1)}.$$

Solution:

Let $P(n)$ be the given statement.

i.e., $$P(n):1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots n)}=\frac{2n}{(n+1)}$$

For $n=1$,

$$P(1):1=\frac{2\times 1}{(1+1)}=\frac{2}{2}=1$$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots k)}=\frac{2k}{(k+1)}$ ...(1)

We will now prove that $P(k+1)$ is also true.
Now, we have

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots(k+1))}$$

$$\Rightarrow\left[1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots k)}\right]+\frac{1}{(1+2+3+\ldots+k+(k+1))}$$

$$\Rightarrow\frac{2k}{(k+1)}+\frac{1}{(1+2+3+\ldots+k+(k+1))} \qquad \ldots[\text{from }(1)]$$

$$\Rightarrow\frac{2k}{(k+1)}+\frac{1}{\frac{(k+1)(k+2)}{2}} \qquad \ldots\left[\because 1+2+3+\ldots+n=\frac{n(n+1)}{2}\right]$$

$$\Rightarrow\frac{2k}{(k+1)}+\frac{2}{(k+1)(k+2)}$$

$$\Rightarrow\frac{2k(k+2)+2}{(k+1)(k+2)}$$

$$\Rightarrow\frac{2(k^2+2k+1)}{(k+1)(k+2)}$$

$$\Rightarrow\frac{2(k+1)^2}{(k+1)(k+2)}$$

$$\Rightarrow\frac{2(k+1)}{(k+2)}$$

$$\Rightarrow\frac{2(k+1)}{(k+1)+1}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:4

$1.2.3+2.3.4+\ldots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$.

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n):1.2.3+2.3.4+\ldots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$

For $n=1$,

$$P(1): 1.2.3 = 6 = \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1.2.3.4}{4} = 6$$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $$1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4} \qquad \ldots(1)$$

We will now prove that $P(k+1)$ is also true.
Now, we have

$$1.2.3 + 2.3.4 + \ldots + (k+1)[(k+1)+1][(k+1)+2]$$
$$\Rightarrow [1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2)] + (k+1)(k+2)(k+3)$$
$$\Rightarrow \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \qquad \ldots[\text{from }(1)]$$
$$\Rightarrow \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$$
$$\Rightarrow \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$
$$\Rightarrow \frac{(k+1)[(k+1)+1][(k+1)+2][(k+1)+3]}{4}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:5

$$1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}.$$

Solution:

Let $P(n)$ be the given statement.

i.e., $$P(n): 1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

For $n=1$,

$$P(1):1.3=3=\frac{(2\times1-1)3^{1+1}+3}{4}=\frac{1.3^2+3}{4}=\frac{12}{4}=3$$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $$1.3+2.3^2+3.3^3+\ldots+k.3^k=\frac{(2k-1)3^{k+1}+3}{4} \qquad \ldots(1)$$

We will now prove that $P(k+1)$ is also true.
Now, we have

$$1.3+2.3^2+3.3^3+\ldots+(k+1).3^{k+1}$$
$$\Rightarrow\left[1.3+2.3^2+3.3^3+\ldots+k.3^k\right]+(k+1).3^{k+1}$$
$$\Rightarrow\frac{(2k-1)3^{k+1}+3}{4}+(k+1).3^{k+1} \qquad \ldots[\text{from }(1)]$$
$$\Rightarrow\frac{(2k-1)3^{k+1}+3+4(k+1).3^{k+1}}{4}$$
$$\Rightarrow\frac{3^{k+1}\left[(2k-1)+4(k+1)\right]+3}{4}$$
$$\Rightarrow\frac{3^{k+1}\left[2k-1+4k+4\right]+3}{4}$$
$$\Rightarrow\frac{3^{k+1}\left[6k+3\right]+3}{4}$$
$$\Rightarrow\frac{3^{k+1}.3\left[2k+1\right]+3}{4}$$
$$\Rightarrow\frac{\left[2k+2-1\right].3^{(k+1)+1}+3}{4}$$
$$\Rightarrow\frac{\left[2(k+1)-1\right]3^{(k+1)+1}+3}{4}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:6

$$1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right].$$

Solution:

Let $P(n)$ be the given statement.

i.e., $$P(n): 1.2 + 2.3 + 3.4 + \ldots + n.(n+1) = \left[\frac{n(n+1)(n+2)}{3}\right]$$

