NCERT Solutions Class 12 Maths Chapter 5 Continuity and Differentiability

Question 1:

Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, $x = -3$ and at $x = 5$.

Solution:

The given function is $f(x) = 5x - 3$

At $x = 0$, $f(0) = 5(0) - 3 = -3$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (5x - 3) = 5(0) - 3 = -3$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continous at $x = 0$.

At $x = -3$, $f(-3) = 5(-3) - 3 = -18$

$\lim_{x \to -3} f(x) = \lim_{x \to -3} (5x - 3) = 5(-3) - 3 = -18$

$\therefore \lim_{x \to -3} f(x) = f(-3)$

Therefore, f is continous at $x = -3$.

At $x = 5$, $f(5) = 5(5) - 3 = 22$

$\lim_{x \to 5} f(x) = \lim_{x \to 5} (5x - 3) = 5(5) - 3 = 22$

$\therefore \lim_{x \to 5} f(x) = f(5)$

Therefore, f is continous at $x = 5$.

Question 2:

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Solution:

The given function is $f(x) = 2x^2 - 1$

At $x = 3$, $f(3) = 2(3)^2 - 1 = 17$

$\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x^2 - 1) = 2(3^2) - 1 = 17$

$\therefore \lim_{x \to 3} f(x) = f(3)$

Therefore, f is continous at $x = 3$.

Question 3:

Examine the following functions for continuity.

(i) $f(x) = x - 5$

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(ii) $f(x) = \dfrac{1}{x-5}, x \neq 5$

(iii) $f(x) = \dfrac{x^2 - 25}{x+5}, x \neq -5$

(iv) $f(x) = |x-5|, x \neq 5$

Solution:

(i) The given function is $f(x) = x - 5$

It is evident that f is defined at every real number k and its value at k is $k - 5$.
It is also observed that

$$\lim_{x \to k} f(x) = \lim_{x \to k} (x - 5) = k - 5 = f(k)$$
$$\therefore \lim_{x \to k} f(x) = f(k)$$

Hence, f is continuous at every real number and therefore, it is a continuous function.

(ii) The given function is $f(x) = \dfrac{1}{x-5}, x \neq 5$
For any real number $k \neq 5$, we obtain

$$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{x-5} = \frac{1}{k-5}$$

Also,

$$f(k) = \frac{1}{k-5} \qquad (\text{As } k \neq 5)$$
$$\therefore \lim_{x \to k} f(x) = f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(iii) The given function is $f(x) = \dfrac{x^2 - 25}{x+5}, x \neq -5$
For any real number $c \neq -5$, we obtain

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2 - 25}{x+5} = \lim_{x \to c} \frac{(x+5)(x-5)}{x+5} = \lim_{x \to c} (x-5) = (c-5)$$

Also,

$$f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5)$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

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Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(iv)   The given function is $f(x) = |x-5| = \begin{cases} 5-x, & \text{if } x < 5 \\ x-5, & \text{if } x \geq 5 \end{cases}$

This function f is defined at all points of the real line. Let c be a point on a real line. Then, $c < 5$, $c = 5$ or $c > 5$

Case I: $c < 5$

Then, $f(c) = 5 - c$

$\lim_{x \to c} f(x) = \lim_{x \to c} (5-x) = 5-c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all real numbers less than 5.

Case II: $c = 5$

Then, $f(c) = f(5) = (5-5) = 0$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5} (5-x) = (5-5) = 0$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5} (x-5) = 0$

$\therefore \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

Therefore, f is continuous at $x = 5$

Case III: $c > 5$

Then, $f(c) = f(5) = c - 5$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x-5) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Solution:

The given function is $f(x) = x^n$

It is observed that f is defined at all positive integers, n, and its value at n is n^n.
Then,

$$\lim_{x \to n} f(n) = \lim_{x \to n}(x^n) = x^n$$

$$\therefore \lim_{x \to n} f(x) = f(n)$$

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by $f(x) = \begin{cases} x, \text{if } x \leq 1 \\ 5, \text{if } x > 1 \end{cases}$ continuous at $x = 0$? At $x = 1$? At $x = 2$?

Solution:

The given function is $f(x) = \begin{cases} x, \text{if } x \leq 1 \\ 5, \text{if } x > 1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.
Then,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0}(x) = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$.

At $x = 1$,

It is evident that f is defined at 1 and its value at 1 is 1.
The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(x) = 1$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(5) = 5$$

$$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

Therefore, f is not continuous at $x = 1$.

At $x = 2$,

It is evident that f is defined at 2 and its value at 2 is 5.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2}(5) = 5$$

$$\therefore \lim_{x \to 1} f(x) = f(2)$$

Therefore, f is continuous at $x = 2$.

Question 6:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$.

Solution:

The given function is $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

$c < 2$

$c > 2$

$c = 2$

Case I: $c < 2$

$f(c) = 2c+3$

Then,

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(2x+3) = 2c+3$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 2$.

Case II: $c > 2$

Then,

$f(c) = 2c-3$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(2x-3) = 2c-3$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 2$

Case III: $c = 2$

Then, the left hand limit of f at $x = 2$ is,

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-}(2x+3) = 2(2)+3 = 7$

The right hand limit of f at $x = 2$ is,

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+}(2x-3) = 2(2)-3 = 1$

It is observed that the left and right hand limit of f at $x = 2$ do not coincide.

Therefore, f is not continuous at $x = 2$.

Hence, $x = 2$ is the only point of discontinuity of f.

Question 7:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

Solution:

The given function is

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < -3$, then $f(c) = -c + 3$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-x + 3) = -c + 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < -3$.

Case II:

If $c = -3$, then $f(-3) = -(-3) + 3 = 6$

$\lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^-} (-x + 3) = -(-3) + 3 = 6$

$\lim\limits_{x \to -3^+} f(x) = \lim\limits_{x \to -3^+} (-2x) = -2(-3) = 6$

$\therefore \lim\limits_{x \to -3} f(x) = f(-3)$

Therefore, f is continuous at $x = -3$.

Case III:

If $-3 < c < 3$, then $f(c) = -2c$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (-2x) = -2c$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is,

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (-2x) = -2(3) = -6$

The right hand limit of f at $x = 3$ is,

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (6x + 2) = 6(3) + 2 = 20$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.
Therefore, f is not continuous at $x = 3$.

Case V:

If $c > 3$, then $f(c) = 6c + 2$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (6x + 2) = 6c + 2$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 3$.
Hence, $x = 3$ is the only point of discontinuity of f.

Question 8:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Solution:

The given function is

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \dfrac{|x|}{x} = \dfrac{-x}{x} = -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \dfrac{|x|}{x} = \dfrac{x}{x} = 1, & \text{if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = -1$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points $x < 0$.

Case II:

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(-1) = -1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(1) = 1$$

It is observed that the left and right hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$.

Case III:

If $c > 0$, then $f(c) = 1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(1) = 1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$.

Hence, $x = 0$ is the only point of discontinuity of f.

Question 9:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$.

Solution:

The given function is $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

It is known that $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$f(x) = \begin{cases} \dfrac{x}{|x|} = \dfrac{x}{-x} = -1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

$\Rightarrow f(x) = -1 \; \forall \; x \in R$

Let c be any real number.

Then, $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(-1) = -1$

Also, $f(c) = -1 = \lim\limits_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^2 + 1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x^2 + 1) = c^2 + 1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$.

Case II:

If $c = 1$, then $f(c) = f(1) = 1 + 1 = 2$

The left hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-}(x^2 + 1) = 1^2 + 1 = 2$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+}(x + 1) = 1 + 1 = 2$

$\therefore \lim\limits_{x \to 1} f(x) = f(1)$

Therefore, f is continuous at $x = 1$.

Case III:

If $c > 1$, then $f(c) = c + 1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x + 1) = c + 1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$.

Hence, the given function f has no point of discontinuity.

Question 11:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.
Case I:

If $c < 2$, then $f(c) = c^3 - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 - 3) = c^3 - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 2$.
Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3) = 2^3 - 3 = 5$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = 2^2 + 1 = 5$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Therefore, f is continuous at $x = 2$.
Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 2$.

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Question 12:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$.

Solution:

The given function is $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.
Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^{10} - 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^{10} - 1) = c^{10} - 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$.

Case II:
If $c = 1$, then the left hand limit of f at $x = 1$ is,
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0$$
The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2) = 1^2 = 1$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.
Therefore, f is not continuous at $x = 1$.

Case III:

If $c > 1$, then $f(c) = c^2$
$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2) = c^2$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$.
Thus from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f.

