NCERT Solutions Class 11 Maths Chapter 5 Complex Numbers and Quadratic Equations

Question 1:

Express the given complex number in the form $a+ib$: $(5i)\left(-\frac{3}{5}i\right)$

Solution:

$$\begin{aligned}(5i)\left(-\frac{3}{5}i\right) &= -5i\times\frac{3}{5}\times i\\ &= -3i^2 \qquad \left[\because i^2=-1\right]\\ &= -3(-1)\\ &= 3\\ &= 3+i0\end{aligned}$$

Question 2:

Express the given complex number in the form $a+ib$: i^9+i^{19}

Solution:

$$\begin{aligned}i^9+i^{19} &= i^{4\times2+1}+i^{4\times4+3}\\ &= \left(i^4\right)^2\times i+\left(i^4\right)^4\times i^3\\ &= 1\times i+1\times(-i) \qquad \left[\because i^4=1,\ i^3=-i\right]\\ &= i+(-i)\\ &= 0\\ &= 0+i0\end{aligned}$$

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Question 3:

Express the given complex number in the form $a+ib$: i^{-39}

Solution:

$$
\begin{aligned}
i^{-39} &= i^{4\times(-9)-3} \\
&= \left(i^4\right)^{-9} \times i^{-3} \\
&= (1)^{-9} \times i^{-3} \qquad \left[\because i^4 = 1\right] \\
&= \frac{1}{i^3} \\
&= \frac{1}{-i} \qquad \left[\because i^3 = -i\right] \\
&= -\frac{1}{i} \times \frac{i}{i} \\
&= -\frac{i}{i^2} \\
&= \frac{-i}{-1} \qquad \left[\because i^2 = -1\right] \\
&= i \\
&= 0 + i1
\end{aligned}
$$

Question 4:

Express the given complex number in the form $a+ib$: $3(7+i7)+i(7+i7)$

Solution:

$$
\begin{aligned}
3(7+i7)+i(7+i7) &= 21+21i+7i+7i^2 \\
&= 21+28i+7\times(-1) \qquad \left[\because i^2 = -1\right] \\
&= 14+i28
\end{aligned}
$$

Question 5:

Express the given complex number in the form $a+ib$: $(1-i)-(-1+i6)$

Solution:

$$(1-i)-(-1+i6)=1-i+1-6i$$
$$=2-i7$$

Question 6:

Express the given complex number in the form $a+ib$: $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$

Solution:

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)=\frac{1}{5}+\frac{2}{5}i-4-\frac{5}{2}i$$
$$=\left(\frac{1}{5}-4\right)+i\left(\frac{2}{5}-\frac{5}{2}\right)$$
$$=\left(-\frac{19}{5}\right)+i\left(-\frac{21}{10}\right)$$
$$=-\frac{19}{5}-i\frac{21}{10}$$

Question 7:

Express the given complex number in the form $a+ib$: $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

Solution:

$$\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)=\frac{1}{3}+\frac{7}{3}i+4+\frac{1}{3}i+\frac{4}{3}-i$$
$$=\left(\frac{1}{3}+4+\frac{4}{3}\right)+i\left(\frac{7}{3}+\frac{1}{3}-1\right)$$
$$=\frac{17}{3}+i\frac{5}{3}$$

Question 8:

Express the given complex number in the form $a+ib$: $(1-i)^4$

Solution:

$$\begin{aligned}(1-i)^4 &= \left[(1-i)^2\right]^2 \\ &= \left[1^2+i^2-2i\right]^2 \\ &= [1-1-2i]^2 \\ &= [2i]^2 \\ &= 4i^2 \qquad (\because i^2=-1) \\ &= -4\end{aligned}$$

Question 9:

Express the given complex number in the form $a+ib$: $\left(\frac{1}{3}+3i\right)^3$

Solution:

$$\begin{aligned}\left(\frac{1}{3}+3i\right)^3 &= \left(\frac{1}{3}\right)^3+(3i)^3+3\left(\frac{1}{3}\right)(3i)\left(\frac{1}{3}+3i\right) \\ &= \frac{1}{27}+27i^3+3i\left(\frac{1}{3}+3i\right) \\ &= \frac{1}{27}+27(-i)+i+9i^2 \qquad (\because i^3=-i) \\ &= \frac{1}{27}-27i+i-9 \qquad (\because i^2=-1) \\ &= \left(\frac{1}{27}-9\right)-26i \\ &= -\frac{242}{27}-i26\end{aligned}$$

Question 10:

Express the given complex number in the form $a+ib$: $\left(-2-\frac{1}{3}i\right)^3$

Solution:

$$\left(-2-\frac{1}{3}i\right)^3=(-1)^3\left(2+\frac{1}{3}i\right)^3$$

$$=-\left[2^3+\left(\frac{i}{3}\right)^3+3(2)\left(\frac{i}{3}\right)\left(2+\frac{i}{3}\right)\right]$$

$$=-\left[8+\frac{i^3}{27}+2i\left(2+\frac{i}{3}\right)\right]$$

$$=-\left[8-\frac{i}{27}+4i+\frac{2}{3}i^2\right] \qquad [\because i^3=-i]$$

$$=-\left[8-\frac{i}{27}+4i-\frac{2}{3}\right] \qquad [\because i^2=-1]$$

$$=-\left[\frac{22}{3}+\frac{107i}{27}\right]$$

$$=-\frac{22}{3}-i\frac{107}{27}$$

Question 11:

Find the multiplicative inverse of the complex number $4-3i$

Solution:

Let $z=4-3i$

Then, $\overline{z}=4+3i$ and

$$|z|^2=4^2+(-3)^2$$
$$=16+9$$
$$=25$$

Therefore, the multiplicative inverse of $4-3i$ is given by

$$z^{-1}=\frac{\overline{z}}{|z|^2}$$
$$=\frac{4+3i}{25}$$
$$=\frac{4}{25}+i\frac{3}{25}$$