For $n = 1$,

$$P(1): 1.2 = 2 = \left[\frac{1(1+1)(1+2)}{3}\right] = \frac{1.2.3}{3} = 2$$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $$1.2 + 2.3 + 3.4 + \ldots + k.(k+1) = \left[\frac{k(k+1)(k+2)}{3}\right] \quad \ldots(1)$$

We will now prove that $P(k+1)$ is also true.
Now, we have

$$1.2 + 2.3 + 3.4 + \ldots + (k+1)\left[(k+1)+1\right]$$
$$\Rightarrow \left[1.2 + 2.3 + 3.4 + \ldots + k.(k+1)\right] + (k+1)(k+2)$$
$$\Rightarrow \left[\frac{k(k+1)(k+2)}{3}\right] + (k+1)(k+2) \quad \ldots\left[\text{from } (1)\right]$$
$$\Rightarrow \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$
$$\Rightarrow \frac{(k+1)(k+2)(k+3)}{3}$$
$$\Rightarrow \frac{(k+1)\left[(k+1)+1\right]\left[(k+1)+2\right]}{3}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:7

$$1.3 + 3.5 + 5.7 + \ldots + (2n-1)(2n+1) = \frac{n(4n^2 + 6n - 1)}{3}.$$

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n): 1.3 + 3.5 + 5.7 + \ldots + (2n-1)(2n+1) = \dfrac{n(4n^2 + 6n - 1)}{3}$

For $n = 1$,

$$P(1): 1.3 = 3 = \frac{1(4.1^2 + 6.1 - 1)}{3} = \frac{9}{3} = 3$$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $1.3 + 3.5 + 5.7 + \ldots + (2k-1)(2k+1) = \dfrac{k(4k^2 + 6k - 1)}{3}$ ...(1)

We will now prove that $P(k+1)$ is also true.
Now, we have

$$1.3 + 3.5 + 5.7 + \ldots + [2(k+1) - 1][2(k+1) + 1]$$
$$\Rightarrow [1.3 + 3.5 + 5.7 + \ldots + (2k-1)(2k+1)] + (2k+1)(2k+3)$$
$$\Rightarrow \left[\frac{k(4k^2 + 6k - 1)}{3}\right] + (4k^2 + 8k + 3) \qquad \ldots[\text{from } (1)]$$
$$\Rightarrow \frac{k(4k^2 + 6k - 1) + 3(4k^2 + 8k + 3)}{3}$$
$$\Rightarrow \frac{4k^3 + 6k^2 - k + 12k^2 + 24k + 9}{3}$$
$$\Rightarrow \frac{4k^3 + 18k^2 + 23k + 9}{3}$$
$$\Rightarrow \frac{4k^3 + 14k^2 + 9k + 4k^2 + 14k + 9}{3}$$
$$\Rightarrow \frac{k(4k^2 + 14k + 9) + (4k^2 + 14k + 9)}{3}$$
$$\Rightarrow \frac{(k+1)(4k^2 + 14k + 9)}{3}$$
$$\Rightarrow \frac{(k+1)(4k^2 + 8k + 4 + 6k + 6 - 1)}{3}$$
$$\Rightarrow \frac{(k+1)[4(k^2 + 2k + 2) + 6(k+1) - 1]}{3}$$
$$\Rightarrow \frac{(k+1)[4(k+1)^2 + 6(k+1) - 1]}{3}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:8

$1.2+2.2^2+3.2^3+\ldots+n.2^n=(n-1)2^{n+1}+2.$

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n): 1.2+2.2^2+3.2^3+\ldots+n.2^n=(n-1)2^{n+1}+2$

For $n=1$,

$P(1): 1.2=2=(1-1)2^{1+1}+2=0+2=2$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $1.2+2.2^2+3.2^3+\ldots+k.2^n=(k-1)2^{k+1}+2 \qquad \ldots(1)$