Question 13:

Is the function defined by $f(x) = \begin{cases} x + 5, & \text{if } x \leq 1 \\ x - 5, & \text{if } x > 1 \end{cases}$ a continous function?

Solution:

The given function is $f(x) = \begin{cases} x + 5, & \text{if } x \leq 1 \\ x - 5, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.
Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c + 5$
$$\lim_{x \to c} f(x) = \lim_{x \to c} (x + 5) = c + 5$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < 1$.

Case II:

If $c = 1$, then $f(1) = 1 + 5 = 6$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+5) = 1 + 5 = 6$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-5) = 1 - 5 = -4$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$.

Case III:

If $c > 1$, then $f(c) = c - 5$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x-5) = c - 5$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$.

From the above observation it can be concluded that, $x = 1$ is the only point of discontinuity of f.

Question 14:

Discuss the continuity of the function f, where f is defined by
$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$$

Solution:

The given function is
$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$$

The given function f is defined at all the points of the interval $[0,10]$.

Let c be a point in the interval $[0,10]$.

Case I:

If $0 \le c < 1$, then $f(c) = 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (3) = 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in the interval $[0,1)$.

Case II:

If $c = 1$, then $f(3) = 3$

The left hand limit of f at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3) = 3$

The right hand limit of f at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4) = 4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$.

Case III:

If $1 < c < 3$, then $f(c) = 4$

$\lim_{x \to c} f(x) = \lim_{x \to c} (4) = 4$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at in the interval $(1,3)$.

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of f at $x = 3$ is,

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$

The right hand limit of f at $x = 3$ is,

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is discontinuous at $x = 3$.

Case V:

If $3 < c \le 10$, then $f(c) = 5$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(5) = 5$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval $(3, 10]$.

Hence, f is discontinuous at $x = 1$ and $x = 3$.

Question 15:

Discuss the continuity of the function f, where f is defined by $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 2c$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(2x) = 2c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < 0$.

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(2x) = 2(0) = 0$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(0) = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

Case III:

If $0 < c < 1$, then $f(x) = 0$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous in the interval $(0,1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4(1) = 4$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.
Therefore, f is not continuous at $x = 1$.

Case V:

If $c < 1$, then $f(c) = 4c$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$.
Hence, f is not continuous only at $x = 1$.

Question 16:

Discuss the continuity of the function f, where f is defined by $f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points.
Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-2) = -2$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < -1$.

Case II:

If $c = -1$, then $f(c) = f(-1) = -2$

The left hand limit of f at $x = -1$ is,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2$$

The right hand limit of f at $x = -1$ is,

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x) = 2(-1) = -2$$

$$\therefore \lim_{x \to -1} f(x) = f(-1)$$

Therefore, f is continuous at $x = -1$

Case III:

If $-1 < c < 1$, then $f(c) = 2c$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous in the interval $(-1, 1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 2(1) = 2$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2(1) = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2) = 2$$

$$\therefore \lim_{x \to 1} f(x) = f(c)$$

Therefore, f is continuous at $x = 2$.

Case V:

If $c > 1$, then $f(c) = 2$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (2) = 2$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$.

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Question 17:

Find the relationship between a and b so that the function f defined by $f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$ is continous at $x = 3$.

Solution:

The given function is $f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$

For f to be continuous at $x = 3$, then

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) \qquad \dots (1)$$

Also,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1) = 3a + 1$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3) = 3b + 3$$

$$f(3) = 3a + 1$$

Therefore, from (1), we obtain

$$3a + 1 = 3b + 3 = 3a + 1$$

$$\Rightarrow 3a + 1 = 3b + 3$$

$$\Rightarrow 3a = 3b + 2$$

$$\Rightarrow a = b + \frac{2}{3}$$

Therefore, the required relationship is given by, $a = b + \frac{2}{3}$.

Question 18:

For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$ is continous at $x = 0$? What about continuity at $x = 1$?

Solution:

The given function is

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

If f is continuous at $x = 0$, then

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x \to 0^-} \lambda(x^2 - 2x) = \lim_{x \to 0^+} (4x + 1) = \lambda(0^2 - 2 \times 0)$$

$$\Rightarrow \lambda(0^2 - 2 \times 0) = 4(0) + 1 = 0$$

$$\Rightarrow 0 = 1 = 0 \qquad\qquad [\text{which is not possible}]$$

Therefore, there is no value of λ for which f is continuous at $x = 0$.

At $x = 1$

$$f(1) = 4x + 1 = 4(1) + 1 = 5$$

$$\lim_{x \to 1} (4x + 1) = 4(1) + 1 = 5$$

$$\therefore \lim_{x \to 1} f(x) = f(1)$$

Therefore, for any values of λ, f is continuous at $x = 1$.

Question 19:

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point. Here $[x]$ denotes the greatest integer less than or equal to x.

Solution:

The given function is $g(x) = x - [x]$
It is evident that g is defined at all integral points.
Let n be an integer.

Then,
$$g(n) = n - [n] = n - n = 0$$
The left hand limit of g at $x = n$ is,
$$\lim_{x \to n^-} g(x) = \lim_{x \to n^-} (x - [x]) = \lim_{x \to n^-} (x) - \lim_{x \to n^-} [x] = n - (n-1) = 1$$

The right hand limit of g at $x = n$ is,
$$\lim_{x \to n^+} g(x) = \lim_{x \to n^+} (x - [x]) = \lim_{x \to n^+} (x) - \lim_{x \to n^+} [x] = n - n = 0$$

It is observed that the left and right hand limit of g at $x = n$ do not coincide.
Therefore, g is not continuous at $x = n$.
Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Solution:

The given function is $f(x) = x^2 - \sin x + 5$
It is evident that f is defined at $x = \pi$.
At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$
Consider $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5)$

Put $x = \pi + h$, it is evident that if $x \to \pi$, then $h \to 0$
$$\therefore \lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x) + 5$$
$$= \lim_{h \to 0} \left[(\pi + h)^2 - \sin(\pi + h) + 5 \right]$$
$$= \lim_{h \to 0} (\pi + h)^2 - \lim_{h \to 0} \sin(\pi + h) + \lim_{h \to 0} 5$$
$$= (\pi + 0)^2 - \lim_{h \to 0} [\sin \pi \cos h + \cos \pi \sin h] + 5$$
$$= \pi^2 - \lim_{h \to 0} \sin \pi \cos h - \lim_{h \to 0} \cos \pi \sin h + 5$$
$$= \pi^2 - \sin \pi \cos 0 - \cos \pi \sin 0 + 5$$
$$= \pi^2 - 0(1) - (-1)0 + 5$$
$$= \pi^2 + 5$$
$$= f(\pi)$$

Therefore, the given function f is continuous at $x = \pi$.

Question 21:

Discuss the continuity of the following functions.

(i) $f(x) = \sin x + \cos x$

(ii) $f(x) = \sin x - \cos x$

(iii) $f(x) = \sin x \times \cos x$

Solution:

It is known that if g and h are two continuous functions, then $g + h, g - h$ and g, h are also continuous.

Let $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$g(c) = \sin c$

$$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$$

$$= \lim_{h \to 0} \sin(c + h)$$

$$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$$

$$= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)$$

$$= \sin c \cos 0 + \cos c \sin 0$$

$$= \sin c (1) + \cos c (0)$$

$$= \sin c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, $g(x) = \sin x$ is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$

$$= \lim_{h \to 0} \cos(c+h)$$

$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$

$$= \lim_{h \to 0} (\cos c \cos h) - \lim_{h \to 0} (\sin c \sin h)$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c (1) - \sin c (0)$$

$$= \cos c$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.
Therefore, it can be concluded that,

(i) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function.

(ii) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function.

(iii) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function.

Question 22:
Discuss the continuity of the cosine, cosecant, secant, and cotangent functions.

Solution:
It is known that if g and h are two continuous functions, then

(i) $\dfrac{h(x)}{g(x)}, g(x) \neq 0$ is continuous.

(ii) $\dfrac{1}{g(x)}, g(x) \neq 0$ is continuous.

(iii) $\dfrac{1}{h(x)}, h(x) \neq 0$ is continuous.

Let $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

It is evident that $g(x) = \sin x$ is defined for every real number.
Let c be a real number. Put $x = c + h$
If $x \to c$, then $h \to 0$

$$g(c) = \sin c$$

$$\begin{aligned}
\lim_{x \to c} g(x) &= \lim_{x \to c} \sin x \\
&= \lim_{h \to 0} \sin(c+h) \\
&= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] \\
&= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h) \\
&= \sin c \cos 0 + \cos c \sin 0 \\
&= \sin c (1) + \cos c (0) \\
&= \sin c
\end{aligned}$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, $g(x) = \sin x$ is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$h(c) = \cos c$$

$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c+h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} (\cos c \cos h) - \lim_{h \to 0} (\sin c \sin h) \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c (1) - \sin c (0) \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

Therefore, it can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x}, \sin x \neq 0$$ is continuous.

$$\Rightarrow \operatorname{cosec} x, x \neq n\pi \ (n \in Z)$$ is continuous.