Question 12:

Find the multiplicative inverse of the complex number $\sqrt{5}+3i$

Solution:

Let $z=\sqrt{5}+3i$

Then, $\bar{z}=\sqrt{5}-3i$ and

$$\begin{aligned}|z|^2&=\left(\sqrt{5}\right)^2+3^2\\&=5+9\\&=14\end{aligned}$$

Therefore, the multiplicative inverse of $\sqrt{5}+3i$ is given by

$$\begin{aligned}z^{-1}&=\frac{\bar{z}}{|z|^2}\\&=\frac{\sqrt{5}-3i}{14}\\&=\frac{\sqrt{5}}{14}-\frac{3}{14}i\end{aligned}$$

Question 13:

Find the multiplicative inverse of the complex number $-i$

Solution:

Let $z=-i$

Then, $\bar{z}=i$ and

$$\begin{aligned}|z|^2&=1^2\\&=1\end{aligned}$$

Therefore, the multiplicative inverse of $-i$ is given by

$$\begin{aligned}z^{-1}&=\frac{\bar{z}}{|z|^2}\\&=\frac{i}{1}\\&=i\end{aligned}$$

Question 14:

Express the following expression in the form $a+ib$:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-\sqrt{2}i\right)}$$

Solution:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}i\right)-\left(\sqrt{3}-\sqrt{2}i\right)}=\frac{(3)^2-\left(i\sqrt{5}\right)^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i} \qquad \left[\because (a+b)(a-b)=a^2-b^2\right]$$

$$=\frac{9-5i^2}{2\sqrt{2}i}$$

$$=\frac{9-5(-1)}{2\sqrt{2}i} \qquad \left[\because i^2=-1\right]$$

$$=\frac{9+5}{2\sqrt{2}i}$$

$$=\frac{14}{2\sqrt{2}i}\times\frac{i}{i}$$

$$=\frac{7i}{\sqrt{2}i^2}$$

$$=\frac{7i}{\sqrt{2}(-1)} \qquad \left[\because i^2=-1\right]$$

$$=\frac{-7i}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{-7\sqrt{2}i}{2}$$

$$=0+i\frac{-7\sqrt{2}}{2}$$

EXERCISE 5.2

Question 1:

Find the modulus and argument of the complex number $z=-1-i\sqrt{3}$

Solution:

$z=-1-i\sqrt{3}$

Let $r\cos\theta=-1$ and $r\sin\theta=-\sqrt{3}$

On squaring and adding, we obtain

$$(r\cos\theta)^2+(r\sin\theta)^2=(-1)^2+(-\sqrt{3})^2$$

$$\Rightarrow r^2(\cos^2\theta+\sin^2\theta)=1+3 \qquad [\because \cos^2\theta+\sin^2\theta=1]$$

$$\Rightarrow r^2=4$$

$$\Rightarrow r=\sqrt{4}=2 \qquad [\because \text{Conventionally}, r>0]$$

Therefore, Modulus $=2$

Hence, $2\cos\theta=-1$ and $2\sin\theta=-\sqrt{3}$

$$\Rightarrow \cos\theta=-\frac{1}{2} \text{ and } \sin\theta=-\frac{\sqrt{3}}{2}$$

Since both the values of $\sin\theta$ and $\cos\theta$ are negative in III quadrant,

Argument $=-\left(\pi-\frac{\pi}{3}\right)=\frac{-2\pi}{3}$

Thus, the modulus and argument of the complex number $-1-i\sqrt{3}$ are 2 and $\frac{-2\pi}{3}$ respectively.

Question 2:

Find the modulus and argument of the complex number $z=-\sqrt{3}+i$

Solution:

$z=-\sqrt{3}+i$

Let $r\cos\theta=-\sqrt{3}$ and $r\sin\theta=1$

On squaring and adding, we obtain

$r^2\cos^2\theta + r^2\sin^2\theta = \left(-\sqrt{3}\right)^2 + 1^2$

$\Rightarrow r^2 = 3+1=4$ $\quad [\because \cos^2\theta+\sin^2\theta=1]$

$\Rightarrow r=\sqrt{4}=2$ $\quad [\because \text{Conventionally, } r>0]$

Therefore, Modulus $=2$

Hence, $2\cos\theta=-\sqrt{3}$ and $2\sin\theta=1$

$\Rightarrow \cos\theta=-\frac{\sqrt{3}}{2}$ and $\sin\theta=\frac{1}{2}$

Since, θ lies in the quadrant II, $\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

Thus, the modulus and argument of the complex number $-\sqrt{3}+i$ are 2 and $\frac{5\pi}{6}$ respectively.

Question 3:

Convert the given complex number in polar form: $1-i$

Solution:

$z=1-i$

Let $r\cos\theta=1$ and $r\sin\theta=-1$

On squaring and adding, we obtain

$r^2\cos^2\theta + r^2\sin^2\theta = 1^2+(-1)^2$

$\Rightarrow r^2\left(\cos^2\theta+\sin^2\theta\right)=1+1$

$\Rightarrow r^2=2$

$\Rightarrow r=\sqrt{2}$ $\quad [\because \text{Conventionally, } r>0]$

Therefore,

$\sqrt{2}\cos\theta=1$ and $\sqrt{2}\sin\theta=-1$

$\Rightarrow \cos\theta=\frac{1}{\sqrt{2}}$ and $\sin\theta=-\frac{1}{\sqrt{2}}$

Since, θ lies in the quadrant IV, $\theta=-\frac{\pi}{4}$

Hence,

$$
\begin{aligned}
1-i &= r\cos\theta + ir\sin\theta \\
&= \sqrt{2}\cos\left(-\frac{\pi}{4}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{4}\right) \\
&= \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]
\end{aligned}
$$

Thus, this is the required polar form.

