NCERT Solutions Class 11 Maths Chapter 9 Sequences and Series

Question 1:

Write the first five terms of the sequences whose n^{th} term is $a_n = n(n+2)$.

Solution:

$a_n = n(n+2)$

Substituting $n = 1,2,3,4,5$

$a_1 = 1(1+2) = 3$

$a_2 = 2(2+2) = 8$

$a_3 = 3(3+2) = 15$

$a_4 = 4(4+2) = 24$

$a_5 = 5(5+2) = 35$

Therefore, the required terms are $3,8,15,24$ and 35.

Question 2:

Write the first five terms of the sequences whose n^{th} term is $a_n = \dfrac{n}{n+1}$.

Solution:

$a_n = \dfrac{n}{n+1}$

Substituting $n = 1,2,3,4,5$

$a_1 = \dfrac{1}{1+1} = \dfrac{1}{2}$

$a_2 = \dfrac{2}{2+1} = \dfrac{2}{3}$

$a_3 = \dfrac{3}{3+1} = \dfrac{3}{4}$

$a_4 = \dfrac{4}{4+1} = \dfrac{4}{5}$

$a_5 = \dfrac{5}{5+1} = \dfrac{5}{6}$

Therefore, the required terms are $\dfrac{1}{2},\dfrac{2}{3},\dfrac{3}{4},\dfrac{4}{5}$ and $\dfrac{5}{6}$.

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Question 3:

Write the first five terms of the sequences whose n^{th} term is $a_n = 2^n$.

Solution:

$a_n = 2^n$

Substituting $n = 1, 2, 3, 4, 5$

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

Therefore, the required terms are $2, 4, 8, 16$ and 32.

Question 4:

Write the first five terms of the sequences whose n^{th} term is $a_n = \frac{2n-3}{6}$.

Solution:

$a_n = \frac{2n-3}{6}$

Substituting $n = 1, 2, 3, 4, 5$

$$a_1 = \frac{2(1)-3}{6} = \frac{-1}{6}$$

$$a_2 = \frac{2(2)-3}{6} = \frac{1}{6}$$

$$a_3 = \frac{2(3)-3}{6} = \frac{1}{2}$$

$$a_4 = \frac{2(4)-3}{6} = \frac{5}{6}$$

$$a_5 = \frac{2(5)-3}{6} = \frac{7}{6}$$

Therefore, the required terms are $\frac{-1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$ and $\frac{7}{6}$.

Question 5:

Write the first five terms of the sequences whose n^{th} term is $a_n = (-1)^{n-1} 5^{n+1}$.

Solution:

$$a_n = (-1)^{n-1} 5^{n+1}$$

Substituting $n = 1, 2, 3, 4, 5$

$$a_1 = (-1)^{1-1} 5^{1+1} = 5^2 = 25$$
$$a_2 = (-1)^{2-1} 5^{2+1} = -5^3 = -125$$
$$a_3 = (-1)^{3-1} 5^{3+1} = 5^4 = 625$$
$$a_4 = (-1)^{4-1} 5^{4+1} = -5^5 = -3125$$
$$a_5 = (-1)^{5-1} 5^{5+1} = 5^6 = 15625$$

Therefore, the required terms are $25, -125, 625, -3125$ and 15625.

Question 6:

Write the first five terms of the sequences whose n^{th} term is $a_n = n\frac{n^2+5}{4}$.

Solution:

$$a_n = n\frac{n^2+5}{4}$$

Substituting $n = 1, 2, 3, 4, 5$

$$a_1 = 1\frac{1^2+5}{4} = \frac{3}{2}$$
$$a_2 = 2\frac{2^2+5}{4} = \frac{9}{2}$$
$$a_3 = 3\frac{3^2+5}{4} = \frac{21}{2}$$
$$a_4 = 4\frac{4^2+5}{4} = 21$$
$$a_5 = 5\frac{5^2+5}{4} = \frac{75}{2}$$

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Therefore, the required terms are $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}, 21$ and $\frac{75}{2}$.

Question 7:

Find the 17^{th} and 24^{th} term of the sequences whose n^{th} term is $a_n = 4n - 3$.

Solution:

$a_n = 4n - 3$

Substituting $n = 17$

$a_{17} = 4(17) - 3 = 68 - 3 = 65$

Substituting $n = 24$

$a_{24} = 4(24) - 3 = 96 - 3 = 93$

Question 8:

Write the 7^{th} term of the sequences whose n^{th} term is $a_n = \frac{n^2}{2^n}$.

Solution:

$a_n = \frac{n^2}{2^n}$

Substituting $n = 7$

$a_7 = \frac{7^2}{2^7} = \frac{49}{128}$

Question 9:

Write the 9^{th} term of the sequences whose n^{th} term is $a_n = (-1)^{n-1} n^3$.

Solution:

$a_n = (-1)^{n-1} n^3$

Substituting $n = 9$

$a_9 = (-1)^{9-1} 9^3 = 729$

Question 10:

Write the 20^{th} term of the sequences whose n^{th} term is $a_n = \frac{n(n-2)}{n+3}$.

Solution:

$$a_n = \frac{n(n-2)}{n+3}$$

Substituting $n = 20$

$$a_{20} = \frac{20(20-2)}{20+3} = \frac{360}{23}$$

Question 11:

Write the first five terms of the following sequence and obtain the corresponding series:

$a_1 = 3, a_n = 3a_{n-1} + 2$ for all $n > 1$.

Solution:

$a_1 = 3, a_n = 3a_{n-1} + 2$ for all $n > 1$.

$$a_2 = 3a_1 + 2 = 3(3) + 2 = 11$$
$$a_3 = 3a_2 + 2 = 3(11) + 2 = 35$$
$$a_4 = 3a_3 + 2 = 3(35) + 2 = 107$$
$$a_5 = 3a_4 + 2 = 3(107) + 2 = 323$$

Hence, the first five terms of the sequence are $3, 11, 35, 107$ and 323.

The corresponding series is $3 + 11 + 35 + 107 + 323 + \ldots$

Question 12:

Write the first five terms of the following sequence and obtain the corresponding series:

$a_1 = -1, a_n = \frac{a_{n-1}}{n}$ for all $n \geq 2$.

Solution:

$$a_1 = -1,\ a_n = \frac{a_{n-1}}{n};\ n \geq 2$$

$$a_2 = \frac{a_1}{2} = \frac{-1}{2}$$
$$a_3 = \frac{a_2}{3} = \frac{-1}{6}$$
$$a_4 = \frac{a_3}{4} = \frac{-1}{24}$$
$$a_5 = \frac{a_4}{5} = \frac{-1}{120}$$

Hence, the first five terms of the sequence are $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}$ and $\frac{-1}{120}$.

The corresponding series is $(-1) + \left(\frac{-1}{2}\right) + \left(\frac{-1}{6}\right) + \left(\frac{-1}{24}\right) + \left(\frac{-1}{120}\right) + \ldots$

Question 13:

Write the first five terms of the following sequence and obtain the corresponding series:

$a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$.

Solution:

$$a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$$
$$\Rightarrow a_3 = a_2 - 1 = 2 - 1 = 1$$
$$a_4 = a_3 - 1 = 1 - 1 = 0$$
$$a_5 = a_4 - 1 = 0 - 1 = -1$$

Hence, the first five terms of the sequence are $2, 2, 1, 0$ and -1

The corresponding series is $2 + 2 + 1 + 0 + (-1) + \ldots$

Question 14:

The Fibonacci sequence is defined by $1 = a_1 = a_2$ and $a_n = a_{n-1} + a_{n-2}, n > 2$. Find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$

Solution:

$$1 = a_1 = a_2$$
$$a_n = a_{n-1} + a_{n-2}, n > 2$$

Therefore,

$a_3 = a_2 + a_1 = 1 + 1 = 2$
$a_4 = a_3 + a_2 = 2 + 1 = 3$
$a_5 = a_4 + a_3 = 3 + 2 = 5$
$a_6 = a_5 + a_4 = 5 + 3 = 8$

For $n = 1, \frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = \frac{1}{1} = 1$

For $n = 2, \frac{a_{n+1}}{a_n} = \frac{a_3}{a_2} = \frac{2}{1} = 2$

For $n = 3, \frac{a_{n+1}}{a_n} = \frac{a_4}{a_3} = \frac{3}{2}$

For $n = 4, \frac{a_{n+1}}{a_n} = \frac{a_5}{a_4} = \frac{5}{3}$

For $n = 5, \frac{a_{n+1}}{a_n} = \frac{a_6}{a_5} = \frac{8}{5}$

EXERCISE 9.2

Question 1:
Find the sum of odd integers from 1 to 2001.

Solution:
The odd integers from 1 to 2001 are $1,3,5,7,9,\ldots,1999,2001$.
This sequence forms an A.P.
Here, first term, $a=1$
Common difference, $d=2$
Last term $l=a_n=2001$
Therefore,

$$\begin{aligned}
a_n &= a+(n-1)d \\
1+(n-1)(2) &= 2001 \\
1+2n-2 &= 2001 \\
2n &= 2001+1 \\
n &= \frac{2002}{2} \\
n &= 1001
\end{aligned}$$

Now,

$$\begin{aligned}
S_n &= \frac{n}{2}\left[2a+(n-1)d\right] \\
S_{1001} &= \frac{1001}{2}\left[2\times1+(1001-1)\times2\right] \\
&= \frac{1001}{2}\left[2+1000\times2\right] \\
&= \frac{1001}{2}\times2002 \\
&= 1001\times1001 \\
&= 1002001
\end{aligned}$$

Thus, the sum of odd numbers from 1 to 2001 is 1002001.

Question 2:
Find the sum of all-natural numbers lying between 100 and 1000, which are multiples of 5.

Solution:
The natural numbers lying between 100 and 1000, which are multiples of 5 are

$$105, 110, 115, \ldots, 995$$

Here, first term, $a = 105$
Common difference, $d = 5$
Last term $l = a_n = 995$
Therefore,

$$\begin{aligned} a_n &= a + (n-1)d \\ 105 + (n-1)5 &= 995 \\ (n-1)5 &= 995 - 105 \\ (n-1) &= \frac{890}{5} \\ n &= 178 + 1 \\ n &= 179 \end{aligned}$$

Now,

$$\begin{aligned} S_n &= \frac{179}{2}\left[2(105) + (179-1)(5)\right] \\ &= \frac{179}{2}\left[2(105) + (178)(5)\right] \\ &= 179\left[105 + 89(5)\right] \\ &= 179\left[105 + 445\right] \\ &= 179 \times 550 \\ &= 98450 \end{aligned}$$

Thus, the sum of all-natural numbers lying between 100 and 1000, which are multiples of 5 is 98450.

Question 3:

In an A.P, the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20^{th} term is -112.

Solution:

It is given that the first term of the A.P. $a = 2$
Let d be the common difference of the A.P.

Hence, the A.P is $2, 2+d, 2+2d, 2+3d \ldots$

We know that, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Sum of the first five terms

$$\begin{aligned}S_5 &= \frac{5}{2}\left[2\times 2+(5-1)d\right]\\ &= \frac{5}{2}[4+4d]\\ &= 10+10d\end{aligned}$$

Sum of the first ten terms

$$\begin{aligned}S_{10} &= \frac{10}{2}\left[2\times 2+(10-1)d\right]\\ &= 5[4+9d]\\ &= 20+45d\end{aligned}$$

Hence,
Sum of the next five terms

$$\begin{aligned}S_{10}-S_5 &= (20+45d)-(10+10d)\\ &= 20+45d-10-10d\\ &= 10+35d\end{aligned}$$

According to the given condition,

$$\begin{aligned}&\Rightarrow 10+10d = \frac{1}{4}(10+35d)\\ &\Rightarrow 40+40d = 10+35d\\ &\Rightarrow 40d-35d = 10-40\\ &\Rightarrow 5d = -30\\ &\Rightarrow d = -6\end{aligned}$$

Therefore,

$$\begin{aligned}a_{20} &= a+(20-1)d\\ &= 2+(19)(-6)\\ &= 2-114\\ &= -112\end{aligned}$$

Thus, the 20^{th} term of the A.P is -112.

Question 4:

How many terms of the A.P. $-6, \frac{-11}{2}, -5, \ldots$ are needed to give the sum -25?

Solution:

It is given that $-6, \frac{-11}{2}, -5, \ldots$ are in A.P.
Let the sum of n terms of the given A.P be $S_n = -25$.

Here, first term $a = -6$

Common difference $d = \frac{-11}{2} + 6 = \frac{-11+12}{2} = \frac{1}{2}$

We know that, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Therefore,

$$\Rightarrow -25 = \frac{n}{2}\left[2\times(-6) + (n-1)\left(\frac{1}{2}\right)\right]$$
$$\Rightarrow -50 = n\left[-12 + \frac{n}{2} - \frac{1}{2}\right]$$
$$\Rightarrow -50 = n\left[\frac{n}{2} - \frac{25}{2}\right]$$
$$\Rightarrow -100 = n[n-25]$$
$$\Rightarrow n^2 - 25n + 100 = 0$$
$$\Rightarrow n^2 - 5n - 20n + 100 = 0$$
$$\Rightarrow n(n-5) - 20(n-5) = 0$$
$$\Rightarrow (n-5)(n-20) = 0$$
$$\Rightarrow n = 20, 5$$

Question 5:

In an A.P, if the p^{th} term is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, prove that the sum of first pq terms is $\frac{1}{2}(pq+1)$, where $p \neq q$.

Solution:

It is known that general term of an A.P is $a_n = a + (n-1)d$
Therefore,

$$p^{th} term = a_p = a + (p-1)d = \frac{1}{q} \qquad \ldots(1)$$

$$q^{th} term = a_q = a + (q-1)d = \frac{1}{p} \qquad \ldots(2)$$

Subtracting (2) from (1), we obtain

$$\Rightarrow (p-1)d - (q-1)d = \frac{1}{q} - \frac{1}{p}$$

$$\Rightarrow (p-1-q+1)d = \frac{p-q}{pq}$$

$$\Rightarrow (p-q)d = \frac{p-q}{pq}$$

$$\Rightarrow (p-q)d = \frac{p-q}{pq}$$

$$\Rightarrow d = \frac{1}{pq}$$

Putting the value of d in (1), we obtain

$$\Rightarrow a + (p-1)\frac{1}{pq} = \frac{1}{q}$$

$$\Rightarrow a = \frac{1}{q} - \frac{1}{q} + \frac{1}{pq} = \frac{1}{pq}$$

Therefore,

$$S_{pq} = \frac{pq}{2}\left[2a + (pq-1)d\right]$$

$$= \frac{pq}{2}\left[\frac{2}{pq} + (pq-1)\frac{1}{pq}\right]$$

$$= \frac{pq}{2}\left[\frac{2}{pq} + 1 - \frac{1}{pq}\right]$$

$$= \frac{pq}{2}\left[\frac{1}{pq} + 1\right]$$

$$= \frac{1}{2} + \frac{pq}{2}$$

$$= \frac{1}{2}(pq+1)$$

Thus, the sum of first pq terms of the A.P is $\frac{1}{2}(pq+1)$.