We will now prove that $P(k+1)$ is also true.
Now, we have

$$1.2+2.2^2+3.2^3+\ldots+(k+1).2^{k+1}$$
$$\Rightarrow \left[1.2+2.2^2+3.2^3+\ldots+k.2^n\right]+(k+1).2^{k+1}$$
$$\Rightarrow (k-1)2^{k+1}+2+(k+1).2^{k+1} \qquad \ldots[\text{from }(1)]$$
$$\Rightarrow \left[(k-1)+(k+1)\right]2^{k+1}+2$$
$$\Rightarrow 2k.2^{(k+1)}+2$$
$$\Rightarrow k.2^{(k+1)+1}+2$$
$$\Rightarrow \left[(k+1)-1\right].2^{(k+1)+1}+2$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:9

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}.$$

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n):\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

For $n=1$,

$P(1):\frac{1}{2}=1-\frac{1}{2^1}=1-\frac{1}{2}=\frac{1}{2}$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$ ...(1)

We will now prove that $P(k+1)$ is also true.

Now, we have

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^{k+1}}$$

$$\Rightarrow\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}\right]+\frac{1}{2^{k+1}}$$

$$\Rightarrow 1-\frac{1}{2^k}+\frac{1}{2^{k+1}} \qquad \ldots[\text{from }(1)]$$

$$\Rightarrow 1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$$

$$\Rightarrow 1-\frac{1}{2^k}\cdot\frac{1}{2}$$

$$\Rightarrow 1-\frac{1}{2^{k+1}}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:10

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}.$$

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n):\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$

For $n=1$,

$P(1):\frac{1}{2.5}=\frac{1}{10}=\frac{1}{(6.1+4)}=\frac{1}{10}$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{(6k+4)}$ ...(1)

We will now prove that $P(k+1)$ is also true.

Now, we have

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{[3(k+1)-1][3(k+1)+2]}$$

$$\Rightarrow\left[\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3k-1)(3k+2)}\right]+\frac{1}{(3k+2)(3k+5)}$$

$$\Rightarrow\frac{k}{(6k+4)}+\frac{1}{(3k+2)(3k+5)} \qquad \ldots[\text{from } (1)]$$

$$\Rightarrow\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$\Rightarrow\frac{1}{(3k+2)}\left[\frac{k}{2}+\frac{1}{(3k+5)}\right]$$

$$\Rightarrow\frac{1}{(3k+2)}\left[\frac{k(3k+5)+2}{2(3k+5)}\right]$$

$$\Rightarrow\frac{1}{(3k+2)}\left[\frac{3k^2+5k+2}{2(3k+5)}\right]$$

$$\Rightarrow\frac{1}{(3k+2)}\left[\frac{3k^2+3k+2k+2}{2(3k+5)}\right]$$

$$\Rightarrow\frac{1}{(3k+2)}\left[\frac{3k(k+1)+2(k+1)}{2(3k+5)}\right]$$

$$\Rightarrow\frac{1}{(3k+2)}\left[\frac{(k+1)(3k+2)}{2(3k+5)}\right]$$

$$\Rightarrow\frac{(k+1)}{(6k+10)}$$

$$\Rightarrow\frac{(k+1)}{[(6k+6)+4]}$$

$$\Rightarrow\frac{(k+1)}{[6(k+1)+4]}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:11

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}.$$

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n): \frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

For $n=1$,

$P(1): \frac{1}{1.2.3}=\frac{1}{6}=\frac{1(1+3)}{4(1+1)(1+2)}=\frac{1.4}{4.2.3}=\frac{1}{6}$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$ ...(1)

We will now prove that $P(k+1)$ is also true.
Now, we have

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{(k+1)[(k+1)+1][(k+1)+2]}$$

$$\Rightarrow\left[\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

$$\Rightarrow\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \qquad \ldots[\text{from }(1)]$$

$$\Rightarrow\frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)}{4}+\frac{1}{(k+3)}\right]$$

$$\Rightarrow\frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)^2+4}{4(k+3)}\right]$$

$$\Rightarrow\frac{1}{(k+1)(k+2)}\left[\frac{k\left(k^2+6k+9\right)+4}{4(k+3)}\right]$$

$$\Rightarrow\frac{1}{(k+1)(k+2)}\left[\frac{k^3+6k^2+9k+4}{4(k+3)}\right]$$

$$\Rightarrow\frac{1}{(k+1)(k+2)}\left[\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right]$$

$$\Rightarrow\frac{1}{(k+1)(k+2)}\left[\frac{k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)}{4(k+3)}\right]$$

$$\Rightarrow\frac{1}{(k+1)(k+2)}\left[\frac{(k+4)\left(k^2+2k+1\right)}{4(k+3)}\right]$$

$$\Rightarrow\frac{1}{(k+1)(k+2)}\left[\frac{(k+4)(k+1)^2}{4(k+3)}\right]$$

$$\Rightarrow\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\Rightarrow\frac{(k+1)[(k+1)+3]}{4[(k+1)+1][(k+1)+2]}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:12

$a+ar+ar^2+\ldots+ar^{n-1}=\dfrac{a\left(r^n-1\right)}{r-1}$.