Therefore, cosecant is continuous except at $x = n\pi \ (n \in Z)$

$$\sec x = \frac{1}{\cos x}, \cos x \neq 0$$ is continuous.

$$\Rightarrow \sec x, x \neq (2n+1)\frac{\pi}{2} (n \in Z)$$ is continuous.

Therefore, secant is continuous except at $x = (2n+1)\dfrac{\pi}{2} (n \in Z)$

$\cot x = \dfrac{\cos x}{\sin x}, \sin x \neq 0$ is continuous.

$\Rightarrow \cot x, x \neq n\pi (n \in Z)$ is continuous.

Therefore, cotangent is continuous except at $x = n\pi (n \in Z)$.

Question 23:

Find the points of discontinuity of f, where $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$

The given function f is defined at all the points of the real line.
Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = \dfrac{\sin c}{c}$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \left(\dfrac{\sin x}{x} \right) = \dfrac{\sin c}{c}$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$.

Case II:

If $c > 0$, then $f(c) = c+1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (x+1) = c+1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$.

Case III:

If $c = 0$, then $f(c) = f(0) = 0 + 1 = 1$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{\sin x}{x} \right) = 1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1) = 1$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

Determine if f defined by
$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$
is a continuous function?

Solution:

The given function is
$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin \dfrac{1}{c}$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(x^2 \sin \frac{1}{x} \right) = \left(\lim_{x \to c} x^2 \right) \left(\lim_{x \to c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x \neq 0$.

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \leq \sin\frac{1}{x} \leq 1, x \neq 0$

$$\Rightarrow -x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x \to 0}\left(-x^2\right) \leq \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \leq \lim_{x \to 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \leq 0$$

$$\Rightarrow \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x \to 0^-} f(x) = 0$$

Similarly,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$$

Therefore, f is continuous at $x = 0$.

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f, where f is defined by $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$

The given function f is defined at all the points of the real line.
Let c be a point on the real line.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x \neq 0$.

Case II:

If $c = 0$, then $f(0) = -1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Therefore, f is continuous at $x = 0$.

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 26:

Find the values of k so that the function f is continuous at the indicated point

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases} \quad \text{at} \quad x = \dfrac{\pi}{2}$$

Solution:

The given function is

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

The given function f is continuous at $x = \dfrac{\pi}{2}$, if f is defined at $x = \dfrac{\pi}{2}$ and if the value of the f at $x = \dfrac{\pi}{2}$ equals the limit of f at $x = \dfrac{\pi}{2}$.

It is evident that f is defined at $x = \dfrac{\pi}{2}$ and $f\left(\dfrac{\pi}{2}\right) = 3$

$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \dfrac{k\cos x}{\pi - 2x}$

Put $x = \dfrac{\pi}{2} + h$

Then $x \to \dfrac{\pi}{2} \Rightarrow h \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$= k\lim_{h \to 0} \frac{-\sin h}{-2h} = \frac{k}{2}\lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2} = 3$$

$$\Rightarrow k = 6$$

Therefore, the value of $k = 6$.

Question 27:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases} \quad \text{at } x = 2$$

Solution:

The given function is $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$

The given function f is continuous at $x = 2$, if f is defined at $x = 2$ and if the value of the f at $x = 2$ equals the limit of f at $x = 2$.

It is evident that f is defined at $x = 2$ and $f(2) = k(2)^2 = 4k$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \to 2^-} (kx^2) = \lim_{x \to 2^+} (3) = 4k$$

$$\Rightarrow k \times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

Therefore, the value of $k = \frac{3}{4}$.

Question 28:

Find the values of k so that the function f is continuous at the indicated point

$$f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases} \quad \text{at } x = \pi$$

Solution:

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function f is continuous at $x = \pi$, if f is defined at $x = \pi$ and if the value of the f at $x = \pi$ equals the limit of f at $x = \pi$.

It is evident that f is defined at $x = \pi$ and $f(\pi) = k\pi + 1$

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x \to \pi^-} (kx+1) = \lim_{x \to \pi^+} (\cos x) = k\pi + 1$$

$$\Rightarrow k\pi + 1 = \cos\pi = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the value of $k = -\dfrac{2}{\pi}$.

Question 29:

Find the values of k so that the function f is continuous at the indicated point

$$f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases} \quad \text{at } x = 5.$$

Solution:

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$

The given function f is continuous at $x = 5$, if f is defined at $x = 5$ and if the value of the f at $x = 5$ equals the limit of f at $x = 5$.

It is evident that f is defined at $x = 5$ and $f(5) = kx+1 = 5k+1$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$$

$$\Rightarrow \lim_{x \to 5^-} (kx+1) = \lim_{x \to 5^+} (3x-5) = 5k+1$$

$$\Rightarrow 5k+1 = 3(5)-5 = 5k+1$$

$$\Rightarrow 5k+1 = 15-5 = 5k+1$$

$$\Rightarrow 5k+1 = 10 = 5k+1$$

$$\Rightarrow 5k+1 = 10$$

$$\Rightarrow 5k = 9$$

$$\Rightarrow k = \frac{9}{5}$$

Therefore, the value of $k = \dfrac{9}{5}$.

Question 30:

Find the values of a & b such that the function defined by $f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$, is a continuous function.

Solution:

The given function is $f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$

It is evident that f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at $x=2$ and $x=10$

Since f is continuous at $x=2$, we obtain

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \to 2^-} (5) = \lim_{x \to 2^+} (ax+b) = 5$$

$$\Rightarrow 5 = 2a+b = 5$$

$$\Rightarrow 2a+b = 5 \qquad \ldots(1)$$

Since f is continuous at $x=10$, we obtain

$$\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$$

$$\Rightarrow \lim_{x \to 10^-} (ax+b) = \lim_{x \to 10^+} (21) = 21$$

$$\Rightarrow 10a+b = 21 = 21$$

$$\Rightarrow 10a+b = 21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

$8a = 16$

$\Rightarrow a = 2$

By putting $a = 2$ in equation (1), we obtain

$2(2) + b = 5$

$\Rightarrow 4 + b = 5$

$\Rightarrow b = 1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Solution:

The given function is $f(x) = \cos(x^2)$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = goh$, where $g(x) = \cos x$ and $h(x) = x^2$

$\left[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x) \right]$

It has to be proved first that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Let $g(c) = \cos c$. Put $x = c + h$

If $x \to c$, then $h \to 0$

$\lim_{x \to c} g(x) = \lim_{x \to c} \cos x$

$\qquad = \lim_{h \to 0} \cos(c + h)$

$\qquad = \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$

$\qquad = \lim_{h \to 0} (\cos c \cos h) - \lim_{h \to 0} (\sin c \sin h)$

$\qquad = \cos c \cos 0 - \sin c \sin 0$

$\qquad = \cos c (1) - \sin c (0)$

$\qquad = \cos c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, $g(x) = \cos x$ is a continuous function.

Let $h(x) = x^2$

It is evident that h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$

$\therefore \lim_{x \to k} h(x) = h(k)$

Therefore, h is a continuous function.

It is known that for real valued functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Solution:

The given function is $f(x) = |\cos x|$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = goh$, where $g(x) = |x|$ and $h(x) = \cos x$

$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x) \right]$

It has to be proved first that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as $g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

It is evident that g is defined for every real number.
Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$

$\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$.

Case II:

If $c > 0$, then $g(c) = c$

$$\lim_{x \to c} g(x) = \lim_{x \to c}(x) = c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that $x > 0$.

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-}(-x) = 0$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+}(x) = 0$$

$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+}(x) = g(0)$$

Therefore, g is continuous at all $x = 0$.

From the above three observations, it can be concluded that g is continuous at all points.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$h(c) = \cos c$$

$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0}[\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0}(\cos c \cos h) - \lim_{h \to 0}(\sin c \sin h) \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c(1) - \sin c(0) \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Question 33:

Show that the function defined by $f(x) = |\sin x|$ is a continuous function.

Solution:

The given function is $f(x) = |\sin x|$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = goh$, where $g(x) = |x|$ and $h(x) = \sin x$

$\left[\because (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x) \right]$

It has to be proved first that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as $g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

It is evident that g is defined for every real number.
Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$

$\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} (-x) = -c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$.