Question 4:

Convert the given complex number in polar form: $-1+i$

Solution:

$z = -1+i$

Let $r\cos\theta = -1$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$$
\begin{aligned}
& r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + 1^2 \\
\Rightarrow\ & r^2\left(\cos^2\theta + \sin^2\theta\right) = 1+1 \\
\Rightarrow\ & r^2 = 2 \\
\Rightarrow\ & r = \sqrt{2} \qquad [\because \text{Conventionally}, r > 0]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \sqrt{2}\cos\theta = -1 \text{ and } \sqrt{2}\sin\theta = 1 \\
\Rightarrow\ & \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}
\end{aligned}
$$

Since, θ lies in the quadrant II, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence,

$$
\begin{aligned}
-1+i &= r\cos\theta + ir\sin\theta \\
&= \sqrt{2}\cos\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4} \\
&= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)
\end{aligned}
$$

Thus, this is the required polar form.

Question 5:

Convert the given complex number in polar form: $-1-i$

Solution:

$z=-1-i$

Let $r\cos\theta=-1$ and $r\sin\theta=-1$

On squaring and adding, we obtain

$$r^2\cos^2\theta+r^2\sin^2\theta=(-1)^2+(-1)^2$$
$$\Rightarrow r^2\left(\cos^2\theta+\sin^2\theta\right)=1+1$$
$$\Rightarrow r^2=2$$
$$\Rightarrow r=\sqrt{2} \qquad [\because \text{Conventionally, } r>0]$$

Therefore,

$$\sqrt{2}\cos\theta=-1 \text{ and } \sqrt{2}\sin\theta=-1$$
$$\Rightarrow \cos\theta=-\frac{1}{\sqrt{2}} \text{ and } \sin\theta=-\frac{1}{\sqrt{2}}$$

Since, θ lies in the quadrant III, $\theta=-\left(\pi-\frac{\pi}{4}\right)=-\frac{3\pi}{4}$

Hence,

$$\begin{aligned}-1-i&=r\cos\theta+ir\sin\theta\\&=\sqrt{2}\cos\frac{-3\pi}{4}+i\sqrt{2}\sin\frac{-3\pi}{4}\\&=\sqrt{2}\left(\cos\frac{-3\pi}{4}+i\sin\frac{-3\pi}{4}\right)\end{aligned}$$

Thus, this is the required polar form.

Question 6:

Convert the given complex number in polar form: -3

Solution:

$z=-3$

Let $r\cos\theta=-3$ and $r\sin\theta=0$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-3)^2 + (0)^2$$
$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 9$$
$$\Rightarrow r^2 = 9$$
$$\Rightarrow r = 3 \qquad [\because \text{Conventionally}, r > 0]$$

Therefore,

$$3\cos\theta = -3 \text{ and } 3\sin\theta = 0$$
$$\Rightarrow \cos\theta = -1 \text{ and } \sin\theta = 0$$

Since the θ lies in the quadrant II, $\theta = \pi$

Hence,

$$\begin{aligned} -3 &= r\cos\theta + ir\sin\theta \\ &= 3\cos\pi + i3\sin\pi \\ &= 3(\cos\pi + i\sin\pi) \end{aligned}$$

Thus, this is the required polar form.

Question 7:

Convert the given complex number in polar form: $\sqrt{3} + i$

Solution:

$z = \sqrt{3} + i$

Let $r\cos\theta = \sqrt{3}$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = \left(\sqrt{3}\right)^2 + 1^2$$
$$\Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = 3 + 1$$
$$\Rightarrow r^2 = 4$$
$$\Rightarrow r = \sqrt{4} = 2 \qquad [\text{Conventionally}, r > 0]$$

Therefore,

$$2\cos\theta = \sqrt{3} \text{ and } 2\sin\theta = 1$$
$$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = \frac{1}{2}$$

Since, θ lies in quadrant I, $\theta = \frac{\pi}{6}$

Hence,

$$\sqrt{3}+i=r\cos\theta+ir\sin\theta$$
$$=2\cos\frac{\pi}{6}+i2\sin\frac{\pi}{6}$$
$$=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

Thus, this is the required polar form.

Question 8:

Convert the given complex number in polar form: i

Solution:

$z=i$

Let $r\cos\theta=0$ and $r\sin\theta=1$

On squaring and adding, we obtain

$$r^2\cos^2\theta+r^2\sin^2\theta=0^2+1^2$$
$$\Rightarrow r^2\left(\cos^2\theta+\sin^2\theta\right)=1$$
$$\Rightarrow r^2=1$$
$$\Rightarrow r=\sqrt{1}=1 \qquad [\text{Conventionally, } r>0]$$

Therefore,

$$\cos\theta=0 \text{ and } \sin\theta=1$$

Since, θ lies in quadrant I, $\theta=\frac{\pi}{2}$

Hence,

$$i=r\cos\theta+ir\sin\theta$$
$$=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$$

Thus, this is the required polar form.

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EXERCISE 5.3

Question 1:

Solve the equation $x^2+3=0$

Solution:

The given quadratic equation is $x^2+3=0$

On comparing the given equation with $ax^2+bx+c=0$,

We obtain $a=1, b=0,$ and $c=3$

Therefore, the discriminant of the given equation is

$$\begin{aligned} D &= b^2-4ac \\ &= 0^2-4\times1\times3 \\ &= -12 \end{aligned}$$

Therefore, the required solutions are

$$\begin{aligned} \frac{-b\pm\sqrt{D}}{2a} &= \frac{-0\pm\sqrt{-12}}{2\times1} \\ &= \frac{\pm\sqrt{12}i}{2} \qquad \left[\because \sqrt{-1}=i\right] \\ &= \frac{\pm2\sqrt{3}i}{2} \\ &= \pm\sqrt{3}i \end{aligned}$$

Question 2:

Solve the equation $2x^2+x+1=0$

Solution:

The given quadratic equation is $2x^2+x+1=0$

On comparing the given equation with $ax^2+bx+c=0$,

We obtain $a=2, b=1,$ and $c=1$

Therefore, the discriminant of the given equation is

$$\begin{aligned} D &= b^2-4ac \\ &= 1^2-4\times2\times1 \\ &= -7 \end{aligned}$$