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Question 6:

If the sum of a certain number of terms of the A.P $25, 22, 19, \ldots$ is 116. Find the last term.

Solution:

It is given that $25, 22, 19, \ldots$ are in A.P.
Let the sum of n terms of an A.P be 116.
Here, first term $a = 25$
Common difference $d = 22 - 25 = -3$

We know that, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$
Therefore,

$$\Rightarrow 116 = \frac{n}{2}\left[2(25) + (n-1)(-3)\right]$$
$$\Rightarrow 116 = \frac{n}{2}\left[50 - 3n + 3\right]$$
$$\Rightarrow 232 = n(53 - 3n) = 53n - 3n^2$$
$$\Rightarrow 3n^2 - 53n + 232 = 0$$
$$\Rightarrow 3n^2 - 24n - 29n + 232 = 0$$
$$\Rightarrow 3n(n-8) - 29(n-8) = 0$$
$$\Rightarrow (n-8)(3n-29) = 0$$
$$\Rightarrow n = 8, \frac{29}{3} \qquad \therefore$$

However, n cannot be equal to $\frac{29}{3}$
Hence, $n = 8$

Therefore, last term $l = a_n = a + (n-1)d$

$$a_8 = 25 + (8-1)(-3)$$
$$= 25 + (7)(-3)$$
$$= 25 - 21$$
$$= 4$$

Thus, the last term of the A.P is 4.

Question 7:

Find the sum to n terms of the A.P, whose k^{th} term is $5k + 1$.

Solution:

It is given that k^{th} term of the A.P is $5k+1$.

i.e., $a_k=(5k+1)$

It is known that general term of an A.P is $a_n=a+(n-1)d$

$$a_k=a+(k-1)d$$
$$5k+1=a+(k-1)d$$
$$5k+1=kd+(a-d)$$

Comparing the coefficients of k, we obtain $d=5$ and

$$\Rightarrow a-d=1$$
$$\Rightarrow a-5=1$$
$$\Rightarrow a=6$$

Therefore,

$$S_n=\frac{n}{2}[2a+(n-1)d]$$
$$=\frac{n}{2}[2(6)+(n-1)(5)]$$
$$=\frac{n}{2}[12+5n-5]$$
$$=\frac{n}{2}[5n+7]$$

Question 8:

If the sum of n terms of an A.P is $(pn+qn^2)$, where p and q are constants, find the common difference.

Solution:

We know that $S_n=\frac{n}{2}[2a+(n-1)d]$

According to the question,

$$\frac{n}{2}[2a+(n-1)d]=pn+qn^2$$
$$\frac{n}{2}[2a+nd-d]=pn+qn^2$$
$$an+\frac{1}{2}dn^2-\frac{1}{2}dn=pn+qn^2$$
$$\left(a-\frac{1}{2}d\right)n+\frac{1}{2}dn^2=pn+qn^2$$

Comparing the coefficients of n^2 on both sides, we obtain

$$\Rightarrow \frac{1}{2}d = q$$
$$\Rightarrow d = 2q$$

Thus, the common difference is $2q$.

Question 9:

The sums of n terms of two arithmetic progressions are in the ratio $5n+4:9n+6$. Find the ratio of their 18^{th} terms.

Solution:

Let a_1, a_2 and d_1, d_2 be the first terms and common differences of two arithmetic progressions respectively.

According to the given condition,

$$\frac{\text{Sum of n terms of first A.P}}{\text{Sum of n terms of second A.P}} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \frac{\frac{n}{2}\left[2a_1+(n-1)d_1\right]}{\frac{n}{2}\left[2a_2+(n-1)d_2\right]} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{5n+4}{9n+6} \qquad \ldots(1)$$

Substituting $n=35$ in (1), we obtain

$$\Rightarrow \frac{2a_1+(34)d_1}{2a_2+(34)d_2} = \frac{5(35)+4}{9(35)+6}$$

$$\Rightarrow \frac{a_1+17d_1}{a_2+17d_2} = \frac{179}{321} \qquad \ldots(2)$$

Now,

$$\frac{18^{th}\text{term of first A.P}}{18^{th}\text{term of second A.P}} = \frac{a_1+17d_1}{a_2+17d_2} \qquad \ldots(3)$$

From (2) and (3), we obtain

$$\frac{18^{th}\text{term of first A.P}}{18^{th}\text{term of second A.P}} = \frac{179}{321}$$

Thus, the ratio of their 18^{th} terms is $179:321$.

Question 10:

If the sum of first p terms of an A.P is equal to the sum of first q terms, then find the sum of first $(p+q)$ terms.

Solution:

Let a and d be the first term and the common difference of the A.P. respectively.

We know that $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

Hence,

$$S_p = \frac{p}{2}\left[2a+(p-1)d\right]$$
$$S_q = \frac{q}{2}\left[2a+(q-1)d\right]$$

According to the question,

$$\Rightarrow \frac{p}{2}\left[2a+(p-1)d\right] = \frac{q}{2}\left[2a+(q-1)d\right]$$
$$\Rightarrow p\left[2a+(p-1)d\right] = q\left[2a+(q-1)d\right]$$
$$\Rightarrow 2ap+pd(p-1) = 2aq+qd(q-1)$$
$$\Rightarrow 2a(p-q)+d\left[p(p-1)-q(q-1)\right]=0$$
$$\Rightarrow 2a(p-q)+d\left[p^2-p-q^2+q\right]=0$$
$$\Rightarrow 2a(p-q)+d\left[(p-q)(p+q)-(p-q)\right]=0$$
$$\Rightarrow 2a(p-q)+d\left[(p-q)(p+q-1)\right]=0$$
$$\Rightarrow 2a+d(p+q-1)=0$$
$$\Rightarrow d = \frac{-2a}{p+q-1} \qquad \ldots(1)$$

Now,

$$S_{p+q}=\frac{p+q}{2}\left[2a+(p+q-1)\left(\frac{-2a}{p+q-1}\right)\right] \qquad [from(1)]$$

$$=\frac{p+q}{2}[2a-2a]$$

$$=0$$

Thus, the sum of the first $(p+q)$ terms of the A.P is 0.

Question11:

Sum of the first p,q and r terms of an A.P., are a,b and c respectively. Prove that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$

Solution:

Let a_1 and d be the first term and the common difference of the given arithmetic progression respectively.

We know that $S_n=\frac{n}{2}[2a+(n-1)d]$

According to the given information,

$$S_p=\frac{p}{2}[2a_1+(p-1)d]$$

$$a=\frac{p}{2}[2a_1+(p-1)d]$$

$$\Rightarrow 2a_1+(p-1)d=\frac{2a}{p} \qquad \ldots(1)$$

$$S_q=\frac{q}{2}[2a_1+(q-1)d]$$

$$b=\frac{q}{2}[2a_1+(q-1)d]$$

$$\Rightarrow 2a_1+(q-1)d=\frac{2b}{q} \qquad \ldots(2)$$

$$S_r=\frac{r}{2}[2a_1+(r-1)d]$$

$$c=\frac{r}{2}[2a_1+(r-1)d]$$

$$\Rightarrow 2a_1+(r-1)d=\frac{2c}{r} \qquad \ldots(3)$$

Subtracting (2) from (1)

$$\Rightarrow (p-1)d-(q-1)d=\frac{2a}{p}-\frac{2b}{q}$$

$$\Rightarrow d(p-1-q+1)=\frac{2aq-2bp}{pq}$$

$$\Rightarrow d(p-q)=\frac{2aq-2bp}{pq}$$

$$\Rightarrow d=\frac{2(aq-bp)}{pq(p-q)} \qquad \ldots(4)$$

Subtracting (3) from (2)

$$\Rightarrow (q-1)d-(r-1)d=\frac{2b}{q}-\frac{2c}{r}$$

$$\Rightarrow d(q-1-r+1)=\frac{2b}{q}-\frac{2c}{r}$$

$$\Rightarrow d(q-r)=\frac{2br-2cq}{qr}$$

$$\Rightarrow d=\frac{2(br-qc)}{qr(q-r)} \qquad \ldots(5)$$

Equating both the values of d obtained in (4) and (5), we obtain

$$\Rightarrow \frac{aq-bp}{pq(p-q)}=\frac{br-qc}{qr(q-r)}$$

$$\Rightarrow qr(q-r)(aq-bp)=pq(p-q)(br-qc)$$

$$\Rightarrow r(aq-bp)(q-r)=p(br-qc)(p-q)$$

$$\Rightarrow (aqr-bpr)(q-r)=(bpr-pqc)(p-q)$$

Dividing both sides by pqr, we obtain

$$\Rightarrow \left(\frac{a}{p}-\frac{b}{q}\right)(q-r)=\left(\frac{b}{q}-\frac{c}{r}\right)(p-q)$$

$$\Rightarrow \frac{a}{p}(q-r)-\frac{b}{q}(q-r)=\frac{b}{q}(p-q)-\frac{c}{r}(p-q)$$

$$\Rightarrow \frac{a}{p}(q-r)-\frac{b}{q}(q-r+p-q)+\frac{c}{r}(p-q)=0$$

$$\Rightarrow \frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

Hence, proved.

Question 12:

The ratio of the sum of m and n terms of an A.P is $m^2 : n^2$. Show that the ratio of m^{th} and n^{th} term is $(2m-1):(2n-1)$.

Solution:

Let a and d be the first term and the common difference of the given arithmetic progression respectively.

According to the given condition,

$$\Rightarrow \frac{\text{Sum of m terms}}{\text{Sum of n terms}} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{\frac{m}{2}\left[2a+(m-1)d\right]}{\frac{n}{2}\left[2a+(n-1)d\right]} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n} \qquad \ldots(1)$$

Putting $m = 2m-1$ and $n = 2n-1$, we obtain

$$\frac{2a+(2m-2)d}{2a+(2n-2)d} = \frac{2m-1}{2n-1}$$

$$\Rightarrow \frac{a+(m-1)d}{a+(n-1)d} = \frac{2m-1}{2n-1} \qquad \ldots(2)$$

$$\frac{m^{th} \text{ term of A.P}}{n^{th} \text{ term of A.P}} = \frac{a+(m-1)d}{a+(n-1)d} \qquad \ldots(3)$$

From (2) and (3), we obtain

$$\frac{\text{m}^{th} \text{ term of A.P}}{n^{th} \text{ term of A.P}} = \frac{2m-1}{2n-1}$$

Hence, proved.

Question 13:

If the sum of n terms of an A.P of $3n^2+5n$ and its m^{th} term is 164, find the value of m.

Solution:

Let a and d be the first term and the common difference of the given arithmetic progression respectively.

$$a_m = a + (m-1)d = 164 \qquad \ldots(1)$$

Sum of n terms, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Here,

$$\Rightarrow \frac{n}{2}[2a+nd-d]=3n^2+5n$$
$$\Rightarrow an+\frac{d}{2}n^2-\frac{d}{2}n=3n^2+5n$$
$$\Rightarrow \frac{d}{2}n^2+\left(a-\frac{d}{2}\right)n=3n^2+5n \qquad \ldots(2)$$

Comparing the coefficient of n^2 on both sides in (2), we obtain

$$\Rightarrow \frac{d}{2}=3$$
$$\Rightarrow d=6$$

Comparing the coefficient of n on both sides in (2), we obtain

$$\Rightarrow a-\frac{d}{2}=5$$
$$\Rightarrow a-3=5$$
$$\Rightarrow a=8$$

Therefore, from (1)

$$\Rightarrow 8+(m-1)6=164$$
$$\Rightarrow (m-1)6=164-8$$
$$\Rightarrow (m-1)=\frac{156}{6}$$
$$\Rightarrow m=26+1$$
$$\Rightarrow m=27$$

Thus, the value of $m=27$.

Question 14:

Insert five numbers between 8 and 26 such that resulting sequence is an A.P.

Solution:

Let A_1, A_2, A_3, A_4 and A_5 be the five numbers between 8 and 26 such that;

$8, A_1, A_2, A_3, A_4, A_5, 26$ are in A.P.

Here, $a=8, b=26, n=7$

Hence,

$$\Rightarrow 26 = 8 + (7-1)d$$
$$\Rightarrow 26 = 8 + (6)d$$
$$\Rightarrow 6d = 26 - 8$$
$$\Rightarrow 6d = 18$$
$$\Rightarrow d = 3$$

Therefore,

$$A_1 = a + d = 8 + 3 = 11$$
$$A_2 = a + 2d = 8 + (2)3 = 14$$
$$A_3 = a + 3d = 8 + (3)3 = 17$$
$$A_4 = a + 4d = 8 + (4)3 = 20$$
$$A_5 = a + 5d = 8 + (5)3 = 23$$

Thus, the required five numbers between 8 and 26 are $11, 14, 17, 20$ and 23.

Question 15:

If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M between a and b, then find the value of n.

Solution:

A.M of a and $b = \frac{a+b}{2}$

According to the question,

$$\Rightarrow \frac{a+b}{2} = \frac{a^n + b^n}{a^{n-1} + b^{n-1}}$$
$$\Rightarrow (a+b)\left(a^{n-1} + b^{n-1}\right) = 2\left(a^n + b^n\right)$$
$$\Rightarrow a^n + ab^{n-1} + ba^{n-1} + b^n = 2a^n + 2b^n$$
$$\Rightarrow ab^{n-1} + ba^{n-1} = a^n + b^n$$
$$\Rightarrow ab^{n-1} - b^n = a^n - ba^{n-1}$$
$$\Rightarrow b^{n-1}(a-b) = a^{n-1}(a-b)$$
$$\Rightarrow b^{n-1} = a^{n-1}$$
$$\Rightarrow \left(\frac{a}{b}\right)^{n-1} = 1 = \left(\frac{a}{b}\right)^0$$
$$\Rightarrow n - 1 = 0$$
$$\Rightarrow n = 1$$

Question 16:

Between 1 and 31, m numbers have been inserted in such a way that the resulting sequence is an A.P and the ratio of 7^{th} and $(m-1)^{th}$ numbers is $5:9$. Find the value of m.

Solution:

Let $A_1, A_2, \dots, A_m$ be numbers such that $1, A_1, A_2, \dots, A_m, 31$ are in A.P.

Here, $a=1, b=31, n=m+2$

Therefore,

$$\Rightarrow 31 = 1 + (m+2-1)d$$
$$\Rightarrow 30 = (m+1)d$$
$$\Rightarrow d = \frac{30}{m+1} \qquad \dots(1)$$

Hence,

$$A_1 = a + d$$
$$A_2 = a + 2d$$
$$A_3 = a + 3d$$
$$A_7 = a + 7d$$
$$A_{m-1} = a + (m-1)d$$

According to the given condition,

$$\Rightarrow \frac{a+7d}{a+(m-1)d} = \frac{5}{9}$$
$$\Rightarrow \frac{1+7\left(\frac{30}{m+1}\right)}{1+(m-1)\left(\frac{30}{m+1}\right)} = \frac{5}{9} \qquad [from\ (1)]$$
$$\Rightarrow \frac{m+1+7(30)}{m+1+30(m-1)} = \frac{5}{9}$$
$$\Rightarrow \frac{m+1+210}{m+1+30m-30} = \frac{5}{9}$$
$$\Rightarrow \frac{m+211}{31m-29} = \frac{5}{9}$$
$$\Rightarrow 9m + 1899 = 155m - 145$$
$$\Rightarrow 155m - 9m = 1899 + 145$$
$$\Rightarrow 146m = 2044$$
$$\Rightarrow m = 14$$

Thus, the value of $m = 14$.