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n): a+ar+ar^2+\ldots+ar^{n-1}=\dfrac{a(r^n-1)}{r-1}$

For $n=1$,

$P(1): a=\dfrac{a(r^1-1)}{r-1}=\dfrac{a(r-1)}{r-1}=a$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $a+ar+ar^2+\ldots+ar^{k-1}=\dfrac{a(r^k-1)}{r-1}$ $\quad\ldots(1)$

We will now prove that $P(k+1)$ is also true.
Now, we have

$$a+ar+ar^2+\ldots+ar^{(k+1)-1}$$
$$\Rightarrow\left[a+ar+ar^2+\ldots+ar^{k-1}\right]+ar^k$$
$$\Rightarrow\frac{a(r^k-1)}{r-1}+ar^k \qquad \ldots[\text{from }(1)]$$
$$\Rightarrow\frac{a(r^k-1)+ar^k(r-1)}{r-1}$$
$$\Rightarrow\frac{ar^k-a+ar^{k+1}-ar^k}{r-1}$$
$$\Rightarrow\frac{ar^{k+1}-a}{r-1}$$
$$\Rightarrow\frac{a(r^{k+1}-1)}{r-1}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:13

$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$.

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$

For $n=1$,

$P(1):\left(1+\frac{3}{1}\right)=4=(1+1)^2=2^2=4$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)=(k+1)^2 \qquad \ldots(1)$

We will now prove that $P(k+1)$ is also true.
Now, we have

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{[2(k+1)+1]}{(k+1)^2}\right)$$

$$\Rightarrow\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left(1+\frac{(2k+3)}{(k+1)^2}\right)$$

$$\Rightarrow(k+1)^2\left(1+\frac{(2k+3)}{(k+1)^2}\right) \qquad \ldots[\text{from }(1)]$$

$$\Rightarrow(k+1)^2\left(\frac{(k+1)^2+(2k+3)}{(k+1)^2}\right)$$

$$\Rightarrow(k+1)^2+(2k+3)$$

$$\Rightarrow k^2+2k+1+2k+3$$

$$\Rightarrow k^2+4k+4$$

$$\Rightarrow(k+2)^2$$

$$\Rightarrow(k+1+1)^2$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:14

$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1).$

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$

For $n=1$,

$P(1):\left(1+\frac{1}{1}\right)=2=(1+1)=2$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)=(k+1) \qquad \ldots(1)$

We will now prove that $P(k+1)$ is also true.

Now, we have

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k+1}\right)$$

$$\Rightarrow\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$$\Rightarrow(k+1)\left(1+\frac{1}{k+1}\right)$$

$$\Rightarrow(k+1)\left(\frac{k+1+1}{k+1}\right) \qquad \ldots[\text{from }(1)]$$

$$\Rightarrow[(k+1)+1]$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:15

$1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$.

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n): 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

For $n = 1$,

$P(1): 1^2 = 1 = \frac{1(2.1-1)(2.1+1)}{3} = \frac{1.1.3}{3} = 1$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$ ...(1)

We will now prove that $P(k+1)$ is also true.

Now, we have

$$1^2+3^2+5^2+\ldots+\left[2(k+1)-1\right]^2$$

$$\Rightarrow\left[1^2+3^2+5^2+\ldots+(2k-1)^2\right]+(2k+1)^2$$

$$\Rightarrow\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2 \qquad \ldots\left[\text{from (1)}\right]$$

$$\Rightarrow\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$\Rightarrow\frac{(2k+1)\left[k(2k-1)+3(2k+1)\right]}{3}$$

$$\Rightarrow\frac{(2k+1)\left[2k^2-k+6k+3\right]}{3}$$

$$\Rightarrow\frac{(2k+1)\left[2k^2+5k+3\right]}{3}$$

$$\Rightarrow\frac{(2k+1)\left[2k^2+2k+3k+3\right]}{3}$$

$$\Rightarrow\frac{(2k+1)\left[2k(k+1)+3(k+1)\right]}{3}$$

$$\Rightarrow\frac{(2k+1)(k+1)(2k+3)}{3}$$

$$\Rightarrow\frac{(k+1)\left[2(k+1)-1\right]\left[2(k+1)+1\right]}{3}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:16

$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$.