Therefore, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times 2}$$
$$= \frac{-1 \pm \sqrt{7}i}{4} \qquad \left[\because \sqrt{-1} = i\right]$$

Question 3:

Solve the equation $x^2 + 3x + 9 = 0$

Solution:

The given quadratic equation is $x^2 + 3x + 9 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,

We obtain $a = 1, b = 3,$ and $c = 9$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac$$
$$= 3^2 - 4 \times 1 \times 9$$
$$= -27$$

Hence, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2 \times 1}$$
$$= \frac{-3 \pm 3\sqrt{-3}}{2}$$
$$= \frac{-3 \pm 3\sqrt{3}i}{2} \qquad \left[\because \sqrt{-1} = i\right]$$

Question4:

Solve the equation $-x^2 + x - 2 = 0$

Solution:

The given quadratic equation is $-x^2 + x - 2 = 0$

On comparing the given equation with $ax^2 + bx + c = 0$,

We obtain $a = -1, b = 1$ and $c = -2$

Therefore, the discriminant of the given equation is

$$\begin{aligned} D &= b^2 - 4ac \\ &= 1^2 - 4\times(-1)\times(-2) \\ &= -7 \end{aligned}$$

Hence, the required solutions are

$$\begin{aligned} \frac{-b\pm\sqrt{D}}{2a} &= \frac{-1\pm\sqrt{-7}}{2\times(-1)} \\ &= \frac{-1\pm\sqrt{7}i}{-2} \qquad \left[\sqrt{-1}=i\right] \end{aligned}$$

Question 5:

Solve the equation $x^2+3x+5=0$

Solution:

The given quadratic equation is $x^2+3x+5=0$

On comparing the given equation with $ax^2+bx+c=0$,

We obtain $a=1, b=3,$ and $c=5$

Therefore, the discriminant of the given equation is

$$\begin{aligned} D &= b^2 - 4ac \\ &= 3^2 - 4\times1\times5 \\ &= -11 \end{aligned}$$

Hence, the required solutions are

$$\begin{aligned} \frac{-b\pm\sqrt{D}}{2a} &= \frac{-3\pm\sqrt{-11}}{2\times1} \\ &= \frac{-3\pm\sqrt{11}i}{2} \qquad \left[\sqrt{-1}=i\right] \end{aligned}$$

Question 6:

Solve the equation $x^2-x+2=0$

Solution:

The given quadratic equation is $x^2-x+2=0$

On comparing the given equation with $ax^2+bx+c=0$,

We obtain $a=1, b=-1,$ and $c=2$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac$$
$$= (-1)^2 - 4 \times 1 \times 2$$
$$= -7$$

Hence, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-1) \pm \sqrt{-7}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{7}i}{2} \qquad \left[\because \sqrt{-1} = i\right]$$

Question 7:

Solve the equation $\sqrt{2}x^2 - x + \sqrt{2} = 0$

Solution:

The given quadratic equation is $\sqrt{2}x^2 - x + \sqrt{2} = 0$

On comparing the given equation with $ax^2 + bx + c = 0,$

We obtain $a = \sqrt{2}, b = -1,$ and $c = \sqrt{2}$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac$$
$$= (-1)^2 - 4 \times \sqrt{2} \times \sqrt{2}$$
$$= -7$$

Hence, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-1) \pm \sqrt{-7}}{2 \times \sqrt{2}}$$
$$= \frac{1 \pm \sqrt{7}i}{2\sqrt{2}} \qquad \left[\because \sqrt{-1} = i\right]$$

Question 8:

Solve the equation $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Solution:

The given quadratic equation is $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

On comparing the given equation with $ax^2 + bx + c = 0,$

We obtain $a=\sqrt{3}, b=-\sqrt{2},$ and $c=3\sqrt{3}$

Therefore, the discriminant of the given equation is

$$\begin{aligned} D &= b^2-4ac \\ &= \left(-\sqrt{2}\right)^2-4\times\left(\sqrt{3}\right)\times\left(3\sqrt{3}\right) \\ &= -34 \end{aligned}$$

Hence, the required solutions are

$$\begin{aligned} \frac{-b\pm\sqrt{D}}{2a} &= \frac{-\left(-\sqrt{2}\right)\pm\sqrt{-34}}{2\times\sqrt{3}} \\ &= \frac{\sqrt{2}\pm\sqrt{34}i}{2\sqrt{3}} \qquad \left[\because \sqrt{-1}=i\right] \end{aligned}$$

Question 9:

Solve the equation $x^2+x+\frac{1}{\sqrt{2}}=0$

Solution:

The given quadratic equation is $x^2+x+\frac{1}{\sqrt{2}}=0$

This equation can also be written as $\sqrt{2}x^2+\sqrt{2}x+1=0$

On comparing the given equation with $ax^2+bx+c=0,$

We obtain $a=\sqrt{2}, b=\sqrt{2}$ and $c=1$

Therefore, the discriminant of the given equation is

$$\begin{aligned} D &= b^2-4ac \\ &= \left(\sqrt{2}\right)^2-4\times\left(\sqrt{2}\right)\times 1 \\ &= 2-4\sqrt{2} \end{aligned}$$

Hence, the required solutions are

$$\frac{-b\pm\sqrt{D}}{2a}=\frac{-\sqrt{2}\pm\sqrt{2-4\sqrt{2}}}{2\times\sqrt{2}}$$

$$=\frac{-\sqrt{2}\pm\sqrt{2\left(1-2\sqrt{2}\right)}}{2\sqrt{2}}$$

$$=\left(\frac{-\sqrt{2}\pm\sqrt{2}\left(\sqrt{2\sqrt{2}-1}\right)i}{2\sqrt{2}}\right) \qquad \left[\because \sqrt{-1}=i\right]$$

$$=\frac{-1\pm\left(\sqrt{2\sqrt{2}-1}\right)i}{2}$$

Question 10:

Solve the equation $x^2+\frac{x}{\sqrt{2}}+1=0$

Solution:

The given quadratic equation is $x^2+\frac{x}{\sqrt{2}}+1=0$

This equation can also be written as $\sqrt{2}x^2+x+\sqrt{2}=0$

On comparing the given equation with $ax^2+bx+c=0$,

We obtain $a=\sqrt{2}, b=1$ and $c=\sqrt{2}$

Therefore, the discriminant of the given equation is

$$\begin{aligned}D&=b^2-4ac\\&=(1)^2-4\times\left(\sqrt{2}\right)\times\left(\sqrt{2}\right)\\&=1-8\\&=-7\end{aligned}$$

Hence, the required solutions are

$$\frac{-b\pm\sqrt{D}}{2a}=\frac{-1\pm\sqrt{-7}}{2\times\sqrt{2}}$$

$$=\frac{-1\pm\sqrt{7}i}{2\sqrt{2}} \qquad \left[\because \sqrt{-1}=i\right]$$

MISCELLANEOUS EXERCISE

Question 1:

Evaluate: $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

Solution:

$$\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3=\left[i^{4\times4+2}+\frac{1}{i^{4\times6+1}}\right]^3$$

$$=\left[\left(i^4\right)^4\times i^2+\frac{1}{\left(i^4\right)^6\times i}\right]^3$$

$$=\left[i^2+\frac{1}{i}\right]^3 \qquad \left[\because i^4=1\right]$$

$$=\left[-1+\frac{1}{i}\times\frac{i}{i}\right]^3 \qquad \left[\because i^2=-1\right]$$

$$=\left[-1+\frac{i}{i^2}\right]^3$$

$$=\left[-1-i\right]^3$$

$$=(-1)^3\left[1+i\right]^3$$

$$=-\left[1^3+i^3+3\times1\times i\left(1+i\right)\right]$$

$$=-\left[1+i^3+3i+3i^2\right]$$

$$=-\left[1-i+3i-3\right]$$

$$=-\left[-2+2i\right]$$

$$=2-2i$$

Question 2:

For any two complex numbers z_1 and z_2, prove that

$\operatorname{Re}\left(z_1z_2\right)=\operatorname{Re}z_1\operatorname{Re}z_2-\operatorname{Im}z_1\operatorname{Im}z_2$

Solution:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1y_2 \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 \qquad [\because i^2 = -1] \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \end{aligned}$$

$$\Rightarrow \operatorname{Re}(z_1z_2) = x_1x_2 - y_1y_2$$
$$\Rightarrow \operatorname{Re}(z_1z_2) = \operatorname{Re} z_1 \operatorname{Re} z_2 - \operatorname{Im} z_1 \operatorname{Im} z_2$$

Hence, proved.

Question 3:

Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form

Solution:

$$\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)=\left[\frac{(1+i)-2(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right]$$

$$=\left[\frac{1+i-2+8i}{1+i-4i-4i^2}\right]\left[\frac{3-4i}{5+i}\right]$$

$$=\left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right]$$

$$=\left[\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}\right]$$

$$=\frac{33+31i}{28-10i}$$

$$=\frac{33+31i}{2(14-5i)}$$

$$=\frac{(33+31i)}{2(14-5i)}\times\frac{(14+5i)}{(14+5i)} \qquad \left[\begin{array}{l}\text{On multiplying numerator and}\\ \text{denominator by } (14+5i)\end{array}\right]$$

$$=\frac{462+165i+434i+155i^2}{2\left[(14)^2-(5i)^2\right]}$$

$$=\frac{307+599i}{2(196-25i^2)}$$

$$=\frac{307+599i}{2(221)}$$

$$=\frac{307}{442}+\frac{599}{442}i$$

This is the required standard form.

Question 4:

If $x-iy=\sqrt{\dfrac{a-ib}{c-id}}$ prove that $\left(x^2+y^2\right)^2=\dfrac{a^2+b^2}{c^2+d^2}$

Solution:

$$x-iy=\sqrt{\frac{a-ib}{c-id}}=\sqrt{\frac{a-ib}{c-id}\times\frac{c+id}{c+id}} \qquad \left[\begin{array}{l}\text{On multiplying numerator and}\\ \text{denominator by} (c+id)\end{array}\right]$$

$$=\sqrt{\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}}$$

$$(x-iy)^2=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

$$x^2-y^2-2ixy=\frac{(ac+bd)}{c^2+d^2}+i\frac{(ad-bc)}{c^2+d^2}$$

On comparing real and imaginary parts, we obtain

$$x^2 - y^2 = \frac{ac+bd}{c^2+d^2}, -2xy = \frac{ad-bc}{c^2+d^2} \qquad \ldots(1)$$

Since,

$$\left(x^2+y^2\right)^2 = \left(x^2-y^2\right)^2 + \left(2xy\right)^2$$
$$= \left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{ad-bc}{c^2+d^2}\right)^2 \qquad \left[\text{Using } (1)\right]$$
$$= \frac{a^2c^2+b^2d^2+2acbd+a^2d^2+b^2c^2-2abcd}{\left(c^2+d^2\right)^2}$$
$$= \frac{a^2c^2+b^2d^2+a^2d^2+b^2c^2}{\left(c^2+d^2\right)^2}$$
$$= \frac{a^2\left(c^2+d^2\right)+b^2\left(c^2+d^2\right)}{\left(c^2+d^2\right)^2}$$
$$= \frac{\left(a^2+b^2\right)\left(c^2+d^2\right)}{\left(c^2+d^2\right)^2}$$
$$= \frac{\left(a^2+b^2\right)}{\left(c^2+d^2\right)}$$

Hence, proved.