Question 17:

A man starts repaying a loan as first instalment of ₹ 100. If he increases the instalment by ₹ 5 every month, what amount will he repay in the 30^{th} instalment?

Solution:

The first installment of the load is ₹ 100.
The second installment of the load is ₹ 105 and so on.
The amount that the man repays every month becomes an A.P.

The A.P is $100, 105, 110, \ldots$

Here, $a = 100, d = 5$

$$\begin{aligned} A_{30} &= a + (30 - 1)d \\ &= 100 + 29(5) \\ &= 100 + 145 \\ &= 245 \end{aligned}$$

Thus, the amount to be paid in the 30^{th} instalment is ₹ 245.

Question 18:

The difference between any two consecutive interior angles of a polygon is $5°$. If the smallest angle is $120°$, find the number of the sides of the polygon.

Solution:

The angles of the polygon will form an A.P with $d = 5°$ and $a = 120°$.

We know that the sum of all angles of a polygon with n sides is $180°(n-2)$
Therefore,

$$\Rightarrow S_n = 180°(n-2)$$

$$\Rightarrow \frac{n}{2}\left[2a+(n-1)d\right] = 180^\circ(n-2)$$

$$\Rightarrow \frac{n}{2}\left[240^\circ+(n-1)5^\circ\right] = 180^\circ(n-2)$$

$$\Rightarrow n\left[240+(n-1)5\right] = 360(n-2)$$

$$\Rightarrow 240n+5n^2-5n = 360n-720$$

$$\Rightarrow 5n^2-125n+720 = 0$$

$$\Rightarrow n^2-25n+144 = 0$$

$$\Rightarrow n^2-16n-9n+144 = 0$$

$$\Rightarrow n(n-16)-9(n-16) = 0$$

$$\Rightarrow (n-9)(n-16) = 0$$

$$\Rightarrow n = 9 \text{ or } 16$$

EXERCISE 9.3

Question 1:

Find the 20^{th} and n^{th} terms of the G.P. $\frac{5}{2},\frac{5}{4},\frac{5}{8},\ldots$

Solution:

The given G.P is $\frac{5}{2},\frac{5}{4},\frac{5}{8},\ldots\ldots\ldots\ldots\ldots$

Here, $a=\frac{5}{2}$ and $r=\frac{5/4}{5/2}=\frac{1}{2}$

Therefore,

$$a_{20}=ar^{20-1}=\frac{5}{2}\left(\frac{1}{2}\right)^{19}=\frac{5}{(2)(2)^{19}}=\frac{5}{(2)^{20}}$$

$$a_n=ar^{n-1}=\frac{5}{2}\left(\frac{1}{2}\right)^{n-1}=\frac{5}{(2)(2)^{n-1}}=\frac{5}{(2)^{n}}$$

Question 2:

Find the 12^{th} term of a G.P whose 8^{th} term is 192 and the common ratio is 2.

Solution:

Let a be the first term of the G.P

It is given that common ratio, $r=2$

Eighth term of the G.P, $a_8=192$

Therefore,

$$\Rightarrow a_8=ar^{8-1}=ar^7$$

$$\Rightarrow ar^7=192$$

$$\Rightarrow a(2)^7=192$$

$$\Rightarrow a(2)^7=(2)^6(3)$$

$$\Rightarrow a=\frac{(2)^6(3)}{(2)^7}=\frac{3}{2}$$

Hence,

$$\begin{aligned} a_{12} &= ar^{12-1} \\ &= ar^{11} \\ &= \left(\frac{3}{2}\right)(2)^{11} \\ &= (3)(2)^{10} \\ &= 3072 \end{aligned}$$

Question 3:

The $5^{th}, 8^{th}$ and 11^{th} terms of a G.P are p, q and s respectively. Show that $q^2 = ps$.

Solution:

Let a be the first term and r be the common ratio of the G.P.
According to the question,

$$a_5 = ar^{5-1} = ar^4 = p \qquad \ldots(1)$$

$$a_8 = ar^{8-1} = ar^7 = q \qquad \ldots(2)$$

$$a_{11} = ar^{11-1} = ar^{10} = s \qquad \ldots(3)$$

Dividing (2) by (1), we obtain

$$\frac{ar^7}{ar^4} = \frac{q}{p}$$

$$r^3 = \frac{q}{p} \qquad \ldots(4)$$

Dividing (3) by (2), we obtain

$$\frac{ar^{10}}{ar^7} = \frac{s}{q}$$

$$r^3 = \frac{s}{q} \qquad \ldots(5)$$

Equating the values of r^3 obtained in (4) and (5), we obtain

$$\Rightarrow \frac{q}{p} = \frac{s}{q}$$

$$\Rightarrow q^2 = ps$$

Hence proved.

Question 4:

The 4^{th} term of a G.P is square of its second term, and the first term is -3. Determine its 7^{th} term.

Solution:

Let a be the first term and r be the common ratio of the G.P.

It is known that $a_n = ar^{n-1}$

Therefore,

$$a_2 = ar^1 = (-3)r$$
$$a_4 = ar^3 = (-3)r^3$$

According to the question,

$$\Rightarrow (-3)r^3 = [(-3)r]^2$$
$$\Rightarrow -3r^3 = 9r^2$$
$$\Rightarrow r = -3$$

Hence,

$$\begin{aligned} a_7 &= ar^{7-1} \\ &= ar^6 \\ &= (-3)(-3)^6 \\ &= (-3)^7 \\ &= -2187 \end{aligned}$$

Thus, the seventh term of the G.P is -2187.

Question 5:

Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4 \ldots$ is 128?

(b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729?

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?

Solution:

(a) The given sequence is $2, 2\sqrt{2}, 4 \ldots$

Here, $a = 2$ and $r = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Let the n^{th} term of the sequence be 128

$$\Rightarrow a_n = ar^{n-1} = 128$$
$$\Rightarrow (2)(\sqrt{2})^{n-1} = 128$$
$$\Rightarrow (2)(2)^{\frac{n-1}{2}} = (2)^7$$
$$\Rightarrow (2)^{\frac{n-1}{2}+1} = (2)^7$$

Hence,

$$\Rightarrow \frac{n-1}{2} + 1 = 7$$
$$\Rightarrow \frac{n-1}{2} = 6$$
$$\Rightarrow n - 1 = 12$$
$$\Rightarrow n = 13$$

Thus, the 13^{th} term of the sequence be 128.

(b) The given sequence is $\sqrt{3}, 3, 3\sqrt{3}, \ldots$

Here, $a = \sqrt{3}$ and $r = \frac{3}{\sqrt{3}} = \sqrt{3}$

Let the n^{th} term of the sequence be 729.

$$\Rightarrow a_n = ar^{n-1} = 729$$
$$\Rightarrow (\sqrt{3})(\sqrt{3})^{n-1} = 729$$
$$\Rightarrow (3)^{\frac{1}{2}}(3)^{\frac{n-1}{2}} = (3)^6$$
$$\Rightarrow (3)^{\frac{1}{2}+\frac{n-1}{2}} = (3)^6$$

Hence,

$$\Rightarrow \frac{1}{2} + \frac{n-1}{2} = 6$$
$$\Rightarrow \frac{1+n-1}{2} = 6$$
$$\Rightarrow n = 12$$

Thus, the 12^{th} term of the sequence is 729.

(c) The given sequence is $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

Here, $a=\frac{1}{3}$ and $r=\frac{1/9}{1/3}=\frac{1}{3}$

Let the n^{th} term of the sequence be $\frac{1}{19683}$.

$$\Rightarrow a_n = ar^{n-1}=\frac{1}{19683}$$
$$\Rightarrow \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1}=\frac{1}{19683}$$
$$\Rightarrow \left(\frac{1}{3}\right)^{n}=\left(\frac{1}{3}\right)^{9}$$
$$\Rightarrow n=9$$

Thus, the 9^{th} term of the sequence be $\frac{1}{19683}$.

Question 6:

For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P?

Solution:

The given numbers are $-\frac{2}{7}, x, -\frac{7}{2}$

Hence,

Common ratio $r=\frac{x}{-2/7}=-\frac{7x}{2}$

Also, Common ratio $r=\frac{-7/2}{x}=-\frac{7}{2x}$

Therefore,

$$\Rightarrow -\frac{7x}{2}=-\frac{7}{2x}$$
$$\Rightarrow 14x^2=14$$
$$\Rightarrow x^2=1$$
$$\Rightarrow x=\pm\sqrt{1}$$
$$\Rightarrow x=\pm 1$$

Thus, for $x=\pm 1$, the given numbers will be in G.P.

Question 7:

Find the sum up to 20 terms in the G.P $0.15, 0.015, 0.0015\ldots$

Solution:

The given G.P is $0.15, 0.015, 0.0015\ldots$

Here, $a = 0.15$ and $r = \frac{0.015}{0.15} = 0.1$

It is known that $S_n = \frac{a(1-r^n)}{1-r}$

Therefore,

$$\begin{aligned} S_{20} &= \frac{0.15\left[1-(0.1)^{20}\right]}{1-0.1} \\ &= \frac{0.15}{0.9}\left[1-(0.1)^{20}\right] \\ &= \frac{1}{6}\left[1-(0.1)^{20}\right] \end{aligned}$$

Question 8:

Find the sum of n terms in the G.P $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$

Solution:

The given G.P is $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$

Here, $a = \sqrt{7}$ and $r = \frac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}$

It is known that $S_n = \frac{a(1-r^n)}{1-r}$

Therefore,

$$S_n = \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}}$$

$$= \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{\sqrt{7}\left(\sqrt{3}+1\right)\left[1-\left(\sqrt{3}\right)^n\right]}{1-3}$$

$$= \frac{-\sqrt{7}\left(\sqrt{3}+1\right)\left[1-\left(\sqrt{3}\right)^n\right]}{2}$$

$$= \frac{\sqrt{7}\left(\sqrt{3}+1\right)}{2}\left[(3)^{\frac{n}{2}}-1\right]$$

Question 9:

Find the sum of n terms in the G.P $1, -a, a^2, -a^3, \ldots$ (if $a \neq -1$).

Solution:

The given G.P is $1, -a, a^2, -a^3, \ldots$

Here, $a_1 = 1$ and $r = -a$

It is known that $S_n = \frac{a\left(1-r^n\right)}{1-r}$

Therefore,

$$S_n = \frac{1\left[1-(-a)^n\right]}{1-(-a)}$$

$$= \frac{\left[1-(-a)^n\right]}{1+a}$$

Question 10:

Find the sum of n terms in the G.P. $x^3, x^5, x^7, \ldots$ (if $x \neq \pm 1$).

Solution:

The given G.P is $x^3, x^5, x^7, \ldots$

Here, $a = x^3$ and $r = x^2$

It is known that $S_n = \dfrac{a(1-r^n)}{1-r}$

Therefore,

$$\begin{aligned} S_n &= \frac{x^3\left[1-\left(x^2\right)^n\right]}{1-x^2} \\ &= \frac{x^3\left(1-x^{2n}\right)}{1-x^2} \end{aligned}$$

Question 11:

Evaluate $\sum_{k=1}^{11}\left(2+3^k\right)$

Solution:

$$\begin{aligned} \sum_{k=1}^{11}\left(2+3^k\right) &= \sum_{k=1}^{11}(2)+\sum_{k=1}^{11}\left(3^k\right) \\ &= 22+\sum_{k=1}^{11}3^k \qquad \ldots(1) \end{aligned}$$

Now,

$$\sum_{k=1}^{11}3^k = 3^1+3^2+3^3+\ldots+3^{11}$$

The terms of this sequence $3, 3^2, 3^3, \ldots, 3^{11}$ forms a G.P.

Therefore,

$$\begin{aligned} S_n &= \frac{3\left[(3)^{11}-1\right]}{3-1} \\ &= \frac{3}{2}\left(3^{11}-1\right) \end{aligned}$$

Hence,

$$\sum_{k=1}^{11}3^k = \frac{3}{2}\left(3^{11}-1\right)$$

Substituting this value in (1), we obtain

$$\sum_{k=1}^{11}\left(2+3^k\right)=22+\frac{3}{2}\left(3^{11}-1\right)$$

Question 12:

The sum of first three terms of a G.P is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

Solution:

Let $\frac{a}{r}, a, ar$ be the first three terms of the G.P.
It is given that

$$\frac{a}{r}+a+ar=\frac{39}{10} \qquad \ldots(1)$$

$$\left(\frac{a}{r}\right)(a)(ar)=1 \qquad \ldots(2)$$

From (2), we obtain

$$a^3=1$$

$$\Rightarrow a=1 \qquad \text{(considering real roots)}$$

Substituting $a=1$ in (1), we obtain

$$\Rightarrow \frac{1}{r}+1+r=\frac{39}{10}$$

$$\Rightarrow 1+r+r^2=\frac{39}{10}r$$

$$\Rightarrow 10+10r+10r^2-39r=0$$

$$\Rightarrow 10r^2-29r+10=0$$

$$\Rightarrow 10r^2-25r-4r+10=0$$

$$\Rightarrow 5r(2r-5)-2(2r-5)=0$$

$$\Rightarrow (5r-2)(2r-5)=0$$

$$\Rightarrow r=\frac{2}{5},\frac{5}{2}$$

Thus, the three terms of the G.P are $\frac{5}{2}, 1, \frac{2}{5}$

Question 13:

How many terms of the G.P $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?

Solution:

The given G.P is $3, 3^2, 3^3, \ldots$
Let n terms of this G.P be required to obtain a sum as 120.

Here, $a = 3$ and $r = 3$

Therefore,

$$\Rightarrow S_n = \frac{3(3^n - 1)}{3 - 1}$$
$$\Rightarrow \frac{3(3^n - 1)}{2} = 120$$
$$\Rightarrow \frac{120 \times 2}{3} = 3^n - 1$$
$$\Rightarrow 3^n - 1 = 80$$
$$\Rightarrow 3^n = 81$$
$$\Rightarrow 3^n = 3^4$$
$$\Rightarrow n = 4$$

Thus, four terms of the given G.P are required to obtain the sum of 120.

Question 14:

The sum of first three terms of a G.P is 16 and sum of the next three terms is 128. Determine the first term, common ratio and sum to n terms of the G.P.