Case II:

If $c > 0$, then $g(c) = c$

$\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} (x) = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$.

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} (-x) = 0$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} (x) = 0$

$\therefore \lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^+} (x) = g(0)$

Therefore, g is continuous at all $x = 0$.

From the above three observations, it can be concluded that g is continuous at all points.
Let $h(x) = \sin x$

It is evident that $h(x) = \sin x$ is defined for every real number.
Let c be a real number. Put $x = c + k$
If $x \to c$, then $k \to 0$

$$h(c) = \sin c$$

$$\lim_{x \to c} h(x) = \lim_{x \to c} \sin x$$

$$= \lim_{k \to 0} \sin(c + k)$$

$$= \lim_{k \to 0} [\sin c \cos k + \cos c \sin k]$$

$$= \lim_{k \to 0} (\sin c \cos k) + \lim_{k \to 0} (\cos c \sin k)$$

$$= \sin c \cos 0 + \cos c \sin 0$$

$$= \sin c (1) + \cos c (0)$$

$$= \sin c$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \sin x$ is a continuous function.

It is known that for real valued functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

Solution:

The given function is $f(x) = |x| - |x + 1|$.

The two functions, g and h are defined as $g(x) = |x|$ and $h(x) = |x + 1|$.

Then, $f = g - h$

The continuity of g and h are examined first.

$g(x) = |x|$ can be written as $g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

It is evident that g is defined for every real number.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$

$$\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that $x < 0$.

Case II:

If $c > 0$, then $g(c) = c$

$\lim_{x \to c} g(x) = \lim_{x \to c}(x) = c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$.

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-}(-x) = 0$

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+}(x) = 0$

$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+}(x) = g(0)$

Therefore, g is continuous at all $x = 0$.

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = |x + 1|$ can be written as $h(x) = \begin{cases} -(x+1), & \text{if } x < -1 \\ x + 1, & \text{if } x \geq -1 \end{cases}$

It is evident that h is defined for every real number.
Let c be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$

$\lim_{x \to c} h(x) = \lim_{x \to c}[-(x+1)] = -(c+1)$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x < -1$.

Case II:

If $c > -1$, then $h(c) = c + 1$

$\lim_{x \to c} h(x) = \lim_{x \to c}(x+1) = c + 1$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x > -1$.

Case III:

If $c = -1$, then $h(c) = h(-1) = -1 + 1 = 0$

$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-}[-(x+1)] = -(-1+1) = 0$

$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+}(x+1) = (-1+1) = 0$

$\therefore \lim_{x \to -1^-} h(x) = \lim_{x \to -1^+} h(x) = h(-1)$

Therefore, h is continuous at $x = -1$.

From the above three observations, it can be concluded that h is continuous at all points.

It concludes that g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.

EXERCISE 5.2

Question 1:

Differentiate the function with respect to *x*.

$$\sin\left(x^2 + 5\right)$$

Solution:

Let $f(x) = \sin\left(x^2 + 5\right)$, $u(x) = x^2 + 5$ and $v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v\left(x^2 + 5\right) = \tan\left(x^2 + 5\right) = f(x)$

Thus, f is a composite of two functions.

Put $t = u(x) = x^2 + 5$

Then, we get

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos\left(x^2 + 5\right)$$

$$\frac{dt}{dx} = \frac{d}{dx}\left(x^2 + 5\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

By chain rule of derivative,

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos\left(x^2 + 5\right) \times 2x = 2x\cos\left(x^2 + 5\right)$$

Alternate method:

$$\frac{d}{dx}\left[\sin\left(x^2 + 5\right)\right] = \cos\left(x^2 + 5\right) \cdot \frac{d}{dx}\left(x^2 + 5\right)$$

$$= \cos\left(x^2 + 5\right) \cdot \left[\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(5)\right]$$

$$= \cos\left(x^2 + 5\right) \cdot \left[2x + 0\right]$$

$$= 2x\cos\left(x^2 + 5\right)$$

Question 2:

Differentiate the function with respect to *x*

$$\cos(\sin x)$$

Solution:

Let $f(x) = \cos(\sin x)$, $u(x) = \sin x$ and $v(t) = \cos t$

Then, $(vou)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

Here, f is a composite function of two functions.

Put $t = u(x) = \sin x$

$$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$$

$$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

By chain rule,

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$$

Alternate method:

$$\frac{d}{dx}\left[\cos(\sin x)\right] = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x)$$

$$= -\sin(\sin x) \times \cos x$$

$$= -\cos x \sin(\sin x)$$.

Question 3:
Differentiate the function with respect to x

$$\sin(ax + b)$$

Solution:

Let $f(x) = \sin(ax + b), u(x) = ax + b$ and $v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v(ax + b) = \sin(ax + b) = f(x)$

Here, f is a composite function of two functions u and v.

Put, $t = u(x) = ax + b$

Thus,

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Hence, by chain rule, we get

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a \cos(ax + b)$$

Alternate method:

$$\frac{d}{dx}\Big[\sin(ax+b)\Big] = \cos(ax+b).\frac{d}{dx}(ax+b)$$
$$= \cos(ax+b).\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$= \cos(ax+b).(a+0)$$
$$= a\cos(ax+b)$$

Question 4:
Differentiate the function with respect to x

$$\sec\Big(\tan\big(\sqrt{x}\big)\Big)$$

Solution:

Let $f(x) = \sec\Big(\tan\big(\sqrt{x}\big)\Big), u(x) = \sqrt{x}, v(t) = \tan t$ and $w(s) = \sec s$

Then, $(wovou)(x) = w\Big[v(u(x))\Big] = w\Big[v\big(\sqrt{x}\big)\Big] = w\big(\tan\sqrt{x}\big) = \sec\big(\tan\sqrt{x}\big) = f(x)$

Here, f is a composite function of three functions u, v and w.

Put, $s = v(t) = \tan t$ and $t = u(x) = \sqrt{x}$

Then,

$$\frac{dw}{ds} = \frac{d}{ds}(\sec s)$$
$$= \sec s \tan s$$
$$= \sec(\tan t).\tan(\tan t) \qquad [s = \tan t]$$
$$= \sec\big(\tan\sqrt{x}\big).\tan\big(\tan\sqrt{x}\big) \qquad [t = \sqrt{x}]$$

Now,

$$\frac{ds}{dt} = \frac{d}{dt}(\tan t) = \sec^2 t = \sec^2 \sqrt{x}$$

$$\frac{dt}{dx} = \frac{d}{dx}\big(\sqrt{x}\big) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}.x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we get

$$\frac{d}{dx}\Big[\sec\big(\tan\sqrt{x}\big)\Big] = \frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

$$= \sec\big(\tan\sqrt{x}\big)\cdot\tan\big(\tan\sqrt{x}\big)\cdot\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\,\sec\big(\tan\sqrt{x}\big)\tan\big(\tan\sqrt{x}\big)$$

$$= \frac{\sec^2\sqrt{x}\,\sec\big(\tan\sqrt{x}\big)\tan\big(\tan\sqrt{x}\big)}{2\sqrt{x}}$$

Alternate method:

$$\frac{d}{dx}\Big[\sec\big(\tan\sqrt{x}\big)\Big] = \sec\big(\tan\sqrt{x}\big)\cdot\tan\big(\tan\sqrt{x}\big)\cdot\frac{d}{dx}\big(\tan\sqrt{x}\big)$$

$$= \sec\big(\tan\sqrt{x}\big)\cdot\tan\big(\tan\sqrt{x}\big)\cdot\sec^2\big(\sqrt{x}\big)\cdot\frac{d}{dx}\big(\sqrt{x}\big)$$

$$= \sec\big(\tan\sqrt{x}\big)\cdot\tan\big(\tan\sqrt{x}\big)\cdot\sec^2\big(\sqrt{x}\big)\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{\sec\big(\tan\sqrt{x}\big)\cdot\tan\big(\tan\sqrt{x}\big)\cdot\sec^2\big(\sqrt{x}\big)}{2\sqrt{x}}$$

Question 5:
Differentiate the function with respect to x

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Solution:

Given, $f(x) = \dfrac{\sin(ax+b)}{\cos(cx+d)}$, where $g(x) = \sin(ax+b)$ and $h(x) = \cos(cx+d)$

$\therefore f = \dfrac{g'h - gh'}{h^2}$

Consider $g(x) = \sin(ax+b)$

Let $u(x) = ax+b, v(t) = \sin t$

Then $(vou)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = g(x)$

$\therefore g$ is a composite function of two functions, u and v.