Question 5:

Convert the following in the polar form:

(i) $\dfrac{1+7i}{(2-i)^2}$ (ii) $\dfrac{1+3i}{1-2i}$

Solution:

(i) Here,

$$z = \frac{1+7i}{(2-i)^2}$$
$$= \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{4-1-4i}$$
$$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{3^2+4^2}$$
$$= \frac{3+25i-28}{25} = \frac{-25+25i}{25}$$
$$= -1+i$$

Let $r\cos\theta = -1$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + 1^2$$
$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 1+1$$
$$r^2 = 2$$
$$r = \sqrt{2} \qquad [\text{Conventionally, } r>0]$$

Therefore,

$$\sqrt{2}\cos\theta = -1 \text{ and } \sqrt{2}\sin\theta = 1$$
$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}$$

Since, θ lies in the quadrant II, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$
Hence,

$$z = r\cos\theta + ir\sin\theta$$
$$= \sqrt{2}\cos\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4}$$
$$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

This is the required polar form.

(ii) Here,

$$z = \frac{1+3i}{1-2i}$$
$$= \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i}$$
$$= \frac{1+2i+3i+6i^2}{1^2+2^2} = \frac{1+5i-6}{5}$$
$$= \frac{-5+5i}{5} = -1+i$$

Let $r\cos\theta = -1$ and $r\sin\theta = 1$

On squaring and adding, we obtain

$$r^2\cos^2\theta + r^2\sin^2\theta = (-1)^2 + 1^2$$
$$r^2(\cos^2\theta + \sin^2\theta) = 1 + 1$$
$$r^2 = 2$$
$$r = \sqrt{2} \qquad [\text{Conventionally, } r > 0]$$

Therefore,

$$\sqrt{2}\cos\theta = -1 \text{ and } \sqrt{2}\sin\theta = 1$$
$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}$$

Since, θ lies in the quadrant II, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$
Hence,

$$z = r\cos\theta + ir\sin\theta$$
$$= \sqrt{2}\cos\frac{3\pi}{4} + i\sqrt{2}\sin\frac{3\pi}{4}$$
$$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

This is the required polar form.

Question 6:

Solve the equation $3x^2 - 4x + \frac{20}{3} = 0$

Solution:

The given quadratic equation is $3x^2 - 4x + \frac{20}{3} = 0$
This equation can also be written as $9x^2 - 12x + 20 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain $a = 9, b = -12$ and $c = 20$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac$$
$$= (-12)^2 - 4 \times 9 \times 20$$
$$= 144 - 720$$
$$= -576$$

Hence, the required solutions are

$$\begin{aligned}\frac{-b\pm\sqrt{D}}{2a}&=\frac{-(-12)\pm\sqrt{-576}}{2\times 9}\\&=\frac{12\pm\sqrt{576}i}{18} \qquad\qquad [\because \sqrt{-1}=i]\\&=\frac{12\pm 24i}{18}\\&=\frac{6(2\pm 4i)}{18}\\&=\frac{2\pm 4i}{3}\\&=\frac{2}{3}\pm\frac{4}{3}i\end{aligned}$$

Question 7:

Solve the equation $x^2-2x+\frac{3}{2}=0$

Solution:

The given quadratic equation is $x^2-2x+\frac{3}{2}=0$

This equation can also be written as $2x^2-4x+3=0$

On comparing this equation with $ax^2+bx+c=0$, we obtain $a=2, b=-4$ and $c=3$

Therefore, the discriminant of the given equation is

$$\begin{aligned}D&=b^2-4ac\\&=(-4)^2-4\times 2\times 3\\&=16-24\\&=-8\end{aligned}$$

Hence, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-4) \pm \sqrt{-8}}{2 \times 2}$$
$$= \frac{4 \pm 2\sqrt{2}i}{4} \qquad \left[\because \sqrt{-1} = i\right]$$
$$= \frac{2 \pm \sqrt{2}i}{2}$$
$$= 1 \pm \frac{\sqrt{2}}{2}i$$

Question 8:

Solve the equation $27x^2 - 10x + 1 = 0$

Solution:

The given quadratic equation is $27x^2 - 10x + 1 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain
$a = 27, b = -10$ and $c = 1$

Therefore, the discriminant of the given equation is

$$D = b^2 - 4ac$$
$$= (-10)^2 - 4 \times 27 \times 1$$
$$= 100 - 108$$
$$= -8$$

Hence, the required solutions are

$$\frac{-b \pm \sqrt{D}}{2a} = \frac{-(-10) \pm \sqrt{-8}}{2 \times 27}$$
$$= \frac{10 \pm 2\sqrt{2}i}{54} \qquad \left[\because \sqrt{-1} = i\right]$$
$$= \frac{5 \pm \sqrt{2}i}{27}$$
$$= \frac{5}{27} \pm \frac{\sqrt{2}}{27}i$$

Question 9:

Solve the equation $21x^2 - 28x + 10 = 0$

Solution:

The given quadratic equation is $21x^2 - 28x + 10 = 0$

On comparing this equation with $ax^2 + bx + c = 0$, we obtain

$a = 21, b = -28$ and $c = 10$

Therefore, the discriminant of the given equation is

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-28)^2 - 4 \times 21 \times 10 \\ &= 784 - 840 \\ &= -56 \end{aligned}$$

Hence, the required solutions are

$$\begin{aligned} \frac{-b \pm \sqrt{D}}{2a} &= \frac{-(-28) \pm \sqrt{-56}}{2 \times 21} \\ &= \frac{28 \pm \sqrt{56}i}{42} \qquad \left[\because \sqrt{-1} = i\right] \\ &= \frac{28 \pm 2\sqrt{14}i}{42} \\ &= \frac{28}{42} \pm \frac{2\sqrt{14}}{42} i \\ &= \frac{2}{3} \pm \frac{\sqrt{14}}{21} i \end{aligned}$$

Question 10:

If $z_1 = 2 - i, z_2 = 1 + i,$ find $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right|$

Solution:

$z_1 = 2 - i, z_2 = 1 + i$

Therefore,

$$\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|=\left|\frac{(2-i)+(1+i)+1}{(2-i)-(1+i)+1}\right|$$
$$=\left|\frac{4}{2-2i}\right|=\left|\frac{4}{2(1-i)}\right|$$
$$=\left|\frac{2}{1-i}\times\frac{1+i}{1+i}\right|=\left|\frac{2(1+i)}{(1^2-i^2)}\right|$$
$$=\left|\frac{2(1+i)}{1+1}\right| \qquad \left[i^2=-1\right]$$
$$=\left|\frac{2(1+i)}{2}\right|$$
$$=|1+i|=\sqrt{1^2+1^2}$$
$$=\sqrt{2}$$

Thus, the value of $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$ is $\sqrt{2}$.