Solution:

Let the G.P be $a, ar, ar^2, ar^3, \ldots$
According to the given question

$$a + ar + ar^2 = 16 \text{ and } ar^3 + ar^4 + ar^5 = 128$$

Therefore,

$$a(1 + r + r^2) = 16 \qquad \ldots(1)$$
$$ar^3(1 + r + r^2) = 128 \qquad \ldots(2)$$

Dividing (2) by (1), we obtain

$$\Rightarrow \frac{ar^3\left(1+r+r^2\right)}{a\left(1+r+r^2\right)} = \frac{128}{16}$$

$$\Rightarrow r^3 = 8$$

$$\Rightarrow r = 2$$

Substituting $r = 2$ in (1), we obtain

$$\Rightarrow a(1+2+4) = 16$$

$$\Rightarrow a(7) = 16$$

$$\Rightarrow a = \frac{16}{7}$$

Hence,

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$

$$= \frac{\frac{16}{7}\left(2^n - 1\right)}{2-1}$$

$$= \frac{16}{7}\left(2^n - 1\right)$$

Thus, the first term, $a = \frac{16}{7}$, common ratio, $r = 2$ and sum to n terms, $S_n = \frac{16}{7}\left(2^n - 1\right)$.

Question 15:

Given a G.P with $a = 729$ and 7^{th} term 64, determine S_7.

Solution:

It is given that $a = 729$ and $a_7 = 64$
Let r be the common ratio of the G.P.

It is known that $a_n = ar^{n-1}$
Therefore,

$$\Rightarrow a_7 = ar^{7-1} = ar^6$$

$$\Rightarrow 64 = 729r^6$$

$$\Rightarrow r^6 = \left(\frac{2}{3}\right)^6$$

$$\Rightarrow r = \frac{2}{3}$$

Also,

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_7 = \frac{729\left[1-\left(\frac{2}{3}\right)^7\right]}{1-\frac{2}{3}}$$

$$= 3 \times 729\left[1-\left(\frac{2}{3}\right)^7\right]$$

$$= (3)^7\left[\frac{(3)^7-(2)^7}{(3)^7}\right]$$

$$= (3)^7-(2)^7$$

$$= 2187-128$$

$$= 2059$$

Question 16:

Find a G.P for which the sum of first two terms is -4 and the fifth term is four times the third term.

Solution:

Let a be the first term and r be the common ratio of the G.P.

According to the given conditions,

$$a_5 = 4(a_3)$$

$$ar^4 = 4ar^2$$

$$r^2 = 4$$

$$r = \pm 2$$

Also,

$$S_2 = \frac{a(1-r^2)}{1-r} = -4$$

<u>Case I</u>: for $r = 2$

$$\Rightarrow \frac{a\left[1-(2)^2\right]}{1-2} = -4$$

$$\Rightarrow \frac{a[1-4]}{-1} = -4$$

$$\Rightarrow 3a-4$$

$$\Rightarrow a = -\frac{4}{3}$$

<u>Case II</u>: for $r=-2$

$$\Rightarrow \frac{-a\left[1-(-2)^2\right]}{1-(-2)}-4$$

$$\Rightarrow \frac{a(1-4)}{1+2}=-4$$

$$\Rightarrow \frac{-3a}{3}=-4$$

$$\Rightarrow a=4$$

Thus, the required G.P is $\frac{-4}{3},\frac{-8}{3},\frac{-16}{3},\ldots$ or $4,-8,16,-32,64\ldots$

Question 17:

If the $4^{th}, 10^{th}$ and 16^{th} terms of a G.P are x, y and z respectively. Prove that x, y, z are in G.P.

Solution:

Let a be the first term and r be the common ratio of the G.P.
According to the given statement,

$$a_4 = ar^3 = x \qquad \ldots(1)$$

$$a_{10} = ar^9 = y \qquad \ldots(2)$$

$$a_{16} = ar^{15} = z \qquad \ldots(3)$$

Dividing (2) by (1), we obtain

$$\Rightarrow \frac{y}{x}=\frac{ar^9}{ar^3}$$

$$\Rightarrow \frac{y}{x}=r^6$$

Dividing (3) by (2), we obtain

$$\Rightarrow \frac{z}{y}=\frac{ar^{15}}{ar^9}$$

$$\Rightarrow \frac{z}{y}=r^6$$

Hence,

$$\Rightarrow \frac{y}{x}=\frac{z}{y}$$

$$\Rightarrow y^2 = xz$$

$$\Rightarrow y=\sqrt{xz}$$

Thus, x, y, z are in G.P, proved.

Question 18:

Find the sum to n terms of the given sequence is $8, 88, 888, \ldots$

Solution:

The given sequence is $8, 88, 888, \ldots n \text{ terms}$.

$$\begin{aligned}
S_n &= 8 + 88 + 888 + \ldots n \text{ terms} \\
&= \frac{8}{9}[9 + 99 + 999 + \ldots n \text{ terms}] \\
&= \frac{8}{9}\left[(10-1) + (100-1) + (1000-1) + \ldots n \text{ terms}\right] \\
&= \frac{8}{9}\left[\left(10 + 10^2 + 10^3 + \ldots n \text{ terms}\right) - \left(1 + 1 + 1 \ldots n \text{ terms}\right)\right] \\
&= \frac{8}{9}\left[\frac{10\left(10^n - 1\right)}{10 - 1} - n\right] \\
&= \frac{8}{9}\left[\frac{10\left(10^n - 1\right)}{9} - n\right] \\
&= \frac{80}{81}\left(10^n - 1\right) - \frac{8}{9}n
\end{aligned}$$

Question 19:

Find the sum of the products of the corresponding terms of the sequences $2, 4, 8, 16, 32$ and $128, 32, 8, 2, \frac{1}{2}$.

Solution:

The given sequences are $2, 4, 8, 16, 32$ and $128, 32, 8, 2, \frac{1}{2}$.
Accordingly, the required sum,

$$\begin{aligned}
S &= 2 \times 128 + 4 \times 32 + 8 \times 8 + 16 \times 2 + 32 \times \frac{1}{2} \\
&= 64\left[4 + 2 + 1 + \frac{1}{2} + \frac{1}{2^2}\right] \qquad \ldots(1)
\end{aligned}$$

It can be seen $4, 2, 1, \frac{1}{2}, \frac{1}{2^2}$ is a G.P.

Here, $a = 4$ and $r = \frac{1}{2}$

It is known that, $S_n = \frac{a\left(1 - r^n\right)}{1 - r}$

Therefore,

$$S_5 = \frac{4\left[1 - \left(\frac{1}{2}\right)^5\right]}{1 - \frac{1}{2}}$$

$$= \frac{4\left[1 - \frac{1}{32}\right]}{\frac{1}{2}}$$

$$= \frac{31}{4}$$

Hence,

$$\left[4 + 2 + 1 + \frac{1}{2} + \frac{1}{2^2}\right] = \frac{31}{4}$$

Putting this value in (1), we obtain

$$S = 64 \times \frac{31}{4}$$

$$= 16 \times 31$$

$$= 496$$

Thus, the required sum is 496.

Question 20:

Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots ar^{n-1}$ and A, AR, AR^2, AR^{n-1} form a G.P and find its common ratio.

Solution:

The given sequences are $a, ar, ar^2, \ldots ar^{n-1}$ and A, AR, AR^2, AR^{n-1}.

We need to prove that the sequence: $aA, arAR, ar^2AR^2, \ldots ar^{n-1}AR^{n-1}$ form a G.P.

Let us find the ratio of the sequence

$$\Rightarrow \frac{a_2}{a_1} = \frac{arAR}{aA} = rR$$

$$\Rightarrow \frac{a_3}{a_2} = \frac{ar^2AR^2}{arAR} = rR$$

Thus, the above sequence forms a G.P with common ratio rR.

Question 21:

Find four numbers forming a G.P in which the third term is greater than the first term by 9, the second term is greater than the 4^{th} by 18.

Solution:

Let a be the first term and r be the common ratio of the G.P.

Hence,

$$a_1 = a, a_2 = ar, a_3 = ar^2, a_4 = ar^3$$

According to the given condition,

$$\Rightarrow a_3 = a_1 + 9$$
$$\Rightarrow ar^2 = a + 9$$
$$\Rightarrow ar^2 - a = 9$$
$$\Rightarrow a\left(r^2 - 1\right) = 9 \qquad \ldots(1)$$

$$\Rightarrow a_2 = a_4 + 18$$
$$\Rightarrow ar = ar^3 + 18$$
$$\Rightarrow ar^3 - ar = -18$$
$$\Rightarrow ar\left(r^2 - 1\right) = -18 \qquad \ldots(2)$$

Dividing (2) by (1), we obtain

$$\frac{ar\left(r^2-1\right)}{a\left(r^2-1\right)} = \frac{-18}{9}$$
$$\Rightarrow r = -2$$

Substituting $r = -2$ in (1), we obtain

$$\Rightarrow a\left[(-2)^2 - 1\right] = 9$$
$$\Rightarrow a[4-1] = 9$$
$$\Rightarrow 3a = 9$$
$$\Rightarrow a = \frac{9}{3}$$
$$\Rightarrow a = 3$$

Thus, the first four numbers of the G.P. are $3, 3(-2), 3(-2)^2$ and $3(-2)^3$

i.e., $3, -6, 12, -24$.

Question 22:

If p^{th}, q^{th} and r^{th} terms of a G.P are a, b and c respectively. Prove that $a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1$.

Solution:

Let A be the first term and R be the common ratio of the G.P.
According to the given condition,

$$AR^{p-1} = a$$
$$AR^{q-1} = b$$
$$AR^{r-1} = c$$

Therefore,

$$\begin{aligned} a^{q-r} \cdot b^{r-p} \cdot c^{p-q} &= A^{q-r} \times R^{(p-1)(q-r)} \times A^{r-p} \times R^{(q-1)(r-p)} \times A^{p-q} \times R^{(r-1)(p-q)} \\ &= A^{q-r+r-p+p-q} \times R^{(pr-pr-q+r)+(rq-r+p-pq)+(pr-p-qr+q)} \\ &= A^0 \times R^0 \\ &= 1 \end{aligned}$$

Hence proved.

Question 23:

If the first and n^{th} term of the G.P is a and b respectively, if P is the product of n terms, prove that $P^2 = (ab)^n$.

Solution:

It is given that the first term of the G.P is a and the last term is b.
Let the common ratio be r

Hence, the G.P is $a, ar, ar^2, ar^3, \ldots ar^{n-1}$ and

$$b = ar^{n-1} \qquad \ldots(1)$$

Now, the product of n terms

$$\begin{aligned} P &= (a) \times (ar) \times (ar^2) \times \ldots \times (ar^{n-1}) \\ &= (a \times a \times a \ldots n \text{ times})(r \times r^2 \times \ldots \times r^{n-1}) \\ &= a^n r^{1+2+\ldots+(n-1)} \qquad \ldots(2) \end{aligned}$$

Here, $1, 2, \ldots (n-1)$ is an A.P.
Therefore,

$$1+2+\ldots+(n-1)=\frac{n-1}{2}[2+(n-1-1)\times 1]$$
$$=\frac{n-1}{2}[2+n-2]$$
$$=\frac{n(n-1)}{2}$$

Substituting this value in (2), we obtain

$$P=a^n r^{\frac{n(n-1)}{2}}$$
$$P^2=a^{2n}r^{n(n-1)}$$
$$=\left[a^2 r^{(n-1)}\right]^n$$
$$=\left[a\times ar^{n-1}\right]^n$$
$$P^2=(ab)^n \qquad [\text{using } (1)]$$

Thus, $P^2=(ab)^n$ proved.

Question 24:

Show that the ratio of the sum of first n terms of a G.P to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

Solution:

Let a be the first term and r be the common ratio of the G.P.

Sum of first n terms, $S=\frac{a(1-r^n)}{(1-r)}$

Since, there are n terms from $(n+1)^{th}$ to $(2n)^{th}$ term

Hence, sum of the n terms from $(n+1)^{th}$ to $(2n)^{th}$ terms $S_n=\frac{a_{n+1}(1-r^n)}{1-r}$

It is known that $a_n=ar^{n-1}$

Therefore,

$$a_{n+1}=ar^{n+1-1}$$
$$=ar^n$$

Now,

$$S_n=\frac{ar^n(1-r^n)}{1-r}$$

Thus, the required ratio

$$\frac{S}{S_n}=\frac{\left(\frac{a(1-r^n)}{(1-r)}\right)}{\left(\frac{ar^n(1-r^n)}{1-r}\right)}$$

$$=\frac{a(1-r^n)}{(1-r)}\times\frac{(1-r)}{ar^n(1-r^n)}$$

$$=\frac{1}{r^n}$$

Hence proved.

Question 25:

If a, b, c and d are in G.P. Show that: $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$

Solution:

If a, b, c and d are in G.P.

Therefore,

$$bc = ad \qquad \ldots(1)$$

$$b^2 = ac \qquad \ldots(2)$$

$$c^2 = bd \qquad \ldots(3)$$

We need to prove that, $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$

Since,

$$\begin{aligned} RHS &= (ab+bc+cd)^2 \\ &= (ab+ad+cd)^2 && [\text{Using } (1)] \\ &= [ab+d(a+c)]^2 \\ &= a^2b^2+2abd(a+c)+d^2(a+c)^2 \\ &= a^2b^2+2a^2bd+2acbd+d^2(a^2+2ac+c^2) \\ &= a^2b^2+2a^2c^2+2b^2c^2+d^2a^2+2d^2b^2+d^2c^2 && [\text{Using } (1) \text{ and } (2)] \\ &= a^2b^2+a^2c^2+a^2c^2+b^2c^2+b^2c^2+d^2a^2+d^2b^2+d^2b^2+d^2c^2 \\ &= a^2b^2+a^2c^2+a^2d^2+b^2\times b^2+b^2c^2+b^2d^2+c^2b^2+c^2\times c^2+c^2d^2 \end{aligned}$$

Using (2) and (3) and rearranging the terms

$$\begin{aligned} RHS &= a^2\left(b^2+c^2+d^2\right)+b^2\left(b^2+c^2+d^2\right)+c^2\left(b^2+c^2+d^2\right) \\ &= \left(a^2+b^2+c^2\right)\left(b^2+c^2+d^2\right) \\ &= LHS \end{aligned}$$

Thus, $\left(a^2+b^2+c^2\right)\left(b^2+c^2+d^2\right)=\left(ab+bc+cd\right)^2$ proved.

Question 26:

Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Solution:

Let G_1 and G_2 be two numbers between 3 and 81 such that the series $3, G_1, G_2, 81$ forms a G.P.

Let a be the first term and r be the common ratio of the G.P.

Therefore, $a=3$ and $a_4=81$

$$\begin{aligned} &\Rightarrow 3r^3=81 \\ &\Rightarrow r^3=27 \\ &\Rightarrow r=3 \qquad \text{(considering real roots only)} \end{aligned}$$

Hence,

$$\begin{aligned} G_1 &= ar = 3\times 3 = 9 \\ G_2 &= ar^2 = 3\times(3)^2 = 27 \end{aligned}$$

Thus, the required two numbers are 9 and 27.