Put, $t = u(x) = ax+b$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Thus, by chain rule, we get

$$g' = \frac{dg}{dx} = \frac{dv}{dt}.\frac{dt}{dx} = \cos(ax + b).a = a\cos(ax + b)$$

Consider $h(x) = \cos(cx + d)$

Let $p(x) = cx + d, q(y) = \cos y$

Then, $(qop)(x) = q(p(x)) = q(cx + d) = \cos(cx + d) = h(x)$

$\therefore h$ is a composite function of two functions, p and q.

Put, $y = p(x) = cx + d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx + d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx + d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Using chain rule, we get

$$h' = \frac{dh}{dx} = \frac{dq}{dy}.\frac{dy}{dx}$$

$$= -\sin(cx + d) \times c$$

$$= -c\sin(cx + d)$$

Therefore,

$$f' = \frac{a\cos(ax + b).\cos(cx + d) - \sin(ax + b)\{-c\sin(cx + d)\}}{[\cos(cx + d)]^2}$$

$$= \frac{a\cos(ax + b)}{\cos(cx + d)} + c\sin(ax + b).\frac{\sin(cx + d)}{\cos(cx + d)} \times \frac{1}{\cos(cx + d)}$$

$$= a\cos(ax + b)\sec(cx + d) + c\sin(ax + b)\tan(cx + d)\sec(cx + d)$$

Question 6:
Differentiate the function with respect to x

$$\cos x^3.\sin^2(x^5)$$

Solution:
Given, $\cos x^3.\sin^2(x^5)$

$$\frac{d}{dx}\left[\cos x^3 . \sin^2\left(x^5\right)\right] = \sin^2\left(x^5\right) \times \frac{d}{dx}\left(\cos x^3\right) + \cos x^3 \times \frac{d}{dx}\left[\sin^2\left(x^5\right)\right]$$

$$= \sin^2\left(x^5\right) \times \left(-\sin x^3\right) \times \frac{d}{dx}\left(x^3\right) + \cos x^3 \times 2\sin\left(x^5\right).\frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3 \sin^2\left(x^5\right) \times 3x^2 + 2\sin x^5 \cos x^3 . \cos x^5 \times \frac{d}{dx}\left(x^5\right)$$

$$= -3x^2 \sin x^3 . \sin^2\left(x^5\right) + 2\sin x^5 \cos x^5 \cos x^3 \times 5x^4$$

$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2\left(x^5\right)$$

Question 7:
Differentiate the function with respect to x

$$2\sqrt{\cot\left(x^2\right)}$$

Solution:

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right] = 2.\frac{1}{2\sqrt{\cot\left(x^2\right)}} \times \frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \times -\cos ec^2\left(x^2\right) \times \frac{d}{dx}\left(x^2\right)$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \times \frac{-1}{\sin^2\left(x^2\right)} \times \left(2x\right)$$

$$= \frac{-2x}{\sin x^2 \sqrt{\cos x^2} \sin x^2}$$

$$= \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{2\sin x^2 \cos x^2}}$$

$$= \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{\sin 2x^2}}$$

Question 8:
Differentiate the function with respect to x

$$\cos\left(\sqrt{x}\right)$$

Solution:

Let $f(x) = \cos\left(\sqrt{x}\right)$

Also, let $u(x) = \sqrt{x}$ and, $v(t) = \cos t$

Then,

$$(v o u)(x) = v\big(u(x)\big)$$
$$= v\big(\sqrt{x}\big)$$
$$= \cos \sqrt{x}$$
$$= f(x)$$

Since, f is a composite function of u and v.

$$t = u(x) = \sqrt{x}$$

Then,

$$\frac{dt}{dx} = \frac{d}{dx}\big(\sqrt{x}\big) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2} x^{\frac{-1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$\frac{dv}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

And,
$$= -\sin\big(\sqrt{x}\big)$$

Using chain rule, we get

$$\frac{dt}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

$$= -\sin\big(\sqrt{x}\big) \cdot \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}} \sin\big(\sqrt{x}\big)$$

$$= -\frac{\sin\big(\sqrt{x}\big)}{2\sqrt{x}}$$

Alternate method:

$$\frac{d}{dx}\Big[\cos\big(\sqrt{x}\big)\Big] = -\sin\big(\sqrt{x}\big) \cdot \frac{d}{dx}\big(\sqrt{x}\big)$$

$$= -\sin\big(\sqrt{x}\big) \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= -\sin\sqrt{x} \times \frac{1}{2} x^{\frac{-1}{2}}$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9:

Prove that the function f given by

$f(x) = |x-1|, x \in \mathbf{R}$ is not differentiable at $x = 1$.

Solution:

Given, $f(x) = |x-1|, x \in \mathbf{R}$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$\lim\limits_{h \to 0^-} \dfrac{f(c) - f(c-h)}{h}$ and $\lim\limits_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x = 1$,

Consider LHD at $x = 1$

$$\lim\limits_{h \to 0^-} \frac{f(1) - f(1-h)}{h} = \lim\limits_{h \to 0^-} \frac{f|1-1| - |1-h-1|}{h}$$

$$= \lim\limits_{h \to 0^-} \frac{0 - |h|}{h}$$

$$= \lim\limits_{h \to 0^-} \frac{-h}{h} \qquad (h < 0 \Rightarrow |h| = -h)$$

$$= -1$$

Consider RHD at $x = 1$

$$\lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0^+} \frac{f|1+h-1| - |1-1|}{h}$$

$$= \lim\limits_{h \to 0^+} \frac{|h| - 0}{h}$$

$$= \lim\limits_{h \to 0^+} \frac{h}{h} \qquad (h > 0 \Rightarrow |h| = h)$$

$$= 1$$

Since LHD and RHD at $x = 1$ are not equal,

Therefore, f is not differentiable at $x = 1$.

Question 10:

Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$ is not differentiable at $x = 1$ and $x = 2$.

Solution:

Given, $f(x) = [x], 0 < x < 3$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$\lim\limits_{h \to 0^-} \dfrac{f(c) - f(c-h)}{h}$ and $\lim\limits_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h}$ are finite and equal.

At $x = 1$,

Consider the LHD at $x = 1$

$$\lim\limits_{h \to 0^-} \frac{f(1) - f(1-h)}{h} = \lim\limits_{h \to 0^-} \frac{[1] - [1-h]}{h}$$
$$= \lim\limits_{h \to 0^-} \frac{1 - 0}{h}$$
$$= \lim\limits_{h \to 0^-} \frac{1}{h}$$
$$= \infty$$

Consider RHD at $x = 1$

$$\lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0^+} \frac{[1+h] - [1]}{h}$$
$$= \lim\limits_{h \to 0^+} \frac{1 - 1}{h}$$
$$= \lim\limits_{h \to 0^+} 0$$
$$= 0$$

Since LHD and RHD at $x = 1$ are not equal,

Hence, f is not differentiable at $x = 1$.

To check the differentiability of the given function at $x = 2$,

Consider LHD at $x = 2$

$$\lim\limits_{h \to 0^-} \frac{f(2) - f(2-h)}{h} = \lim\limits_{h \to 0^-} \frac{[2] - [2-h]}{h}$$
$$= \lim\limits_{h \to 0^-} \frac{2 - 1}{h}$$
$$= \lim\limits_{h \to 0^-} \frac{1}{h}$$
$$= \infty$$

Now, consider RHD at $x = 2$

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^+} \frac{2-2}{h}$$

$$= \lim_{h \to 0^+} 0$$

$$= 0$$

Since, LHD and RHD at $x = 2$ are not equal.

Hence, f is not differentiable at $x = 2$.