Solution:

Let $P(n)$ be the given statement.

i.e., $$P(n):\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

For $n=1$,

$$P(1):\frac{1}{1.4}=\frac{1}{4}=\frac{1}{(3.1+1)}=\frac{1}{4}$$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{(3k+1)}$ ...(1)

We will now prove that $P(k+1)$ is also true.
Now, we have

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{[3(k+1)-2][3(k+1)+1]}$$

$$\Rightarrow\left[\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right]+\frac{1}{(3k+1)(3k+4)}$$

$$\Rightarrow\frac{k}{(3k+1)}+\frac{1}{(3k+1)(3k+4)} \qquad \ldots[\text{from }(1)]$$

$$\Rightarrow\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$\Rightarrow\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$\Rightarrow\frac{3k^2+3k+k+1}{(3k+1)(3k+4)}$$

$$\Rightarrow\frac{3k(k+1)+(k+1)}{(3k+1)(3k+4)}$$

$$\Rightarrow\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$\Rightarrow\frac{(k+1)}{[3(k+1)+1]}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:17

$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$.

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$

For $n=1$,

$P(1): \frac{1}{3.5}=\frac{1}{15}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}=\frac{1}{15}$, which is true.

Assume that $P(k)$ is true for some positive integer k

i.e., $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \qquad \ldots(1)$

We will now prove that $P(k+1)$ is also true.
Now, we have

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{[2(k+1)+1][2(k+1)+3]}$$

$$\Rightarrow \left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{(2k+3)(2k+5)}$$

$$\Rightarrow \frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \qquad \ldots[\text{from } (1)]$$

$$\Rightarrow \frac{k(2k+5)+3}{3(2k+3)(2k+5)}$$

$$\Rightarrow \frac{2k^2+5k+3}{3(2k+3)(2k+5)}$$

$$\Rightarrow \frac{2k^2+2k+3k+3}{3(2k+3)(2k+5)}$$

$$\Rightarrow \frac{2k(k+1)+3(k+1)}{3(2k+3)(2k+5)}$$

$$\Rightarrow \frac{(2k+3)(k+1)}{3(2k+3)(2k+5)}$$

$$\Rightarrow \frac{(k+1)}{3(2k+5)}$$

$$\Rightarrow \frac{(k+1)}{3[2(k+1)+3]}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:18

$1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$.

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n):1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$

We note that $P(n)$ is true for $n=1$,
Since,

$$P(1):1<\frac{1}{8}(2.1+1)^2=\frac{9}{8}=1\frac{1}{8}$$

Assume that $P(k)$ is true for some positive integer k

i.e., $1+2+3+\ldots+k<\frac{1}{8}(2k+1)^2 \qquad \ldots(1)$

We will now prove that $P(k+1)$ is true whenever $P(k)$ is true
Now, we have

$$1+2+3+\ldots+k<\frac{1}{8}(2k+1)^2 \qquad [\text{from }(1)]$$

$$\begin{aligned}1+2+3+\ldots+k+(k+1)&<\frac{1}{8}(2k+1)^2+(k+1)\\&<\frac{1}{8}\left[(2k+1)^2+8(k+1)\right]\\&<\frac{1}{8}\left[4k^2+4k+1+8k+8\right]\\&<\frac{1}{8}\left[4k^2+12k+9\right]\\&<\frac{1}{8}[2k+3]^2\\&<\frac{1}{8}\left[2(k+1)+1\right]^2\end{aligned}$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:19

$n(n+1)(n+5)$ is a multiple of 3.

Solution:

We can write

$$P(n): n(n+1)(n+5) \text{ is a multiple of } 3.$$

We note that

$$P(1): 1(1+1)(1+5) = 1.2.6 = 12 \text{ which is a multiple of } 3.$$

Thus $P(n)$ is true for $n=1$

Let $P(k)$ be true for some natural number *k*,

i.e., $P(k): k(k+1)(k+5)$ is a multiple of 3.