Question 27:

Find the value of n so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Solution:

It is known that G.M. of a and b is $\sqrt{ab}$

By the given condition

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{ab}$$

By squaring both sides, we obtain

$$\Rightarrow \frac{\left(a^{n+1}+b^{n+1}\right)^2}{\left(a^n+b^n\right)^2}=ab$$

$$\Rightarrow a^{2n+2}+2a^{n+1}b^{n+1}+b^{2n+2}=(ab)\left(a^{2n}+2a^nb^n+b^{2n}\right)$$

$$\Rightarrow a^{2n+2}+2a^{n+1}b^{n+1}+b^{2n+2}=a^{2n+1}b+2a^{n+1}b^{n+1}+ab^{2n+1}$$

$$\Rightarrow a^{2n+2}+b^{2n+2}=a^{2n+1}b+ab^{2n+1}$$

$$\Rightarrow a^{2n+2}-a^{2n+1}b=ab^{2n+1}-b^{2n+2}$$

$$\Rightarrow a^{2n+1}(a-b)=b^{2n+1}(a-b)$$

$$\Rightarrow \left(\frac{a}{b}\right)^{2n+1}=1=\left(\frac{a}{b}\right)^0$$

$$\Rightarrow 2n+1=0$$

$$\Rightarrow n=-\frac{1}{2}$$

Thus, the value of $n=-\frac{1}{2}$

Question 28:

The sum of two numbers is 6 times their G.M, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

Solution:

Let the two numbers be a and b.

Then its $G.M.=\sqrt{ab}$

According to the given condition,

$$\Rightarrow a+b=6\sqrt{ab} \qquad \ldots(1)$$

$$\Rightarrow (a+b)^2=36(ab)$$

Also,

$$(a-b)^2=(a+b)^2-4ab$$
$$=36ab-4ab$$
$$=32ab$$
$$a-b=\sqrt{32}\sqrt{ab}$$
$$=4\sqrt{2}\sqrt{ab} \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2a=\left(6+4\sqrt{2}\right)\sqrt{ab}$$

$$a=\left(3+2\sqrt{2}\right)\sqrt{ab}$$

Substituting the value of a in (1), we obtain

$$\begin{aligned} b &= 6\sqrt{ab} - \left(3+2\sqrt{2}\right)\sqrt{ab} \\ &= \left(3-2\sqrt{2}\right)\sqrt{ab} \end{aligned}$$

Hence the ratio of the numbers is

$$\begin{aligned} \frac{a}{b} &= \frac{\left(3+2\sqrt{2}\right)\sqrt{ab}}{\left(3-2\sqrt{2}\right)\sqrt{ab}} \\ &= \frac{\left(3+2\sqrt{2}\right)}{\left(3-2\sqrt{2}\right)} \end{aligned}$$

Thus, the required ratio is $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

Question 29:

If A and G be A.M and G.M, respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

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Solution:

It is given that A and G are A.M and G.M, respectively between two positive numbers. Let these two positive numbers be a and b.

Therefore,

$$A.M = A = \frac{a+b}{2}$$
$$\Rightarrow a+b = 2A \qquad \ldots(1)$$

And,

$$G.M = G = \sqrt{ab}$$
$$\Rightarrow ab = G^2 \qquad \ldots(2)$$

Since,

$$\begin{aligned}(a-b)^2 &= (a+b)^2 - 4ab \\ &= 4A^2 - 4G^2 \qquad [\text{using } (1) \text{ and } (2)] \\ &= 4(A^2 - G^2) \\ &= 4(A+G)(A-G)\end{aligned}$$
$$(a-b) = 2\sqrt{(A+G)(A-G)} \qquad \ldots(3)$$

By adding (1) and (3), we obtain

$$\Rightarrow 2a = 2A + 2\sqrt{(A+G)(A-G)}$$
$$\Rightarrow a = A + \sqrt{(A+G)(A-G)}$$

Substituting the value of a in (1), we obtain

$$b = 2A - \left(A + \sqrt{(A+G)(A-G)}\right)$$
$$= A - \sqrt{(A+G)(A-G)}$$

Thus, the two numbers are $A \pm \sqrt{(A+G)(A-G)}$.

Question 30:

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2^{nd} hour, 4^{th} hour and n^{th} hour?

Solution:

It is given that the number of bacteria doubles every hour.

Hence, the number of bacteria after every hour will form a G.P with $a = 30$ and $r = 2$

Therefore, the number of bacteria at the end of 2^{nd} hour will be

$$\begin{aligned} a_3 &= ar^2 \\ &= 30\times(2)^2 \\ &= 120 \end{aligned}$$

The number of bacteria at the end of 4^{th} hour will be

$$\begin{aligned} a_5 &= ar^4 \\ &= 30\times(2)^4 \\ &= 480 \end{aligned}$$

The number of bacteria at the end of n^{th} hour will be

$$\begin{aligned} a_{n+1} &= ar^n \\ &= 30\times(2)^n \end{aligned}$$

Question 31:

What will ₹ 500 amount to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

Solution:

The amount deposited in the bank is ₹ 500.

At the end of first year, amount in ₹ $=500\left(1+\frac{1}{10}\right)=500(1.1)$

At the end of second year, amount in ₹ $=500(1.1)(1.1)$

At the end of third year, amount in ₹ $=500(1.1)(1.1)(1.1)$

and so on…

Therefore, at the end of 10 years, amount in ₹

$$\begin{aligned} &= 500(1.1)(1.1)(1.1)\ldots 10 \text{ times} \\ &= 500(1.1)^{10} \end{aligned}$$

Question 32:

If A.M and G.M are roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Solution:

Let the roots of the quadratic equations be a and b.

According to the condition,

$$A.M = \frac{a+b}{2} = 8$$
$$\Rightarrow a+b = 16 \qquad \ldots(1)$$

$$G.M = \sqrt{ab} = 5$$
$$\Rightarrow ab = 25 \qquad \ldots(2)$$

The quadratic equation is given by,

$$x^2 - x(\text{Sum of roots}) + (\text{Product of roots}) = 0$$
$$x^2 - x(a+b) + (ab) = 0$$
$$x^2 - 16x + 25 = 0 \qquad [\text{Using } (1) \text{ and } (2)]$$

Thus, the required quadratic equation is $x^2 - 16x + 25 = 0$.

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EXERCISE 9.4

Question 1:

Find the sum to n terms of the series $1\times2+2\times3+3\times4+4\times5+\ldots$

Solution:

The given series is $1\times2+2\times3+3\times4+4\times5+\ldots n$ terms

Hence,

$$a_n = n(n+1)$$

Therefore,

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k(k+1) \\
&= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right) \\
&= \frac{n(n+1)}{2}\left(\frac{2n+4}{3}\right) \\
&= \frac{n}{3}(n+1)(n+2)
\end{aligned}$$

Question 2:

Find the sum to n terms of the series $1\times2\times3+2\times3\times4+3\times4\times5+4\times5+\ldots$

Solution:

The given series is $1\times2\times3+2\times3\times4+3\times4\times5+4\times5+\ldots n$ terms

Hence,

$$\begin{aligned}
a_n &= n(n+1)(n+2) \\
&= (n^2+n)(n+2) \\
&= n^3+3n^2+2n
\end{aligned}$$

Therefore,

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} a_k \\
&= \sum_{k=1}^{n}\left(k^3+3k^2+2k\right) \\
&= \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k \\
&= \left[\frac{n(n+1)}{2}\right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} \\
&= \left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1) \\
S_n &= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 2n+1+2\right] \\
&= \frac{n(n+1)}{2}\left[\frac{n^2+n+4n+6}{2}\right] \\
&= \frac{n(n+1)}{4}\left[n^2+5n+6\right] \\
&= \frac{n(n+1)}{4}\left[n^2+2n+3n+6\right] \\
&= \frac{n(n+1)\left[n(n+2)+3(n+2)\right]}{4} \\
&= \frac{n}{4}(n+1)(n+2)(n+3)
\end{aligned}
$$

Question 3:

Find the sum to n terms of the series $3\times1^2+5\times2^2+7\times3^2+\ldots$

Solution:

The given series is $3\times1^2+5\times2^2+7\times3^2+\ldots n$ terms

Hence,

$$
\begin{aligned}
a_n &= (2n+1)n^2 \\
&= 2n^3+n^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} a_k \\
&= \sum_{k=1}^{n} \left(2k^3 + k^2\right) \\
&= 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 \\
&= 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(n+1)}{2}\left[n(n+1) + \frac{2n+1}{3}\right] \\
&= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+2n+1}{3}\right] \\
&= \frac{n(n+1)}{2}\left[\frac{3n^2+5n+1}{3}\right] \\
&= \frac{n}{6}(n+1)\left(3n^2+5n+1\right)
\end{aligned}
$$

Question 4:

Find the sum to n terms of the series $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots$

Solution:

The given series is $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots n$ terms

Hence,

$$
\begin{aligned}
a_n &= \frac{1}{n(n+1)} \\
&= \frac{1}{n} - \frac{1}{n+1}
\end{aligned}
$$

Therefore,

$$a_1 = \frac{1}{1} - \frac{1}{2}$$
$$a_2 = \frac{1}{2} - \frac{1}{3}$$
$$a_3 = \frac{1}{3} - \frac{1}{4}$$
$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding the above terms column wise, we obtain

$$a_1 + a_2 + \ldots_n = \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}\right] - \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \frac{1}{n+1}\right]$$

Thus,

$$\begin{aligned} S_n &= 1 - \frac{1}{n+1} \\ &= \frac{n+1-1}{n+1} \\ &= \frac{n}{n+1} \end{aligned}$$

Question 5:

Find the sum to n terms of the series $5^2 + 6^2 + 7^2 + \cdots + 20^2$

Solution:

The given series is $5^2 + 6^2 + 7^2 + \cdots + 20^2$

Hence,

$$\begin{aligned} a_n &= (n+4)^2 \\ &= n^2 + 8n + 16 \end{aligned}$$

Therefore,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} a_k \\ &= \sum_{k=1}^{n} (k^2 + 8k + 16) \end{aligned}$$

$$\begin{aligned} S_n &= \sum_{k=1}^{n} k^2 + 8\sum_{k=1}^{n} k + \sum_{k=1}^{n} 16 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2} + 16n \end{aligned}$$

Since, 16th term is $(16+4)^2 = (20)^2$

Then,

$$S_{16} = \frac{16(16+1)(2(16)+1)}{6} + \frac{8(16)(16+1)}{2} + 16(16)$$
$$= \frac{(16)(17)(33)}{6} + \frac{(128)(17)}{2} + 256$$
$$= 1496 + 1088 + 256$$
$$= 2840$$

Thus, $5^2 + 6^2 + 7^2 + \cdots + 20^2 = 2840$

Question 6:

Find the sum to n terms of the series $3 \times 8 + 6 \times 11 + 9 \times 14 + \ldots$

Solution:

The given series is $3 \times 8 + 6 \times 11 + 9 \times 14 + \ldots n$ terms

Hence,

$$a_n = \left(n^{th} \text{ term of } 3, 6, 9 \ldots\right) \times \left(n^{th} \text{ term of } 8, 11, 14 \ldots\right)$$
$$= (3n)(3n+5)$$
$$= 9n^2 + 15n$$

Therefore,

$$S_n = \sum_{k=1}^{n} a_k$$
$$= \sum_{k=1}^{n} \left(9k^2 + 15k\right)$$
$$= 9\sum_{k=1}^{n} k^2 + 15\sum_{k=1}^{n} k$$
$$= 9 \times \frac{n(n+1)(2n+1)}{6} + 15 \times \frac{n(n+1)}{2}$$
$$= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2}$$
$$= \frac{3n(n+1)}{2}(2n+1+5)$$
$$= \frac{3n(n+1)}{2}(2n+6)$$
$$= 3n(n+1)(n+3)$$

Question 7:

Find the sum to n terms of the series $1^2 + \left(1^2 + 2^2\right) + \left(1^2 + 2^2 + 3^2\right) + \ldots$

Solution:

The given series is $1^2+\left(1^2+2^2\right)+\left(1^2+2^2+3^2\right)+\ldots n$ terms
Hence,

$$
\begin{aligned}
a_n &= \left(1^2+2^2+3^2+\ldots+n^2\right)\\
&= \frac{n}{6}(n+1)(2n+1)\\
&= \frac{n}{6}\left(2n^2+3n+1\right)\\
&= \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} a_k\\
&= \sum_{k=1}^{n}\left(\frac{1}{3}k^3+\frac{1}{2}k^2+\frac{1}{6}k\right)\\
&= \frac{1}{3}\sum_{k=1}^{n}k^3+\frac{1}{2}\sum_{k=1}^{n}k^2+\frac{1}{6}\sum_{k=1}^{n}k\\
&= \frac{1}{3}\times\frac{n^2(n+1)^2}{(2)^2}+\frac{1}{2}\times\frac{n(n+1)(2n+1)}{6}+\frac{1}{6}\times\frac{n(n+1)}{2}\\
&= \frac{n(n+1)}{6}\left[\frac{n(n+1)}{2}+\frac{(2n+1)}{2}+\frac{1}{2}\right]\\
&= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+1+1}{2}\right]\\
&= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+2}{2}\right]\\
&= \frac{n(n+1)}{6}\left[\frac{n(n+1)+2(n+1)}{2}\right]\\
&= \frac{n(n+1)}{6}\left[\frac{(n+1)(n+2)}{2}\right]\\
&= \frac{n(n+1)^2(n+2)}{12}
\end{aligned}
$$

Question 8:

Find the sum to n terms of the series whose n^{th} term is given by $n(n+1)(n+4)$

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Solution:

The given n^{th} term is $a_n = n(n+1)(n+4)$

Hence,

$$\begin{aligned} a_n &= n(n+1)(n+4) \\ &= n(n^2+5n+4) \\ &= n^3+5n^2+4n \end{aligned}$$

Therefore,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} a_k \\ &= \sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k \\ &= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} \\ &= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4\right] \\ &= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+20n+10+24}{6}\right] \\ &= \frac{n(n+1)}{2}\left[\frac{3n^2+23n+34}{6}\right] \\ &= \frac{n(n+1)(3n^2+23n+34)}{12} \end{aligned}$$

Question 9:

Find the sum to n terms of the series whose n^{th} term is given by n^2+2^n.

Solution:

The given n^{th} term is $a_n = n^2 + 2^n$

Hence,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} k^2 + 2^k \\ &= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k \qquad \ldots(1) \end{aligned}$$

Consider $\sum_{k=1}^{n} 2^k = 2^1 + 2^2 + 2^3 + \ldots$

The above series $2^1+2^2+2^3+\dots$ is a G.P with both the first term and common ratio equal to 2.

$$\sum_{k=1}^{n} 2^k = \frac{(2)\left[(2)^n - 1\right]}{2-1}$$
$$= 2\left(2^n - 1\right) \qquad \dots(2)$$

From (1) and (2), we obtain

$$S_n = \sum_{k=1}^{n} k^2 + 2\left(2^n - 1\right)$$
$$= \frac{n}{6}(n+1)(2n+1) + 2\left(2^n - 1\right)$$

Question 10:

Find the sum to n terms of the series whose n^{th} term is given by $(2n-1)^2$.