EXERCISE 5.3

Question 1:

Find $\dfrac{dy}{dx}: 2x + 3y = \sin x$

Solution:

Given, $2x + 3y = \sin x$

Differentiating with respect to x, we get

$\dfrac{d}{dy}(2x + 3y) = \dfrac{d}{dx}(\sin x)$

$\Rightarrow \dfrac{d}{dx}(2x) + \dfrac{d}{dx}(3y) = \cos x$

$\Rightarrow 2 + 3\dfrac{dy}{dx} = \cos x$

$\Rightarrow 3\dfrac{dy}{dx} = \cos x - 2$

$\therefore \dfrac{dx}{dy} = \dfrac{\cos x - 2}{3}$

Question 2:

Find $\dfrac{dy}{dx}: 2x + 3y = \sin y$

Solution:

Given, $2x + 3y = \sin y$

Differentiating with respect to x, we get

$\dfrac{d}{dx}(2x) + \dfrac{d}{dx}(3y) = \dfrac{d}{dx}(\sin y)$

$\Rightarrow 2 + 3\dfrac{dy}{dx} = \cos y\dfrac{dy}{dx}$ $\begin{bmatrix}\text{By using chain rule}\end{bmatrix}$

$\Rightarrow 2 = (\cos y - 3)\dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{\cos y - 3}$

Question 3:

Find $\dfrac{dy}{dx}: ax + by^2 = \cos y$

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Solution:

Given, $ax + by^2 = \cos y$

Differentiating with respect to x, we get

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y)$$

$$\Rightarrow a + b\frac{d}{dx}(y^2) = \frac{d}{dx}(\cos y) \qquad \ldots(1)$$

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx} \; and \; \frac{d}{dx}(\cos y) = -\sin y\frac{dy}{dx} \qquad \ldots(2)$$

From (1) and (2), we obtain

$$a + b \times 2y\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by + \sin y)\frac{dy}{dx} = -a$$

$$\therefore \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

Question 4:

Find $\dfrac{dy}{dx}$: $xy + y^2 = \tan x + y$

Solution:

Given, $xy + y^2 = \tan x + y$

Differentiating with respect to x, we get

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y.\frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \qquad [\text{using product rule and chain rule}]$$

$$\Rightarrow y.1 + x\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

Question 5:

Find $\dfrac{dy}{dx}$: $x^2 + xy + y^2 = 100$

Solution:

Given, $x^2 + xy + y^2 = 100$

Differentiating with respect to x, we get

$$\frac{d}{dx}\left(x^2 + xy + y^2\right) = \frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = 0$$

$$\Rightarrow 2x + \left[y.\frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y.1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + (x + 2y)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Question 6:

Find $\dfrac{dy}{dx}$: $x^3 + x^2y + xy^2 + y^3 = 81$

Solution:

Given, $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating with respect to x, we get

$$\frac{d}{dx}\left(x^3 + x^2y + xy^2 + y^3\right) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^2y\right) + \frac{d}{dx}\left(xy^2\right) + \frac{d}{dx}\left(y^3\right) = 0$$

$$\Rightarrow 3x^2 + \left[y\frac{d}{dx}\left(x^2\right) + x^2\frac{dy}{dx}\right] + \left[y^2\frac{d}{dx}(x) + x\frac{d}{dx}\left(y^2\right)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left[y.2x + x^2\frac{dy}{dx}\right] + \left[y^2.1 + x.2y.\frac{dy}{dx}\right] + 3y^2\frac{dx}{dy} = 0$$

$$\Rightarrow \left(x^2 + 2xy + 3y^2\right)\frac{dy}{dx} + \left(3x^2 + 2xy + y^2\right) = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\left(3x^2 + 2xy + y^2\right)}{\left(x^2 + 2xy + 3y^2\right)}$$

Question 7:

Find $\dfrac{dx}{dy}$: $\sin^2 y + \cos xy = \pi$

Solution:

Given, $\sin^2 y + \cos xy = \pi$

Differentiating with respect to x, we get

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}\left(\cos xy\right) = 0 \qquad\qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}\left(\sin y\right) = 2\sin y \cos y \frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}\left(\cos xy\right) = -\sin xy \frac{d}{dx}\left(xy\right) = -\sin xy \left[y\frac{d}{dx}(x) + x\frac{dy}{dx} \right]$$

$$= -\sin xy \left[y.1 + x\frac{dy}{dx} \right] = -y\sin xy - x\sin xy \frac{dy}{dx} \qquad ...(3)$$

From (1), (2) and (3), we obtain

$$2\sin y \cos y \frac{dy}{dx} + \left(-y\sin xy - x\sin xy \frac{dy}{dx} \right) = 0$$

$$\Rightarrow \left(2\sin y \cos y - x\sin xy \right)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \left(\sin 2y - x\sin xy \right)\frac{dx}{dy} = y\sin xy$$

$$\therefore \frac{dx}{dy} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

Find $\dfrac{dy}{dx}$: $\sin^2 x + \cos^2 y = 1$

Solution:

Given, $\sin^2 x + \cos^2 y = 1$

Differentiating with respect to x, we get

$$\frac{d}{dx}\left(\sin^2 x + \cos^2 y\right) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right) + \frac{d}{dx}\left(\cos^2 y\right) = 0$$

$$\Rightarrow 2\sin x.\frac{d}{dx}\left(\sin x\right) + 2\cos y.\frac{d}{dx}\left(\cos y\right) = 0$$

$$\Rightarrow 2\sin x \cos x + 2\cos y\left(-\sin y\right).\frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}$: $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution:

Given,

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}\left(\sin y\right) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad \dots(1)$$

$$\Rightarrow \cos y\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

The function $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$

By quotient rule, we get

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = \frac{\left(1+x^2\right)\frac{d}{dx}(2x) - 2x.\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$= \frac{\left(1+x^2\right).2 - 2x.[0+2x]}{\left(1+x^2\right)^2}$$

$$= \frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}$$

$$= \frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

Also, $\sin y = \frac{2x}{1+x^2}$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{\left(1+x^2\right)^2 - 4x^2}{\left(1+x^2\right)^2}}$$

$$= \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}$$

$$= \frac{1-x^2}{1+x^2}$$

From (1), (2) and (3), we get

$$\frac{1-x^2}{1+x^2} \times \frac{dy}{dx} = \frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 10:

Find $\frac{dy}{dx}$: $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Solution:

Given, $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

$\Rightarrow \tan y = \left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ ...(1)

Since, we know that

$\Rightarrow \tan y = \left(\dfrac{3\tan\dfrac{y}{3} - \tan^3\dfrac{y}{3}}{1 - 3\tan^2\dfrac{y}{3}}\right)$ (2)

Comparing (1) and (2) we get,

$x = \tan\dfrac{y}{3}$

Differentiating with respect to x, we get

$\dfrac{d}{dx}(x) = \dfrac{d}{dx}\left(\tan\dfrac{y}{3}\right)$

$\Rightarrow 1 = \sec^2\dfrac{y}{3} \cdot \dfrac{d}{dx}\left(\dfrac{y}{3}\right)$

$\Rightarrow 1 = \sec^2\dfrac{y}{3} \cdot \dfrac{1}{3} \cdot \dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3}{\sec^2\dfrac{y}{3}} = \dfrac{3}{1 + \tan^2\dfrac{y}{3}}$

$\therefore \dfrac{dy}{dx} = \dfrac{3}{1 + x^2}$

Question 11:

Find $\dfrac{dy}{dx}$: $y = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right), 0 < x < 1$

Solution:

Given, $y = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$

$$\Rightarrow \cos y = \left(\frac{1-x^2}{1+x^2} \right)$$

$$\Rightarrow \frac{1 - \tan^2 \frac{y}{2}}{1 + \tan^2 \frac{y}{2}} = \frac{1-x^2}{1+x^2}$$

Comparing LHS and RHS, we get

$$\tan \frac{y}{2} = x$$

Differentiating with respect to x, we get

$$\sec^2 \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2} \right) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2 \frac{y}{2} \times \frac{1}{2}\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2 \frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1 + \tan^2 \frac{y}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 12:

Find $\dfrac{dy}{dx}: y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2} \right), 0 < x < 1$

Solution:

Given, $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2} \right)$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2} \right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2} \right) \qquad \dots (1)$$

Using chain rule, we get

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{4x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2}$$

Therefore,

$$\frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx} \qquad \ldots(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{(1+x^2)\cdot\frac{d}{dx}(1-x^2)-(1-x^2)\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2} \qquad \text{[using quotient rule]}$$

$$= \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}$$

$$= \frac{-4x}{(1+x^2)^2} \qquad \ldots(3)$$

From equation (1), (2) and (3), we get

$$\frac{2x}{1+x^2}\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Question 13:

Find $\dfrac{dy}{dx}$: $y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right), -1 < x < 1$

Solution:

Given, $y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\cos y = \left(\frac{2x}{1+x^2}\right)$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y . \frac{dy}{dx} = \frac{(1+x^2). \frac{d}{dx}(2x) - 2x. \frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx} = \frac{(1+x^2)\times 2 - 2x \times 2x}{(1+x^2)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2(1-x^2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \sqrt{\frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2}}\,.\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)}$$

$$\Rightarrow \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}\,\frac{dy}{dx} = \frac{-2(1-x^2)}{(1-x^2)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\,.\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Question 14:

Find $\dfrac{dy}{dx}$: $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$

Solution:

Given, $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y = \left(2x\sqrt{1-x^2}\right)$$