We can write

$$k(k+1)(k+5) = 3a \qquad \ldots(1)$$

where $a \in N$.

Now, we will prove that $P(k+1)$ is true whenever $P(k)$ is true.

Now,

$$\begin{aligned}
&(k+1)[(k+1)+1][(k+1)+5] \\
&\Rightarrow (k+1)(k+2)[(k+5)+1] \\
&\Rightarrow (k+1)(k+2)(k+5)+(k+1)(k+2) \\
&\Rightarrow (k+2)[(k+1)(k+5)]+(k+1)(k+2) \\
&\Rightarrow [k(k+1)(k+5)+2(k+1)(k+5)]+(k+1)(k+2) \\
&\Rightarrow [3a+2(k+1)(k+5)]+(k+1)(k+2) \qquad [\text{from } (1)] \\
&\Rightarrow 3a+(k+1)[2(k+5)+(k+2)] \\
&\Rightarrow 3a+(k+1)[2k+10+k+2] \\
&\Rightarrow 3a+(k+1)[3k+12] \\
&\Rightarrow 3a+3(k+1)(k+4) \\
&\Rightarrow 3[a+(k+1)(k+4)]
\end{aligned}$$

From the last line, we see that

$$3[a+(k+1)(k+4)] \text{ is a multiple of } 3.$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:20

$10^{2n-1}+1$ is divisible by 11.

Solution:

We can write

$P(n): 10^{2n-1}+1$ is divisible by 11.

We note that

$P(1): 10^{2.1-1}+1=10+1=11$ which is divisible by 11.

Thus $P(n)$ is true for $n=1$

Let $P(k)$ be true for some natural number *k*,

i.e., $P(k): 10^{2k-1}+1$ is divisible by 11.

We can write

$$10^{2k-1}+1=11a \qquad \ldots(1)$$

where $a \in N$.

Now, we will prove that $P(k+1)$ is true whenever $P(k)$ is true.

Now,

$$10^{2(k+1)-1}+1$$
$$\Rightarrow 10^{2k+1}+1$$
$$\Rightarrow 10^{2}\left(10^{2k-1}\right)+1$$
$$\Rightarrow 10^{2}\left(10^{2k-1}+1-1\right)+1$$
$$\Rightarrow 10^{2}\left(10^{2k-1}+1\right)-10^{2}+1$$
$$\Rightarrow 10^{2}.11a-100+1 \qquad [\text{from}(1)]$$
$$\Rightarrow 10^{2}.11a-99$$
$$\Rightarrow 11(100a-9)$$

From the last line, we see that

$11(100a-9)$ is divisible by 11.

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:21

$x^{2n}-y^{2n}$ is divisible by $x+y$.

Solution:

We can write

$P(n): x^{2n}-y^{2n}$ is divisible by $x+y$.

We note that

$P(1): x^{2.1}-y^{2.1}=x^2-y^2=(x+y)(x-y)$ which is divisible by $x+y$.

Thus $P(n)$ is true for $n=1$

Let $P(k)$ be true for some natural number *k*,

i.e., $P(k): x^{2k}-y^{2k}$ is divisible by $x+y$.

We can write

$$x^{2k}-y^{2k}=a(x+y) \qquad \ldots(1)$$

where $a \in N$.

Now, we will prove that $P(k+1)$ is true whenever $P(k)$ is true.

Now,

$$x^{2(k+1)}-y^{2(k+1)}$$
$$\Rightarrow x^{2k+2}-y^{2k+2}$$
$$\Rightarrow x^2\left(x^{2k}\right)-y^2\left(y^{2k}\right)$$
$$\Rightarrow x^2\left(x^{2k}-y^{2k}+y^{2k}\right)-y^2\left(y^{2k}\right)$$
$$\Rightarrow x^2\left(x^{2k}-y^{2k}\right)+x^2y^{2k}-y^2\left(y^{2k}\right)$$
$$\Rightarrow x^2.a(x+y)+y^{2k}\left(x^2-y^2\right) \qquad [\text{from}(1)]$$
$$\Rightarrow x^2.a(x+y)+y^{2k}(x+y)(x-y)$$
$$\Rightarrow (x+y)\left[ax^2+(x-y)y^{2k}\right]$$

From the last line, we see that

$(x+y)\left[ax^2+(x-y)y^{2k}\right]$ is divisible by $x+y$.