Solution:

The given n^{th} term is $a_n = (2n-1)^2$

Hence,

$$a_n = (2n-1)^2$$
$$= 4n^2 - 4n + 1$$

Therefore,

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} a_k \\
&= \sum_{k=1}^{n} \left(4k^2 - 4k + 1\right) \\
&= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \\
&= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n \\
&= n\left[\frac{2\left(2n^2+3n+1\right)}{3} - 2(n+1) + 1\right] \\
&= n\left[\frac{4n^2+6n+2-6n-6+3}{3}\right] \\
&= n\left[\frac{4n^2-1}{3}\right] \\
&= \frac{n}{3}(2n+1)(2n-1)
\end{aligned}
$$

MISCELLANEOUS EXERCISE

Question 1:

Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P is equal to twice the m^{th} term.

Solution:

Let a and d be the first term and common difference of the A.P respectively.

It is known that the k^{th} term of an A.P. is given by $a_k = a+(k-1)d$

Therefore,

$$a_{m+n} = a+(m+n-1)d$$
$$a_{m-n} = a+(m-n-1)d$$
$$a_m = a+(m-1)d$$

Hence,

$$\begin{aligned} a_{m+n}+a_{m-n} &= a+(m+n-1)d+a+(m-n-1)d \\ &= 2a+(m+n-1+m-n-1)d \\ &= 2a+(2m-2)d \\ &= 2a+2(m-1)d \\ &= 2\left[a+(m-1)d\right] \\ &= 2a_m \end{aligned}$$

Thus, the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P is equal to twice the m^{th} term.

Question 2:

Let the sum of three numbers in A.P is 24 and their product is 440. Find the numbers.

Solution:

Let the three numbers in A.P be $(a-d),a$ and $(a+d)$.

According to the given information,

$$(a-d)+(a)+(a-d)=24$$
$$\Rightarrow 3a = 24$$
$$\Rightarrow a = 8 \qquad \ldots(1)$$

And

$$(a-d)(a)(a-d)=440$$
$$\Rightarrow (8-d)(8)(8+d)=440 \qquad \ldots[\text{from } (1)]$$
$$\Rightarrow (8-d)(8+d)=55$$
$$\Rightarrow 64-d^2=55$$
$$\Rightarrow d^2=9$$
$$\Rightarrow d=\pm 3$$

Therefore,
when $d=3$, the numbers are $5,8$ and 11
when $d=-3$, the numbers are 11, 8 and 5.

Thus, the three numbers are 5,8 and 11.

Question 3:

Let the sum of $n, 2n, 3n$ terms of an A.P be S_1, S_2, S_3 respectively. Show that $S_3 = 3(S_2 - S_1)$.

Solution:

Let a and d be the first term and common difference of the A.P respectively.
Therefore,

$$S_1 = \frac{n}{2}[2a+(n-1)d] \qquad \ldots(1)$$
$$S_2 = \frac{2n}{2}[2a+(2n-1)d]$$
$$= n[2a+(2n-1)d] \qquad \ldots(2)$$
$$S_3 = \frac{3n}{2}[2a+(3n-1)d] \qquad \ldots(3)$$

By subtracting (1) and (2), we obtain

$$
\begin{aligned}
S_2 - S_1 &= n\left[2a+(2n-1)d\right]-\frac{n}{2}\left[2a+(n-1)d\right] \\
&= n\left[\frac{4a+4nd-2d-2a-nd+d}{2}\right] \\
&= n\left[\frac{2a+3nd-d}{2}\right] \\
&= \frac{n}{2}\left[2a+(3n-1)d\right] \\
3(S_2 - S_1) &= \frac{3n}{2}\left[2a+(3n-1)d\right] \\
&= S_3 \qquad \left[\text{from } (3)\right]
\end{aligned}
$$

Hence, $S_3 = 3(S_2 - S_1)$ proved.

Question 4:

Find the sum of all numbers between 200 and 400 which are divisible by 7.

Solution:

The numbers lying between 200 and 400 which are divisible by 7 are $203, 210, 217, \ldots 399$

Here, $a = 203, d = 7$ and $a_n = 399$

Therefore,

$$
\begin{aligned}
a_n &= a+(n-1)d \\
399 &= 203+(n-1)7 \\
(n-1)7 &= 196 \\
n-1 &= 28 \\
\Rightarrow n &= 29
\end{aligned}
$$

Hence,

$$
\begin{aligned}
S_{29} &= \frac{29}{2}(203+399) \\
&= \frac{29}{2}(602) \\
&= 29 \times 301 \\
&= 8729
\end{aligned}
$$

Thus, the required sum is 8729.

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Question 5:
Find the sum of all integers from 1 and 100 that are divisible by 2 or 5.

Solution:
The integers from 1 and 100 that are divisible by 2 are $2,4,6,\ldots100$
This forms an A.P with both the first term and common difference equal to 2.
Therefore,

$$\begin{aligned}
a_n &= a+(n-1)d \\
100 &= 2+(n-1)2 \\
100 &= 2+2n-2 \\
\Rightarrow 2n &= 100 \\
\Rightarrow n &= 50
\end{aligned}$$

Therefore,

$$\begin{aligned}
2+4+6+\ldots+100 &= \frac{50}{2}\left[2(2)+(50-1)(2)\right] \\
&= \frac{50}{2}[4+98] \\
&= 25\times102 \\
&= 2550
\end{aligned}$$

The integers from 1 to 100 that are divisible by 5 are $5,10,15,\ldots100$
This forms an A.P with both the first term and common difference equal to 5.
Therefore,

$$\begin{aligned}
a_n &= a+(n-1)d \\
100 &= 5+(n-1)5 \\
100 &= 5+5n-5 \\
\Rightarrow 5n &= 100 \\
\Rightarrow n &= 20
\end{aligned}$$

Therefore,

$$\begin{aligned}
5+10+15+\ldots+100 &= \frac{20}{2}\left[2(5)+(20-1)(5)\right] \\
&= 10\left[10+(19)5\right] \\
&= 10[10+95] \\
&= 10\times105 \\
&= 1050
\end{aligned}$$

The integers, which are divisible by both 2 and 5 are $10,20,30,\ldots100$
This forms an A.P with both the first term and common difference equal to 10.
Therefore,

$$100 = 10 + (n-1)10$$
$$\Rightarrow 10n = 100$$
$$\Rightarrow n = 10$$

Hence,

$$10 + 20 + \ldots + 100 = \frac{10}{2}\left[2(10) + (10-1)(10)\right]$$
$$= 5[20 + 90]$$
$$= 5 \times 110$$
$$= 550$$

Therefore, required sum $= 2550 + 1050 - 550 = 3050$

Thus, the sum of all integers from 1 and 100 that are divisible by 2 or 5 is 3050.

Question 6:

Find the sum of all two-digit numbers which when divided by 4, yields 1 as remainder.

Solution:

The two-digit numbers which when divided by 4, yields 1 as remainder are $13, 17, 21, \ldots 97$.

This forms an A.P with first term 13 and common difference 4.

Let n be the number of terms of the A.P.

It is known that the n^{th} term of an A.P. is given by $a^n = a + (n-1)d$

Therefore,

$$97 = 13 + (n-1)(4)$$
$$4(n-1) = 97 - 13$$
$$n - 1 = \frac{84}{4}$$
$$\Rightarrow n = 22$$

Sum of n terms of an A.P, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Therefore,

$$\begin{aligned} S_{22} &= \frac{22}{2}\left[2(13)+(22-1)(4)\right] \\ &= 11[26+84] \\ &= 11\times 110 \\ &= 1210 \end{aligned}$$

Thus, the required sum is 1210.

Question 7:

If f is a fraction satisfying $f(x+y)=f(x)\cdot f(y)$ for all $x,y\in N$, such that $f(1)=3$ and $\sum_{x=1}^{n} f(x)=120$, find the value of n.

Solution:

$f(x+y)=f(x)\cdot f(y)$ for all $x,y\in N$ and $f(1)=3$

Taking $x=y=1$ in (1), we obtain

$$\begin{aligned} f(x+y) &= f(x)\cdot f(y) \\ f(1+1) &= f(1)f(1) \\ f(2) &= 3\times 3 \\ &= 9 \end{aligned}$$

Similarly,

$$\begin{aligned} f(x+y) &= f(x)\cdot f(y) \\ f(1+2) &= f(1)f(2) \\ f(3) &= 3\times 9 \\ &= 27 \end{aligned}$$

Also,

$$\begin{aligned} f(x+y) &= f(x)\cdot f(y) \\ f(1+3) &= f(1)f(3) \\ f(4) &= 3\times 27 \\ &= 81 \end{aligned}$$

Therefore, $f(1),f(2),f(3)$ i.e., 3,9,27 forms a G.P. with both the first term and common ratio equal to 3.

It is known that $S_n=\dfrac{a\left(r^n-1\right)}{r-1}$

It is given that $\sum_{k=1}^{n} f(x) = 120$

Therefore,

$$\Rightarrow 120 = \frac{3(3^n - 1)}{3-1}$$
$$\Rightarrow 120 = \frac{3}{2}(3^n - 1)$$
$$\Rightarrow 3^n - 1 = 80$$
$$\Rightarrow 3^n = 81$$
$$\Rightarrow 3^n = 3^4$$
$$\Rightarrow n = 4$$

Thus, the value of $n = 4$.

Question 8:
The sum of some terms of G.P is 315 whose first term and common ratio are 5 and 2 respectively. Find the last term and the number of terms.

Solution:
Let the sum of n terms of the G.P be 315.

It is known that $S_n = \frac{a(r^n - 1)}{r-1}$

It is given that $a = 5$ and common ratio $r = 2$

Hence,

$$315 = \frac{5(2^n - 1)}{2-1}$$
$$\Rightarrow 2^n - 1 = 63$$
$$\Rightarrow 2^n = 64 = (2)^6$$
$$\Rightarrow n = 6$$

Therefore, last term of the G.P. is 6th term

$$a_6 = ar^{6-1}$$
$$= ar^5$$
$$= 5 \times (2)^5$$
$$= 5 \times 32$$
$$= 160$$

Thus, the last term of the G.P is 160.

Question 9:

The first term of a G.P is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Solution:

Let a and r be the first term and common ratio of the G.P respectively.

Hence,

$$a = 1$$
$$a_3 = ar^2 = r^2$$
$$a_5 = ar^4 = r^4$$

Therefore,

$$r^2 + r^4 = 90$$
$$r^2 + r^4 - 90 = 0$$
$$r^2 = \frac{-1 + \sqrt{1+360}}{2}$$
$$= \frac{-1 \pm \sqrt{361}}{2}$$
$$= \frac{-1 \pm 19}{2}$$
$$= -10 \text{ or } 9$$
$$r^2 = 9 \qquad [\text{Taking real roots}]$$
$$r = \pm 3$$

Thus, the common ratio of the G.P is ± 3.

Question 10:

The sum of three numbers in G.P is 56. If we subtract $1, 7, 21$ from these numbers in that order, we obtain an A.P. Find the numbers.

Solution:

Let the three numbers in G.P. be a, ar and ar^2.

From the given condition,

$$a + ar + ar^2 = 56$$
$$\Rightarrow a\left(1 + r + r^2\right) = 56 \qquad \ldots(1)$$

Now, $(a-1), (ar-7), \left(ar^2 - 21\right)$ forms an A.P.

Therefore,

$$(ar-7)-(a-1)=(ar^2-21)-(ar-7)$$
$$\Rightarrow ar-a-6=ar^2-ar-14$$
$$\Rightarrow ar^2-2ar+a=8$$
$$\Rightarrow ar^2-ar-ar+a=8$$
$$\Rightarrow a(r^2+1-2r)=8$$
$$\Rightarrow 7a(r^2-2r+1)=56 \qquad \ldots(2)$$

From (1) and (2), we get

$$\Rightarrow 7(r^2-2r+1)=(1+r+r^2)$$
$$\Rightarrow 7r^2-14r+7-1-r-r^2=0$$
$$\Rightarrow 6r^2-15r+6=0$$
$$\Rightarrow 2r^2-5r+2=0$$
$$\Rightarrow 2r^2-4r-r+2=0$$
$$\Rightarrow 2r(r-2)-1(r-2)=0$$
$$\Rightarrow (2r-1)(r-2)=0$$
$$\Rightarrow r=\frac{1}{2},2$$

<u>Case I</u>:

Substituting $r=2$ in (1), we obtain

$$a(1+2+2^2)=56$$
$$\Rightarrow 7a=56$$
$$\Rightarrow a=8$$

Hence, the three numbers are in G.P are 8, 16 and 32.

<u>Case II</u>:

Substituting $r=\frac{1}{2}$ in (1), we obtain

$$a\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2\right)=56$$
$$\Rightarrow \frac{7}{4}a=56$$
$$\Rightarrow a=32$$

Hence, the three numbers are in G.P are 32, 16 and 8.

Thus, the three required numbers are 8, 16 and 32.

Question 11:
A G.P consists of an even number of terms. If the sum of all terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Solution:

Let the G.P be $T_1, T_2, T_3, T_4, \ldots T_{2n}$.
According to the question,

$$\Rightarrow T_1 + T_2 + T_3 + T_4 + \ldots + T_{2n} = 5\left[T_1 + T_3 + T_5 + \ldots + T_{2n-1}\right]$$
$$\Rightarrow T_1 + T_2 + T_3 + T_4 + \ldots + T_{2n} - 5\left[T_1 + T_3 + T_5 + \ldots + T_{2n-1}\right] = 0$$
$$\Rightarrow T_2 + T_4 + \ldots + T_{2n} = 4\left[T_1 + T_3 + \ldots + T_{2n-1}\right]$$

Let the G.P be $a, ar, ar^2, \ldots$

$$\frac{ar\left(r^n - 1\right)}{r-1} = \frac{4 \times a\left(r^n - 1\right)}{r-1}$$
$$\Rightarrow ar = 4a$$
$$\Rightarrow r = 4$$

Thus, the common ratio of the G.P is 4.

Question 12:
The sum of first four terms of an A.P is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

Solution:

Let the A.P be $a, (a+d), (a+2d), (a+3d), \ldots a+(n-2)d, a+(n-1)d$
It is given that, $a = 11$

Sum of the first four terms

$$\Rightarrow a + (a+d) + (a+2d) + (a+3d) = 56$$
$$\Rightarrow 4a + 6d = 56$$
$$\Rightarrow 4 \times 11 + 6d = 56$$
$$\Rightarrow 6d = 56 - 44$$
$$\Rightarrow d = \frac{12}{6}$$
$$\Rightarrow d = 2$$

Sum of last four terms

$$\Rightarrow [a+(n-4)d]+[a+(n-3)d]+[a+(n-2)d]+[a+(n-1)d]=112$$
$$\Rightarrow 4a+(4n-10)d=112$$
$$\Rightarrow 4\times 11+(4n-10)\times 2=112$$
$$\Rightarrow 44+8n-20=112$$
$$\Rightarrow 8n=112-24$$
$$\Rightarrow n=\frac{88}{8}$$
$$\Rightarrow n=11$$

Thus, the number of terms of the A.P is 11.