Differentiating with respect to x, we get

$$\cos y.\frac{dy}{dx} = 2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx} = 2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$$

$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}.\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\dfrac{dy}{dx}$: $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right), 0 < x < \dfrac{1}{\sqrt{2}}$

Solution:

Given, $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

$$\Rightarrow y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$$

$$\Rightarrow \sec y = \left(\frac{1}{2x^2 - 1}\right)$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = \sin \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

EXERCISE 5.4

Question 1:

Differentiating the following wrt x: $\dfrac{e^x}{\sin x}$

Solution:

Let $y = \dfrac{e^x}{\sin x}$

By using the quotient rule, we get

$$\frac{dy}{dx} = \frac{\sin x \dfrac{d}{dx}\left(e^x\right) - e^x \dfrac{d}{dx}\left(\sin x\right)}{\sin^2 x}$$

$$= \frac{\sin x.\left(e^x\right) - e^x.\left(\cos x\right)}{\sin^2 x}$$

$$= \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}$$

Question 2:

Differentiating the following $e^{\sin^{-1} x}$

Solution:

Let $y = e^{\sin^{-1} x}$

By using the quotient rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1} x}\right)$$

$$= e^{\sin^{-1} x}.\frac{d}{dx}\left(\sin^{-1} x\right)$$

$$= e^{\sin^{-1} x}.\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

$$= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in \left(-1,1\right)$$

Question 3:

Differentiating the following wrt $x : e^{x^3}$

Solution:

Let $y = e^{x^3}$

By using the quotient rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right)$$

$$= e^{x^3}.\frac{d}{dx}\left(x^3\right)$$

$$= e^{x^3}.3x^2$$

$$= 3x^2 e^{x^3}$$

Question 4:

Differentiate the following wrt $x: \sin\left(\tan^{-1} e^{-x}\right)$

Solution:

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right).\frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$= \cos\left(\tan^{-1} e^{-x}\right).\frac{1}{1+\left(e^{-x}\right)^2}.\frac{d}{dx}\left(e^{-x}\right)$$

$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}.e^{-x}.\frac{d}{dx}(-x)$$

$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \times (-1)$$

$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following wrt $x: \log\left(\cos e^x\right)$

Solution:

Let $y = \log\left(\cos e^x\right)$

By using the chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^x\right)\right]$$

$$= \frac{1}{\cos e^x} \cdot \frac{d}{dx}\left(\cos e^x\right)$$

$$= \frac{1}{\cos e^x} \cdot \left(-\sin e^x\right) \cdot \frac{d}{dx}\left(e^x\right)$$

$$= \frac{-\sin e^x}{\cos e^x} \cdot e^x$$

$$= -e^x \tan e^x, e^x \neq \left(2n+1\right)\frac{\pi}{2}, n \in \mathbf{N}$$

Question 6:

Differentiate the following wrt $x : e^x + e^{x^2} + \ldots + e^{x^5}$

Solution:

$$\frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right)$$

Differentiating wrt x, we get

$$\frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right) = \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$

$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \times \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \times \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \times \frac{d}{dx}\left(x^5\right)\right]$$

$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$

$$= e^x + 2xe^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}$$

Question 7:

Differentiating the following wrt $x : \sqrt{e^{\sqrt{x}}}, x > 0$

Solution:

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

Differentiating wrt x, we get

$$y^2 = e^{\sqrt{x}}$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(e^{\sqrt{x}})$$

$$\Rightarrow 2y \frac{dy}{dx} = e^{\sqrt{x}} \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow 2y \frac{dy}{dx} = e^{\sqrt{x}} \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x > 0$$

Question 8:

Differentiating the following wrt $x : \log(\log x), x > 1$

Solution:

Let $y = \log(\log x)$

By using the chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\big[\log(\log x)\big]$$

$$= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \cdot \frac{1}{x}$$

$$= \frac{1}{x \log x}, x > 1$$

Question 9:

Differentiating the following wrt $x : \dfrac{\cos x}{\log x}, x > 0$

Solution:

Let $y = \dfrac{\cos x}{\log x}$

By using the quotient rule, we get

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x).\log x - \cos x.\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{-\sin x \log x - \cos x.\frac{1}{x}}{(\log x)^2}$$

$$= -\left[\frac{x \log x.\sin x + \cos x}{x(\log x)^2}\right], x > 0$$

Question 10:

Differentiate the following wrt $x : \cos(\log x + e^x), x > 0$

Solution:

Let $y = \cos(\log x + e^x)$

By using the chain rule, we get

$$\frac{dy}{dx} = -\sin[\log x + e^x].\frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x).\left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)\right]$$

$$= -\sin(\log x + e^x).\left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x), x > 0$$

EXERCISE 5.5

Question 1:

Differentiate the function with respect to x: $\cos x . \cos 2x . \cos 3x$

Solution:

Let $y = \cos x . \cos 2x . \cos 3x$

Taking logarithm on both the sides, we obtain

$\log y = \log(\cos x . \cos 2x . \cos 3x)$

$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{\cos x} . \dfrac{d}{dx}(\cos x) + \dfrac{1}{\cos 2x} . \dfrac{d}{dx}(\cos 2x) + \dfrac{1}{\cos 3x} . \dfrac{d}{dx}(\cos 3x)$

$\Rightarrow \dfrac{dy}{dx} = y\left[-\dfrac{\sin x}{\cos x} - \dfrac{\sin 2x}{\cos 2x} . \dfrac{d}{dx}(2x) - \dfrac{\sin 3x}{\cos 3x} . \dfrac{d}{dx}(3x) \right]$

$\therefore \dfrac{dy}{dx} = -\cos x . \cos 2x . \cos 3x \left[\tan x + 2\tan 2x + 3\tan 3x \right]$

Question 2:

Differentiate the function with respect to x: $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution:

Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm on both the sides, we obtain

$\log y = \log \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

$\Rightarrow \log y = \dfrac{1}{2}\log\left[\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right]$

$\Rightarrow \log y = \dfrac{1}{2}\left[\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\} \right]$

$\Rightarrow \log y = \dfrac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5) \right]$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}\cdot\frac{d}{dx}(x-1) + \frac{1}{x-2}\cdot\frac{d}{dx}(x-2) - \frac{1}{x-3}\cdot\frac{d}{dx}(x-3) \\ -\frac{1}{x-4}\cdot\frac{d}{dx}(x-4) - \frac{1}{x-5}\cdot\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Question 3:

Differentiate the function with respect to x: $(\log x)^{\cos x}$

Solution:

Let $y = (\log x)^{\cos x}$

Taking logarithm on both the sides, we obtain
$$\log y = \cos x.\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(\cos x).\log(\log x) + \cos x.\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = -\sin x\log(\log x) + \cos x.\frac{1}{\log x}.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{d}{dx} = y\left[-\sin x\log(\log x) + \frac{\cos x}{\log x}.\frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x\log(\log x)\right]$$

Question 4:

Differentiate the function with respect to x: $x^x - 2^{\sin x}$

Solution:

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u$ and $2^{\sin x} = v$

$\therefore y = u - v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$

$u = x^x$

Taking logarithm on both the sides, we obtain

$\log u = x \log x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{u} \cdot \dfrac{du}{dx} = \left[\dfrac{d}{dx}(x) \times \log x + x \times \dfrac{d}{dx}(\log x) \right]$

$\Rightarrow \dfrac{du}{dx} = u \left[1 \times \log x + x \times \dfrac{1}{x} \right]$

$\Rightarrow \dfrac{du}{dx} = x^x (\log x + 1)$

$\Rightarrow \dfrac{du}{dx} = x^x (1 + \log x)$

$v = 2^{\sin x}$

Taking logarithm on both the sides, we obtain

$\log v = \sin x . \log 2$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{v} \cdot \dfrac{dv}{dx} = \log 2 . \dfrac{d}{dx}(\sin x)$

$\Rightarrow \dfrac{dv}{dx} = v \log 2 \cos x$

$\Rightarrow \dfrac{dv}{dx} = 2^{\sin x} \cos x \log 2$

$\therefore \dfrac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$

Question 5:

Differentiate the function with respect to x: $(x+3)^2 . (x+4)^3 . (x+5)^4$

Solution:

Let $y = (x+3)^2 . (x+4)^3 . (x+5)^4$

Taking logarithm on both the sides, we obtain

$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$

$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$

Differentiating both sides with respect to x, we obtain

$\frac{1}{y} . \frac{dy}{dx} = 2.\frac{1}{x+3}.\frac{d}{dx}(x+3) + 3.\frac{1}{x+4}.\frac{d}{dx}(x+4) + 4.\frac{1}{x+5}.\frac{d}{dx}(x+5)$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$