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n\in N$.

Question:22

$3^{2n+2}-8n-9$ is divisible by 8.

Solution:

We can write

$P(n):3^{2n+2}-8n-9$ is divisible by 8.

We note that

$P(1):3^{2.1+2}-8.1-9=3^4-8-9=81-17=64$ which is divisible by 8.

Thus $P(n)$ is true for $n=1$

Let $P(k)$ be true for some natural number k,

i.e., $P(k):3^{2k+2}-8k-9$ is divisible by 8.

We can write

$$3^{2k+2}-8k-9=8a \qquad \ldots(1)$$

where $a\in N$.

Now, we will prove that $P(k+1)$ is true whenever $P(k)$ is true.

Now,

$$\begin{aligned}
&3^{2(k+1)+2}-8(k+1)-9\\
&\Rightarrow 3^{2k+4}-8k-8-9\\
&\Rightarrow 3^2.3^{2k+2}-8k-17\\
&\Rightarrow 3^2\left(3^{2k+2}-8k-9+8k+9\right)-8k-17\\
&\Rightarrow 3^2\left(3^{2k+2}-8k-9\right)+3^2\left(8k+9\right)-8k-17\\
&\Rightarrow 3^2.8a+72k+81-8k-17 \qquad [\text{from}(1)]\\
&\Rightarrow 9.8a+64k+64\\
&\Rightarrow 8(9a+8k+8)
\end{aligned}$$

From the last line, we see that

$8(9a+8k+8)$ is divisible by 8.

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:23

41^n-14^n is a multiple of 27.

Solution:

We can write

$P(n): 41^n-14^n$ is a multiple of 27.

We note that

$P(1): 41^1-14^1=41-14=27$ which is a multiple of 27.

Thus $P(n)$ is true for $n=1$

Let $P(k)$ be true for some natural number k,

i.e., $P(k): 41^k-14^k$ is a multiple of 27.

We can write

$$41^k - 14^k = 27a \qquad \ldots(1)$$

where $a \in N$.

Now, we will prove that $P(k+1)$ is true whenever $P(k)$ is true.

Now,

$$
\begin{aligned}
& 41^{k+1} - 14^{k+1} \\
& \Rightarrow 41.41^k - 14.14^k \\
& \Rightarrow 41.\left(41^k - 14^k + 14^k\right) - 14.14^k \\
& \Rightarrow 41.\left(41^k - 14^k\right) + 41.14^k - 14.14^k \\
& \Rightarrow 41.27a + 14^k(41-14) \qquad [\text{from}(1)] \\
& \Rightarrow 41.27a + 14^k.27 \\
& \Rightarrow 27\left(41a + 14^k\right)
\end{aligned}
$$

From the last line, we see that

$27\left(41a + 14^k\right)$ is a multiple of 27.

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.

Question:24

$(2n+7) < (n+3)^2$.

Solution:

Let $P(n)$ be the given statement.

i.e., $P(n):(2n+7) < (n+3)^2$

We note that $P(n)$ is true for $n=1$,

Since,

$$P(1):(2.1+7) = 9 < (1+3)^2 = 16$$

Assume that $P(k)$ is true for some positive integer k

i.e., $(2k+7) < (k+3)^2 \qquad \ldots(1)$

We will now prove that $P(k+1)$ is true whenever $P(k)$ is true
Now, we have

$$2(k+1)+7=2k+2+7$$

$$2(k+1)+7=(2k+7)+2<(k+3)^2+2 \qquad [\text{from } (1)]$$

$$\begin{aligned}(2k+7)+2&<(k+3)^2+2\\&<k^2+6k+9+2\\&<k^2+6k+11\end{aligned}$$

Now,

$$\begin{aligned}[(k+1)+3]^2&=(k+4)^2\\&=k^2+8k+16\end{aligned}$$

Since,

$$k^2+6k+11<k^2+8k+16$$

Therefore,

$$2(k+1)+7<(k+4)^2$$

$$[2(k+1)+7]<[(k+1)+3]^2$$

Thus $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers i.e., $n \in N$.