Question 13:

If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}(x\neq 0)$, then show that a,b,c and d are in G.P.

Solution:

It is given that $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}(x\neq 0)$,
Therefore,

$$\Rightarrow \frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}$$
$$\Rightarrow (a+bx)(b-cx)=(b+cx)(a-bx)$$
$$\Rightarrow ab-acx+b^2x-bcx^2=ab-b^2x+acx-bcx^2$$
$$\Rightarrow 2b^2x=2acx$$
$$\Rightarrow b^2=ac$$
$$\Rightarrow \frac{b}{a}=\frac{c}{b} \qquad \ldots(1)$$

Also,

$$\Rightarrow \frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$$
$$\Rightarrow (b+cx)(c-dx)=(b-cx)(c+dx)$$
$$\Rightarrow bc-bdx+c^2x-cdx^2=bc+bdx-c^2x-cdx^2$$
$$\Rightarrow 2c^2x=2bdx$$
$$\Rightarrow c^2=bd$$
$$\Rightarrow \frac{c}{b}=\frac{d}{c} \qquad \ldots(2)$$

From (1) and (2), we obtain

$$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$$

Thus, a,b,c and d are in G.P.

Question 14:

Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n=S^n$.

Solution:

Let the G.P be $a,ar,ar^2,ar^3\ldots ar^{n-1}$
According to the given information

Sum of the n terms

$$S=\frac{a\left(r^n-1\right)}{r-1}$$

Product of the n terms

$$P = a^n \times r^{1+2+3+\ldots+n-1}$$
$$= a^n r^{\frac{n(n-1)}{2}} \qquad \left[\because \text{ sum of first } n \text{ natural numbers } = n\frac{(n+1)}{2}\right]$$

Sum of reciprocals of n terms

$$R = \frac{1}{a} + \frac{1}{ar} + \ldots + \frac{1}{ar^{n-1}}$$
$$= \frac{r^{n-1} + r^{n-2} + \ldots r + 1}{ar^{n-1}}$$
$$= \frac{1(r^n - 1)}{(r-1)} \times \frac{1}{ar^{n-1}} \qquad \left[\because 1, r, \ldots r^{n-1} \text{ forms a G.P}\right]$$
$$= \frac{r^n - 1}{ar^{n-1}(r-1)}$$

Therefore,

$$P^2 R^n = a^{2n} r^{n(n-1)} \frac{(r^n - 1)^n}{a^n r^{n(n-1)} (r-1)^n}$$
$$= \frac{a^n (r^n - 1)^n}{(r-1)^n}$$
$$= \left[\frac{a(r^n - 1)}{(r-1)}\right]^n$$
$$= S^n$$

Hence, $P^2 R^n = S^n$ proved

Question 15:

The p^{th}, q^{th} and r^{th} terms of an A.P are a, b, c respectively. Show that

$$(q-r)a + (r-p)b + (p-q)c = 0.$$

Solution:

Let t and d be the first term and common difference of the A.P respectively.

The n^{th} term of an A.P is given by, $a_n = t + (n-1)d$

$$a_p = t + (p-1)d = a \quad \ldots(1)$$
$$a_q = t + (q-1)d = b \quad \ldots(2)$$
$$a_r = t + (r-1)d = c \quad \ldots(3)$$

Subtracting (2) from (1), we obtain

$$\Rightarrow (p-1-q+1)d = a-b$$
$$\Rightarrow (p-q)d = a-b$$
$$\Rightarrow d = \frac{a-b}{p-q} \quad \ldots(4)$$

Subtracting (3) from (2), we obtain

$$\Rightarrow (q-1-r+1)d = b-c$$
$$\Rightarrow (q-r)d = b-c$$
$$\Rightarrow d = \frac{b-c}{q-r} \quad \ldots(5)$$

From (4) and (5), we obtain

$$\Rightarrow \frac{a-b}{p-q} = \frac{b-c}{q-r}$$
$$\Rightarrow (a-b)(q-r) = (b-c)(p-q)$$
$$\Rightarrow aq - ar - bq + br = bp - bq - cp + cq$$
$$\Rightarrow bp - cp + cq - aq + ar - br = 0$$
$$\Rightarrow (-aq + ar) + (bp - br) + (-cp + cq) = 0$$
$$\Rightarrow -a(q-r) - b(r-p) - c(p-q) = 0$$
$$\Rightarrow a(q-r) + b(r-p) + c(p-q) = 0$$

Hence, $(q-r)a + (r-p)b + (p-q)c = 0$ proved.

Question 16:

If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P, prove that a, b, c are in A.P.

Solution:

It is given that $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

Therefore,

$$\Rightarrow b\left(\frac{1}{c}+\frac{1}{a}\right)-a\left(\frac{1}{b}+\frac{1}{c}\right)=c\left(\frac{1}{a}+\frac{1}{b}\right)-b\left(\frac{1}{c}+\frac{1}{a}\right)$$
$$\Rightarrow \frac{b(a+c)}{ac}-\frac{a(b+c)}{bc}=\frac{c(a+b)}{ab}-\frac{b(a+c)}{ac}$$
$$\Rightarrow \frac{b^2a+b^2c-a^2b-a^2c}{abc}=\frac{c^2a+c^2b-b^2a-b^2c}{abc}$$
$$\Rightarrow b^2a-a^2b+b^2c-a^2c=c^2a-b^2a+c^2b-b^2c$$
$$\Rightarrow ab(b-a)+c(b^2-a^2)=a(c^2-b^2)+bc(c-b)$$
$$\Rightarrow ab(b-a)+c(b+a)(b-a)=a(c-b)(c+b)+bc(c-b)$$
$$\Rightarrow (b-a)(ab+cb+ca)=(c-b)(ac+ab+bc)$$
$$\Rightarrow b-a=c-b$$

Thus, a,b,c are in A.P. proved.

Question 17:

If a,b,c,d are in G.P, prove that $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P.

Solution:

It is given that a,b,c,d are in G.P.
Therefore,

$$b^2=ac \qquad \ldots(1)$$
$$c^2=bd \qquad \ldots(2)$$
$$ad=bc \qquad \ldots(3)$$

We need to prove $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P. i.e.,

$$(b^n+c^n)^2=(a^n+b^n)\cdot(c^n+d^n)$$

Consider

$$\begin{aligned}(b^n+c^n)^2&=b^{2n}+2b^nc^n+c^{2n}\\&=(b^2)^n+2b^nc^n+(c^2)^n\\&=(ac)^n+2b^nc^n+(bd)^n && [\text{Using (1) and (2)}]\\&=a^nc^n+b^nc^n+b^nc^n+b^nd^n\\&=a^nc^n+b^nc^n+a^nd^n+b^nd^n && [\text{Using (3)}]\\&=c^n(a^n+b^n)+d^n(a^n+b^n)\\&=(a^n+b^n)(c^n+d^n)\end{aligned}$$

Hence, $\left(b^n+c^n\right)^2=\left(a^n+b^n\right)\cdot\left(c^n+d^n\right)$

Thus, $\left(a^n+b^n\right),\left(b^n+c^n\right),\left(c^n+d^n\right)$ are in G.P.

Question 18:

If a and b are the roots of $x^2-3x+p=0$ and c,d are the roots of $x^2-12x+q=0$ where a,b,c,d form a G.P. Prove that $(q+p):(q-p)=17:15$.

Solution:

It is given that a and b are the roots of $x^2-3x+p=0$

Therefore,

$$a+b=3, \qquad ab=p \qquad \ldots(1)$$

Also, c and d are the roots of $x^2-12x+q=0$

$$c+d=12, \qquad cd=q \qquad \ldots(2)$$

It is given that a,b,c,d form a G.P.

Let $a=x,b=xr,c=xr^2,d=xr^3$

Form (1) and (2), we obtain

$$\Rightarrow x+xr=3$$
$$\Rightarrow x(1+r)=3 \qquad \ldots(3)$$

$$\Rightarrow xr^2+xr^3=12$$
$$\Rightarrow xr^2(1+r)=12 \qquad \ldots(4)$$

On dividing (4) by (3), we obtain

$$\frac{xr^2(1+r)}{x(1+r)}=\frac{12}{3}$$
$$\Rightarrow r^2=4$$
$$\Rightarrow r=\pm 2$$

<u>Case I</u>: when $r=2, x=1$

$$ab = x^2 r = 2,$$
$$cd = x^2 r^5 = 32$$

Therefore,

$$\Rightarrow \frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$$
$$\Rightarrow (q+p):(q-p) = 17:15$$

<u>Case II</u>: when $r = -2, x = -3$

$$ab = x^2 r = -18$$
$$cd = x^2 r^5 = -288$$

Therefore,

$$\Rightarrow \frac{q+p}{q-p} = \frac{-288-18}{-288+18} = \frac{-306}{-270} = \frac{17}{15}$$
$$\Rightarrow (q+p):(q-p) = 17:15$$

Thus, $(q+p):(q-p) = 17:15$ proved

Question 19:

The ratio of the A.M and G.M of two positive numbers a and b, is $m:n$. Show that $a:b = \left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$.

Solution:

Let the two numbers be a and b.

$$A.M = \frac{a+b}{2}$$

$$G.M = \sqrt{ab}$$

According to the given condition,

$$\Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$$
$$\Rightarrow \frac{(a+b)^2}{4ab} = \frac{m^2}{n^2}$$
$$\Rightarrow (a+b)^2 = \frac{4abm^2}{n^2}$$
$$\Rightarrow (a+b) = \frac{2m\sqrt{ab}}{n} \qquad \ldots(1)$$

Using this in the identity $(a-b)^2 = (a+b)^2 - 4ab$, we obtain

$$(a-b)^2 = \frac{4abm^2}{n^2} - 4ab$$

$$= \frac{4ab\left(m^2-n^2\right)}{n^2}$$

$$(a-b) = \frac{2\sqrt{ab}\sqrt{m^2-n^2}}{n} \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2a = \frac{2\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$a = \frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

Substituting the value of a in (1), we obtain

$$b = \frac{2\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$= \frac{\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n}\left(\sqrt{m^2-n^2}\right)$$

$$= \frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)$$

Therefore,

$$\frac{a}{b} = \frac{\left(\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)\right)}{\left(\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)\right)}$$

$$= \frac{\left(m+\sqrt{m^2-n^2}\right)}{\left(m-\sqrt{m^2-n^2}\right)}$$

Thus, $a:b = \left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$.

Question 20:

If a,b,c are in A.P; b,c,d are in G.P and $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P. Prove that a,c,e are in G.P.

Solution:

It is given that a,b,c are in A.P

$$b-a=c-b$$

$$b=\frac{a+c}{2} \qquad \ldots(1)$$

It is given that b,c,d are in G.P

$$c^2=bd$$

$$c^2=\left(\frac{a+c}{2}\right)d \qquad [\text{from } (1)]$$

$$d=\frac{2c^2}{a+c} \qquad \ldots(2)$$

Also, $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P.

Therefore,

$$\Rightarrow \frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d}$$

$$\Rightarrow \frac{2}{d}=\frac{1}{c}+\frac{1}{e}$$

$$\Rightarrow \frac{2}{\frac{2c^2}{a+c}}=\frac{1}{c}+\frac{1}{e} \qquad [\text{from } (2)]$$

$$\Rightarrow \frac{2(a+c)}{2c^2}=\frac{1}{c}+\frac{1}{e}$$

$$\Rightarrow \frac{(a+c)}{c^2}=\frac{e+c}{ce}$$

$$\Rightarrow \frac{(a+c)}{c}=\frac{e+c}{e}$$

$$\Rightarrow (a+c)e=(e+c)c$$

$$\Rightarrow ae+ce=ec+c^2$$

$$\Rightarrow c^2=ae$$

Thus, a,c,e are in G.P.

Question 21:

Find the sum of the following series up to n terms:

(i) $5+55+555+\ldots$

(ii) $.6+.66+.666+\ldots$

Solution:

(i) $5+55+555+\ldots$

Let $S_n=5+55+555+\ldots n$ terms

$$
\begin{aligned}
S_n &= \frac{5}{9}[9 + 99 + 999 + \ldots n\text{terms}] \\
&= \frac{5}{9}\left[(10 - 1) + \left(10^2 - 1\right) + \left(10^3 - 1\right) + \ldots n\text{terms}\right] \\
&= \frac{5}{9}\left[\left(10 + 10^2 + 10^3 + \ldots n\text{terms}\right) - (1 + 1 + \ldots n\text{terms})\right] \\
&= \frac{5}{9}\left[\frac{10\left(10^n - 1\right)}{10 - 1} - n\right] \\
&= \frac{5}{9}\left[\frac{10\left(10^n - 1\right)}{9} - n\right] \\
&= \frac{50}{81}\left(10^n - 1\right) - \frac{5n}{9}
\end{aligned}
$$

(ii) $.6+.66+.666+\ldots$

Let $S_n = .6 + .66 + .666 + \ldots n\text{terms} S_n = .6 + .66 + .666 + \ldots n\text{terms}$

$$S_n = 6[0.1 + 0.11 + 0.111 + \ldots n\text{terms}]$$
$$= \frac{6}{9}[0.9 + 0.99 + 0.999 + \ldots n\text{terms}]$$
$$= \frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots n\text{terms}\right]$$
$$= \frac{2}{3}\left[(1+1+1+\ldots n\text{terms})-\frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\ldots n\text{terms}\right)\right]$$
$$= \frac{2}{3}\left[n-\frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right)\right]$$
$$= \frac{2}{3}n-\frac{2}{30}\times\frac{10}{9}\left(1-10^{-n}\right)$$
$$= \frac{2}{3}n-\frac{2}{27}\left(1-10^{-n}\right)$$

Question 22:

Find the 20^{th} term of the series .

Solution:

The given series is

Therefore,

$$a_n = 2n \times (2n+2)$$
$$= 4n^2 + 4n$$

$$a_{20} = 4(20)^2 + 4(20)$$
$$= 4(400) + 80$$
$$= 1680$$

Thus, the 20^{th} term of the series is 1680.

Question 23:

Find the sum of the first nn terms of the series: 3+7+13+21+31+...