Question 11:

If $a+ib=\frac{(x+i)^2}{2x^2+1}$, prove that $a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$

Solution:

$$a+ib=\frac{(x+i)^2}{2x^2+1}$$
$$a+ib=\frac{(x+i)^2}{2x^2+1}=\frac{x^2+i^2+i2x}{2x^2+1}$$
$$=\frac{x^2-1+i2x}{2x^2+1}$$
$$=\frac{x^2-1}{2x^2+1}+i\left(\frac{2x}{2x^2+1}\right)$$

On comparing real and imaginary parts, we obtain

$a=\frac{x^2-1}{2x^2+1}$ and $b=\frac{2x}{2x^2+1}$

Since,

$$a^2+b^2=\left(\frac{x^2-1}{2x^2+1}\right)^2+\left(\frac{2x}{2x^2+1}\right)^2$$
$$=\frac{x^4+1-2x^2+4x^2}{\left(2x^2+1\right)^2}$$
$$=\frac{x^4+1+2x^2}{\left(2x^2+1\right)^2}$$
$$a^2+b^2=\frac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$$

Hence, proved.

Question 12:

Let $z_1=2-i, z_2=-2+i$. Find

(i) $\mathrm{Re}\left(\frac{z_1z_2}{\bar{z}_1}\right)$, (ii) $\mathrm{Im}\left(\frac{1}{z_1\bar{z}_1}\right)$

Solution:

(i) It is given that $z_1=2-i, z_2=-2+i$

$$z_1z_2=(2-i)(-2+i)$$
$$=-4+2i+2i-i^2$$
$$=-4+4i-(-1)$$
$$=-3+4i$$

Now, $\bar{z}_1=2+i$

Hence, $\frac{z_1z_2}{\bar{z}_1}=\frac{-3+4i}{2+i}$

On multiplying numerator and denominator by $(2-i)$, we obtain

$$\frac{z_1z_2}{\bar{z}_1}=\frac{(-3+4i)(2-i)}{(2+i)(2-i)}=\frac{-6+3i+8i-4i^2}{2^2+1^2}=\frac{-6+11i-4(-1)}{5}$$
$$=\frac{-2+11i}{5}=\frac{-2}{5}+\frac{11}{5}i$$

On comparing real parts, we obtain

$$\mathrm{Re}\left(\frac{z_1z_2}{\bar{z}_1}\right)=-\frac{2}{5}$$

(ii) $$\frac{1}{z_1\bar{z}_1}=\frac{1}{(2+i)(2-i)}=\frac{1}{(2)^2+(1)^2}=\frac{1}{5}$$

On comparing imaginary parts, we obtain

$$\operatorname{Im}\left(\frac{1}{z_1\bar{z}_1}\right)=0$$

Question 13:

Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$

Solution:

Let $z=\frac{1+2i}{1-3i}$,

Then,

$$z=\frac{1+2i}{1-3i}\times\frac{1+3i}{1+3i}=\frac{1+2i+3i+6i^2}{1^2+3^2}=\frac{1+5i+6(-1)}{10}$$
$$=\frac{-5+5i}{10}=\frac{-5}{10}+\frac{5}{10}i=\frac{-1}{2}+\frac{1}{2}i$$

Let $z=r\cos\theta+ir\sin\theta$

i.e., $r\cos\theta=-\frac{1}{2}$ and $r\sin\theta=\frac{1}{2}$

On squaring and adding, we obtain

$$r^2\left(\cos^2\theta+\sin^2\theta\right)=\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2$$
$$\Rightarrow r^2=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$
$$\Rightarrow r=\frac{1}{\sqrt{2}} \qquad [\text{Conventionally, } r>0]$$

Therefore,

$$\frac{1}{\sqrt{2}}\cos\theta=-\frac{1}{2} \text{ and } \frac{1}{\sqrt{2}}\sin\theta=\frac{1}{2}$$
$$\Rightarrow \cos\theta=\frac{-1}{\sqrt{2}} \text{ and } \sin\theta=\frac{1}{\sqrt{2}}$$

Since, θ lies in the quadrant II, $\theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

Thus, the modulus and argument of the given complex number are $\frac{1}{\sqrt{2}}$ and $\frac{3\pi}{4}$ respectively.

Question 14:

Find the real numbers x and y if $(x-iy)(3+5i)$ is the conjugate of $-6-24i$

Solution:

Let,

$$\begin{aligned} z &= (x-iy)(3+5i) \\ &= 3x - i3y + i5x - i^2 5y \\ &= 3x + i5x - i3y + 5y \\ &= (3x+5y) + i(5x-3y) \end{aligned}$$

Then, $\bar{z} = (3x+5y) - i(5x-3y)$

It is given that, $\bar{z} = -6-24i$

Therefore, $(3x+5y) - i(5x-3y) = -6-24i$

Equating real and imaginary parts, we obtain

$$3x+5y=-6 \qquad \ldots(1)$$
$$5x-3y=24 \qquad \ldots(2)$$

Multiplying equation (1) by 3 and equation (2) by 5 and then adding them, we obtain

$$9x+15y=-18$$
$$25x-15y=120$$

On adding both equations we get,

$$\begin{aligned} 34x &= 102 \\ x &= \frac{102}{34} \\ x &= 3 \end{aligned}$$

Putting the value of x in equation (1), we obtain

$$3(3)+5y=-6$$
$$5y=-6-9$$
$$y=\frac{-15}{5}$$
$$y=-3$$

Thus, the values of $x=3$ and $y=-3$

Question 15:

Find the modulus of $\frac{1+i}{1-i}-\frac{1-i}{1+i}$

Solution:

$$\frac{1+i}{1-i}-\frac{1-i}{1+i}=\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$$
$$=\frac{1+i^2+2i-1-i^2+2i}{2}$$
$$=\frac{4i}{2}$$
$$=2i$$

Therefore,

$$\left|\frac{1+i}{1-i}-\frac{1-i}{1+i}\right|=|2i|$$
$$=\sqrt{2^2}$$
$$=2$$

Question 16:

If $(x+iy)^3=u+iv$, then show that $\frac{u}{x}+\frac{v}{y}=4\left(x^2-y^2\right)$

Solution:

It is given that $(x+iy)^3=u+iv$

$$\Rightarrow x^3+(iy)^3+3\times x\times iy(x+iy)=u+iv$$
$$\Rightarrow x^3+i^3y^3+3x^2yi+3xy^2i^2=u+iv$$
$$\Rightarrow x^3-iy^3+3x^2yi-3xy^2=u+iv$$
$$\Rightarrow \left(x^3-3xy^2\right)+i\left(3x^2y-y^3\right)=u+iv$$

On equating real and imaginary parts, we obtain

$$u = x^3 - 3xy^2, v = 3x^2y - y^3$$

Therefore,

$$\begin{aligned}\frac{u}{x} + \frac{v}{y} &= \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y} \\ &= \frac{x\left(x^2 - 3y^2\right)}{x} + \frac{y\left(3x^2 - y^2\right)}{y} \\ &= x^2 - 3y^2 + 3x^2 - y^2 \\ &= 4x^2 - 4y^2 \\ &= 4\left(x^2 - y^2\right)\end{aligned}$$

Hence, $\frac{u}{x} + \frac{v}{y} = 4\left(x^2 - y^2\right)$ proved.

Question 17:

If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$

Solution:

Let $\alpha = a + ib$ and $\beta = x + iy$

It is given that, $|\beta| = 1$

Therefore,

$$\sqrt{x^2 + y^2} = 1$$

$$x^2 + y^2 = 1 \qquad \ldots(i)$$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=\left|\frac{(x+iy)-(a+ib)}{1-(a-ib)(x+iy)}\right|$$

$$=\left|\frac{(x-a)+i(y-b)}{1-(ax+iay-ibx+by)}\right|$$

$$=\left|\frac{(x-a)+i(y-b)}{(1-ax-by)+i(bx-ay)}\right|$$

$$=\frac{|(x-a)+i(y-b)|}{|(1-ax-by)+i(bx-ay)|} \qquad \left[\because \left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\right]$$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=\frac{\sqrt{(x-a)^2+(y-b)^2}}{\sqrt{(1-ax-by)^2+(bx-ay)^2}}$$

$$=\frac{\sqrt{x^2+a^2-2ax+y^2+b^2-2by}}{\sqrt{1+a^2x^2+b^2y^2-2ax+2abxy-2by+b^2x^2+a^2y^2-2abxy}}$$

$$=\frac{\sqrt{(x^2+y^2)+a^2+b^2-2ax-2by}}{\sqrt{1+a^2(x^2+y^2)+b^2(y^2+x^2)-2ax-2by}}$$

$$=\frac{\sqrt{1+a^2+b^2-2ax-2by}}{\sqrt{1+a^2+b^2-2ax-2by}} \qquad [\text{Using } (i)]$$

$$=1$$

Thus, $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=1$

Question 18:

Find the number of non-zero integral solutions of the equation $|1-i|^x=2^x$

Solution:

$$|1-i|^x = 2^x$$
$$\left(\sqrt{1^2+(-1)^2}\right) = 2^x$$
$$\left(\sqrt{2}\right)^x = 2^x$$
$$2^{x/2} = 2^x$$
$$\frac{x}{2} = x$$
$$x = 2x$$
$$2x - x = 0$$
$$x = 0$$

Thus, 0 is the only integral solution of the given equation.
Therefore, the number of non-zero integral solutions of the given equation is 0.

Question 19:

If $(a+ib)(c+id)(e+if)(g+ih) = A+iB$, then show that:
$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

Solution:

It is given that, $(a+ib)(c+id)(e+if)(g+ih) = A+iB$

Therefore,

$$|(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$$
$$|(a+ib)|\times|(c+id)|\times|(e+if)|\times|(g+ih)| = |A+iB| \qquad [\because |z_1z_2| = |z_1||z_2|]$$
$$\sqrt{a^2+b^2}\times\sqrt{c^2+d^2}\times\sqrt{e^2+f^2}\times\sqrt{g^2+h^2} = \sqrt{A^2+B^2}$$
$$\sqrt{(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)} = \sqrt{A^2+B^2}$$

On squaring both sides, we obtain

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$$

Hence, proved.

Question 20:

If $\left(\frac{1+i}{1-i}\right)^m = 1$ then find the least positive integral value of m.

Solution:

It is given that $\left(\frac{1+i}{1-i}\right)^m = 1$

$$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$$

$$\left(\frac{(1+i)^2}{1^2+1^2}\right)^m = 1$$

$$\left(\frac{1^2+i^2+2i}{2}\right)^m = 1$$

$$\left(\frac{1-1+2i}{2}\right)^m = 1$$

$$\left(\frac{2i}{2}\right)^m = 1$$

$$i^m = 1$$

$$i^m = i^{4k}$$

Hence, $m = 4k$, where k is some integer.

Since, the least positive integer is 1, $m = 4 \times 1 = 4$

Thus, the least positive integral value of $m = 4$