Solution:

The given series is $3+7+13+21+31+\ldots$, can be written as

$$S = 3+7+13+21+31+\ldots+a_{n-1}+a_n \qquad \ldots(1)$$
$$S = 3+7+13+21+\ldots+a_{n-2}+a_{n-1}+a_n \qquad \ldots(2)$$

On subtracting the equation (2) from (1), we obtain

$$\begin{aligned}
S-S &= [3+7+13+21+31+\ldots+a_{n-1}+a_n]-[3+7+13+21+\ldots+a_{n-2}+a_{n-1}+a_n] \\
0 &= 3+(7+13+21+31+\ldots+a_{n-1}+a_n)-(3+7+13+21+\ldots+a_{n-2}+a_{n-1})-a_n \\
0 &= 3+[(7-3)+(13-7)+(21-13)+\ldots+(a_n-a_{n-1})]-a_n \\
0 &= 3+[4+6+8+\ldots(n-1)\,\text{terms}]-a_n \\
a_n &= 3+[4+6+8+\ldots(n-1)\,\text{terms}] \\
&= 3+\left(\frac{n-1}{2}\right)[2\times 4+(n-1-1)\,2] \\
&= 3+\left(\frac{n-1}{2}\right)[8+2n-4] \\
&= 3+\left(\frac{n-1}{2}\right)[2n+4] \\
&= 3+(n-1)(n+2) \\
&= 3+(n^2+n-2) \\
a_n &= n^2+n+1
\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n} a_k &= \sum_{k=1}^{n} k^2+k+1 \\ &= \sum_{k=1}^{n} k^2+\sum_{k=1}^{n} k+\sum_{k=1}^{n} 1 \\ &= \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}+n \\ &= n\left[\frac{(n+1)(2n+1)+3(n+1)+6}{6}\right]\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n} a_k &= n\left[\frac{2n^2+3n+1+3n+3+6}{6}\right] \\ &= n\left[\frac{2n^2+6n+10}{6}\right] \\ &= \frac{n}{3}\left(n^2+3n+5\right)\end{aligned}$$

Question 24:

If S_1, S_2, S_3 are the sums of first n natural numbers, their squares and their cubes respectively, show that $9S_2^2 = S_3(1+8S_1)$.

Solution:

From the given information,

$$\begin{aligned}S_1 &= \frac{n(n+1)}{2} \\ S_2 &= \frac{n(n+1)(2n+1)}{6} \\ S_3 &= \left[\frac{n(n+1)}{2}\right]^2\end{aligned}$$

Hence,

$$
\begin{aligned}
9{S_2}^2 &= 9\left[\frac{n(n+1)(2n+1)}{6}\right]^2 \\
&= \frac{9}{36}[n(n+1)(2n+1)]^2 \\
&= \frac{1}{4}[n(n+1)(2n+1)]^2 \\
&= \left[\frac{n(n+1)(2n+1)}{2}\right]^2 \qquad \ldots(1)
\end{aligned}
$$

Also,

$$
\begin{aligned}
S_3(1+8S_1) &= \left[\frac{n(n+1)}{2}\right]^2\left[1+\frac{8n(n+1)}{2}\right] \\
&= \left[\frac{n(n+1)}{2}\right]^2\left[1+4n^2+4n\right] \\
&= \left[\frac{n(n+1)}{2}\right]^2(2n+1)^2 \\
&= \left[\frac{n(n+1)(2n+1)}{2}\right]^2 \qquad \ldots(2)
\end{aligned}
$$

Thus, from (1) and (2), we obtain

$$9S_2^2 = S_3(1+8S_1)$$

Question 25:

Find the sum of the following series up to n terms:

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$$

Solution:

The n^{th}term of the given series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ is

$$\begin{aligned} a_n &= \frac{1^3+2^3+3^3+\ldots n^3}{1+3+5+\ldots+(2n-1)} \\ &= \frac{\left[\frac{n(n+1)}{2}\right]^2}{1+3+5+\ldots+(2n-1)} \end{aligned}$$

Here, $1,3,5,\ldots(2n-1)$ is an A.P. with first term a, last term $(2n-1)$ and number of terms n.

Therefore,

$$\begin{aligned} 1+3+5+\cdots+(2n-1) &= \frac{n}{2}[2\times 1+(n-1)\,2] \\ &= n^2 \end{aligned}$$

Hence,

$$\begin{aligned} a_n &= \frac{n^2(n+1)^2}{4n^2} \\ &= \frac{n^2+2n+1}{4} \\ &= \frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4} \end{aligned}$$

Thus,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} a_k \\ Z &= \sum_{k=1}^{n}\left(\frac{1}{4}k^2+\frac{1}{2}k+\frac{1}{4}\right) \\ &= \frac{1}{4}\left[\frac{n\,(n+1)\,(2n+1)}{6}\right]+\frac{1}{2}\left[\frac{n\,(n+1)}{2}\right]+\frac{1}{4}n \\ &= \frac{n\left[(n+1)\,(2n+1)+6\,(n+1)+6\right]}{24} \\ &= \frac{n\left[2n^2+3n+1+6n+6+6\right]}{24} \\ &= \frac{n}{24}\left(2n^2+9n+13\right) \end{aligned}$$

Question 26:

Show that $\frac{1\times2^2+2\times3^2+\ldots+n\times(n+1)^2}{1^2\times2+2^2\times3+\ldots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$.

Solution:

The n^{th} term of the given series $1\times2^2+2\times3^2+\ldots+n\times(n+1)^2$ in numerator

$$\begin{aligned} a_n &= n(n+1)^2 \\ &= n^3+2n^2+n \end{aligned}$$

The n^{th} term of the given series $1^2\times2+2^2\times3+\ldots+n^2\times(n+1)$ in denominator

$$\begin{aligned} a_n &= n^2(n+1) \\ &= n^3+n^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1\times2^2+2\times3^2+\ldots+n\times(n+1)^2}{1^2\times2+2^2\times3+\ldots+n^2\times(n+1)} &= \frac{\sum\limits_{k=1}^{n} a_k}{\sum\limits_{k=1}^{n} a_k} \\ &= \frac{\sum\limits_{k=1}^{n}(k^3+2k^2+k)}{\sum\limits_{k=1}^{n}(k^3+k^2)} \qquad \ldots(1) \end{aligned}$$

Here,

$$\begin{aligned} \sum_{k=1}^{n}(k^3+2k^2+k) &= \frac{n^2(n+1)^2}{4}+\frac{2n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}(2n+1)+1\right] \\ &= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6}\right] \\ &= \frac{n(n+1)}{12}\left[3n^2+11n+10\right] \\ &= \frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right] \\ &= \frac{n(n+1)}{12}\left[3n^2(n+2)+5(n+2)\right] \\ &= \frac{n(n+1)(n+2)(3n+5)}{12} \qquad \ldots(2) \end{aligned}$$

Also,

$$\begin{aligned}\sum_{k=1}^{n}\left(k^3+k^2\right) &= \frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}\\ &= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]\\ &= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]\\ &= \frac{n(n+1)}{12}\left[3n^2+7n+2\right]\\ &= \frac{n(n+1)}{12}\left[3n^2+6n+n+2\right]\\ &= \frac{n(n+1)}{12}\left[3n(n+2)+1(n+2)\right]\\ &= \frac{n(n+1)(n+2)(3n+1)}{12} \qquad \ldots(3)\end{aligned}$$

From (1), $(2)$and (3), we obtain

$$\begin{aligned}\frac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)} &= \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}\\ &= \frac{n(n+1)(n+2)(3n+5)}{n(n+1)(n+2)(3n+1)}\\ &= \frac{3n+5}{3n+1}\end{aligned}$$

Thus, $\frac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)}=\frac{3n+5}{3n+1}$ proved.

Question 27:

A farmer buys a used tractor for ₹ 12000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalment of ₹ 500 plus 12%interest on the unpaid amount. How much will the tractor cost him?

Solution:

It is given that farmer pays ₹ 6000 in cash.

Hence, unpaid amount in ₹ $= 12000 - 6000 = 6000$

According to the given condition, the interest paid annually is

$$(12\% of 6000), (12\% of 5500), (12\% of 5000), \ldots (12\% of 500)$$

Thus, total interest to be paid$500 + 1000 + \ldots + 6000$

$$\begin{aligned} &= 12\% of 6000 + 12\% of 5500 + \ldots + 12\% of 500 \\ &= 12\% of (6000 + 5500 + \ldots + 500) \\ &= 12\% of (500 + 1000 + \ldots + 6000) \end{aligned}$$

Now, the series $500 + 1000 + \ldots + 6000$ forms an A.P. with a and d both equal to 500.

Let the number of terms of the A.P. be n

Therefore,

$$\begin{aligned} &\Rightarrow a_n = a + (n-1)d \\ &\Rightarrow 6000 = 500 + (n-1)(500) \\ &\Rightarrow 6000 - 500 = 500(n-1) \\ &\Rightarrow n - 1 = \frac{5500}{500} \\ &\Rightarrow n = 11 + 1 \\ &\Rightarrow n = 12 \end{aligned}$$

Hence,

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ S_{12} &= \frac{12}{2}[2(500) + (12-1)(500)] \\ &= 6[1000 + 5500] \\ &= 6 \times 6500 \\ &= 39000 \end{aligned}$$

Therefore, total interest to be paid is

$$\begin{aligned} 12\% of\,(500+1000+\ldots+6000) &= 12\% of 39000 \\ &= \frac{12}{100}\times 39000 \\ &= 4680 \end{aligned}$$

Now, total cost of tractor in ₹ $= 12000 + 4680 = 16680$

Thus, the tractor will cost him ₹ 16680.

Question 28:

Shamshad Ali buys a scooter for ₹22000. He pays ₹4000cash and agrees to pay the balance in annual instalment of ₹1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Solution:

It is given that Shamshad Ali buys a scooter for ₹22000 and pays ₹4000 in cash.

Hence, unpaid amount in ₹ $= 22000 - 4000 = 18000$

According to the given condition, the interest paid annually is

$(10\% of 18000)\,,(10\% of 17000)\,,(10\% of 16000)\,,\ldots(10\% of 1000)$

Thus, total interest to be paid

$$\begin{aligned} &= (10\%\text{of}18000) + (10\%\text{of}17000) + \ldots + (10\% of 1000) \\ &= 10\% of\,(18000 + 17000 + \ldots + 1000) \\ &= 10\% of\,(1000 + 2000 + \ldots + 18000) \end{aligned}$$

Now, the series $1000 + 2000 + \ldots + 18000$forms an A.P. with a and d both equal to 1000.

Let the number of terms of the A.P. be n

Therefore,

$$\begin{aligned}
&\Rightarrow a_n = a + (n-1)\,d \\
&\Rightarrow 18000 = 1000 + (n-1)\,(1000) \\
&\Rightarrow 18000 - 1000 = 1000\,(n-1) \\
&\Rightarrow n - 1 = \frac{17000}{1000} \\
&\Rightarrow n = 17 + 1 \\
&\Rightarrow n = 18
\end{aligned}$$

Hence,

$$\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)\,d] \\
S_{18} &= \frac{18}{2}[2\,(1000) + (18-1)\,(1000)] \\
&= 9\,[2000 + 17000] \\
&= 9 \times 19000 \\
&= 171000
\end{aligned}$$

Therefore, total interest to be paid is

$$\begin{aligned}
10\% \text{ of } (1000 + 2000 + \ldots + 18000) &= 10\% of 171000 \\
&= \frac{10}{100} \times 171000 \\
&= 17100
\end{aligned}$$

Now, total cost of scooter in ₹ $= 22000 + 17100 = 39100$

Thus, the scooter will cost him ₹39100.

Question 29:

A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter, find the amount spent on the postage when 8^{th} set of letter is mailed.

Solution:

The number of letters mailed forms a G.P: $4, 4^2, \ldots, 4^8$

Here, $a = 4, r = 4, n = 8$

Hence,

$$\begin{aligned} S_n &= \frac{a\left(r^n - 1\right)}{r - 1} \\ S_8 &= \frac{4\left(4^8 - 1\right)}{4 - 1} \\ &= \frac{4\left(65536 - 1\right)}{3} \\ &= \frac{4 \times 65535}{3} \\ &= 4 \times 21845 \\ &= 87380 \end{aligned}$$

It is given that the cost to mail one letter is 50 paisa

Therefore, cost in ₹ for mailing 87380 letters $= \frac{50}{100} \times 87380 = 43690$

Thus, ₹43690spent when on the postage when 8^{th} set of letter is mailed.

Question 30:

A man deposited ₹10000 in a bank at the rate of 5% simple interest annually. Find the amount in the 15^{th} year since he deposited the amount and also calculate the total amount after 20 years.

Solution:

It is given that the man deposited ₹10000in a bank at the rate of 5% simple interest annually

Hence, annual interest received in ₹ $= \frac{5}{100} \times 10000 = 500$

Amount in ₹ at the end of first year $= 10000 + 500 = 10500$

Amount in ₹ at the end of second year $= 10500 + 500 = 11000$

Amount in ₹ at the end of third year $= 11000 + 500 = 11500$

Therefore, $10000, 10500, 11000, \ldots$ is an A.P.

Hence, the amount in the 15^{th} year

$$\begin{aligned} a_{15} &= 10000 + (15 - 1)(500) \\ &= 10000 + 7000 \\ &= 17000 \end{aligned}$$

Amount after 20 years

$$\begin{aligned} a_{21} &= 10000 + (21 - 1)(500) \\ &= 10000 + 10000 \\ &= 20000 \end{aligned}$$

Thus, the amount in 15^{th} year is ₹17000 and after 20 years amount is ₹20000.

Question 31:

A manufacturer reckons that the value of a machine, which costs him ₹15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Solution:

Cost of machine is ₹15625

Machine depreciates by 20% every year.

Therefore,

its value after every year is 80 of the original cost i.e., $\frac{4}{5}$of the original cost.

Value at end of 5 years $= 15625 \times \underbrace{\frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \ldots \times \frac{4}{5}}_{5 times} = 15625 \times \left(\frac{4}{5}\right)^5 = 5 \times 1024 = 5120$

Thus, value of the machine at the end of 5 years is ₹ 5120.

Question 32:

150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Solution:

Let x be the number of days in which 150 workers finish the work.

According to the given information,

$$150x = 150 + 146 + 142 + \ldots (x+8) \text{ terms}$$

Here, $150 + 146 + 142 + \ldots (x+8)$ terms forms an A.P. with $a = 146, d = -4, n = (x+8)$

Therefore

$$\Rightarrow 150x = \frac{(x+8)}{2}[2(150) + (x+8-1)(-4)]$$
$$\Rightarrow 150x = (x+8)[150 + (x+7)(-2)]$$
$$\Rightarrow 150x = (x+8)(150 - 2x - 14)$$
$$\Rightarrow 150x = (x+8)(136 - 2x)$$
$$\Rightarrow 75x = (x+8)(68 - x)$$
$$\Rightarrow 75x = 68x - x^2 + 544 - 8x$$
$$\Rightarrow x^2 + 15x - 544 = 0$$
$$\Rightarrow x^2 + 32x - 17x - 544 = 0$$
$$\Rightarrow x(x+32) - 17(x+32) = 0$$
$$\Rightarrow (x-17)(x+32) = 0$$
$$\Rightarrow x = 17, -32$$

However, x cannot be negative, hence, $x = 17$

Therefore, the number of days in which the work was completed is $(17+8) = 25$ days.

Thus, the work was completed in 25 days.