$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4.\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$

$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4.\left[\dfrac{\begin{array}{l}2(x+4)(x+5)+3(x+3)(x+5)\\ +4(x+3)(x+4)\end{array}}{(x+3)(x+4)(x+5)}\right]$

$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3.\left[\begin{array}{l}2(x^2+9x+20)+3(x^2+8x+15)\\ +4(x^2+7x+12)\end{array}\right]$

$\therefore \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$

Question 6:

Differentiate the function with respect to x: $\left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Solution:

Let $y = \left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Also, let $u = \left(x+\frac{1}{x}\right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$

$\therefore y = u + v$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(1)

Then, $u = \left(x + \dfrac{1}{x} \right)^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log \left(x + \frac{1}{x} \right)^x$$

$$\Rightarrow \log u = x \log \left(x + \frac{1}{x} \right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x) \times \log \left(x + \frac{1}{x} \right) + x \times \frac{d}{dx} \left[\log \left(x + \frac{1}{x} \right) \right]$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = 1 \times \log \left(x + \frac{1}{x} \right) + x \times \frac{1}{\left(x + \frac{1}{x} \right)} \cdot \frac{d}{dx} \left(x + \frac{1}{x} \right)$$

$$\Rightarrow \frac{du}{dx} = u \left[\log \left(x + \frac{1}{x} \right) + \frac{x}{\left(x + \frac{1}{x} \right)} \times \left(1 - \frac{1}{x^2} \right) \right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x} \right)^x \left[\log \left(x + \frac{1}{x} \right) + \frac{\left(x - \frac{1}{x} \right)}{\left(x + \frac{1}{x} \right)} \right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x} \right)^x \left[\log \left(x + \frac{1}{x} \right) + \frac{x^2 - 1}{x^2 + 1} \right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x} \right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x} \right) \right] \qquad \ldots(2)$$

Now, $v = x^{\left(1 + \frac{1}{x} \right)}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log \left[x^{\left(1 + \frac{1}{x} \right)} \right]$$

$$\Rightarrow \log v = \left(1 + \frac{1}{x} \right) \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}.\frac{dv}{dx} = \left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right] \times \log x + \left(1+\frac{1}{x}\right).\frac{d}{dx}\log x$$

$$\Rightarrow \frac{1}{v}.\frac{dv}{dx} = \left(-\frac{1}{x^2}\right)\log x + \left(1+\frac{1}{x}\right).\frac{1}{x}$$

$$\Rightarrow \frac{1}{v}.\frac{dv}{dx} = -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{-\log x + x + 1}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[\frac{-\log x + x + 1}{x^2}\right] \qquad ...(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right]$$

Question 7:

Differentiate the function with respect to x: $\left(\log x\right)^x + x^{\log x}$

Solution:

Let $y = \left(\log x\right)^x + x^{\log x}$

Also, let $u = \left(\log x\right)^x$ and $v = x^{\log x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

Then, $u = \left(\log x\right)^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log\left[\left(\log x\right)^x\right]$$

$$\Rightarrow \log u = x\log\left(\log x\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} . \frac{du}{dx} = \frac{d}{dx}(x) \times \log(\log x) + x . \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log(\log x) + x . \frac{1}{(\log x)} . \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{x}{(\log x)} . \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{(\log x)}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\frac{\log(\log x) . \log x + 1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x-1} \left[1 + \log x . \log(\log x)\right] \qquad \dots (2)$$

$$v = x^{\log x}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log\left(x^{\log x}\right)$$

$$\Rightarrow \log v = \log x \log x = (\log x)^2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} . \frac{dv}{dx} = \frac{d}{dx}\left[(\log x)^2\right]$$

$$\Rightarrow \frac{1}{v} . \frac{dv}{dx} = 2(\log x) . \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx} = 2v(\log x) . \frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x} \frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x - 1} . \log x \qquad \dots (3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = (\log x)^{x-1}\left[1 + \log x . \log(\log x)\right] + 2x^{\log x - 1} . \log x$$

Question 8:

Differentiate the function with respect to x: $(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution:

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(1)

Then, $u = (\sin x)^x$

Taking logarithm on both the sides, we obtain

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x \log(\sin x)$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{u}.\dfrac{du}{dx} = \dfrac{d}{dx}(x) \times \log(\sin x) + x.\dfrac{d}{dx}\big[\log(\sin x)\big]$

$\Rightarrow \dfrac{du}{dx} = u\left[1 \times \log(\sin x) + x.\dfrac{1}{(\sin x)}.\dfrac{d}{dx}(\sin x)\right]$

$\Rightarrow \dfrac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \dfrac{x}{(\sin x)}.\cos x\right]$

$\Rightarrow \dfrac{du}{dx} = (\sin x)^x\big[x\cot x + \log\sin x\big]$ ...(2)

$v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$\dfrac{dv}{dx} = \dfrac{1}{\sqrt{1 - \left(\sqrt{x}\right)^2}}.\dfrac{d}{dx}\left(\sqrt{x}\right)$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{1}{\sqrt{1-x}}.\dfrac{1}{2\sqrt{x}}$

$\Rightarrow \dfrac{dv}{dx} = \dfrac{1}{2\sqrt{x - x^2}}$ ...(3)

Therefore, from (1), (2) and (3);

$\dfrac{dy}{dx} = (\sin x)^x\big[x\cot x + \log\sin x\big] + \dfrac{1}{2\sqrt{x - x^2}}$

Question 9:

Differentiate the function with respect to x: $x^{\sin x} + (\sin x)^{\cos x}$

Solution:

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also, let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$\therefore y = u + v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} \qquad \ldots (1)$

Then, $u = x^{\sin x}$

Taking logarithm on both the sides, we obtain

$\Rightarrow \log u = \log\left(x^{\sin x}\right)$

$\Rightarrow \log u = \sin x \log x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{u} \cdot \dfrac{du}{dx} = \dfrac{d}{dx}(\sin x) \cdot \log x + \sin x \cdot \dfrac{d}{dx}(\log x)$

$\Rightarrow \dfrac{du}{dx} = u\left[\cos x \log x + \sin x \cdot \dfrac{1}{x}\right]$

$\Rightarrow \dfrac{du}{dx} = x^{\sin x}\left[\cos x \log x + \dfrac{\sin x}{x}\right] \qquad \ldots (2)$

$v = (\sin x)^{\cos x}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log (\sin x)^{\cos x}$$

$$\Rightarrow \log v = \cos x \log (\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cos x) \times \log (\sin x) + \cos x \times \frac{d}{dx}\left[\log (\sin x) \right]$$

$$\Rightarrow \frac{dv}{dx} = v \left[-\sin x . \log (\sin x) + \cos x . \frac{1}{\sin x} . \frac{d}{dx}(\sin x) \right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x} \left[-\sin x \log (\sin x) + \frac{\cos x}{\sin x} \cos x \right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x} \left[-\sin x \log (\sin x) + \cot x \cos x \right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x} \left[\cot x \cos x - \sin x \log (\sin x) \right] \qquad \dots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = x^{\sin x} \left[\cos x \log x + \frac{\sin x}{x} \right] + (\sin x)^{\cos x} \left[\cot x \cos x - \sin x \log (\sin x) \right]$$

Question 10:

Differentiate the function with respect to x: $x^{x \cos x} + \dfrac{x^2 + 1}{x^2 - 1}$

Solution:

Let $y = x^{x \cos x} + \dfrac{x^2 + 1}{x^2 - 1}$

Also, let $u = x^{x \cos x}$ and $v = \dfrac{x^2 + 1}{x^2 - 1}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \dots(1)$$

Then, $u = x^{x \cos x}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log \left(x^{x \cos x} \right)$$

$$\Rightarrow \log u = x \cos x \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x) \cdot \cos x \cdot \log x + x \cdot \frac{d}{dx}(\cos x) \cdot \log x + x \cos x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \cdot \cos x \cdot \log x + x \cdot (-\sin x) \log x + x \cos x \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^{x \cos x} \left[\cos x \log x - x \cdot \sin x \log x + \cos x \right]$$

$$\Rightarrow \frac{du}{dx} = x^{x \cos x} \left[\cos x (1 + \log x) - x \cdot \sin x \log x \right] \qquad \ldots(2)$$

$$v = \frac{x^2 + 1}{x^2 - 1}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log(x^2 + 1) - \log(x^2 - 1)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \frac{dv}{dx} = \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1}$$

$$\Rightarrow \frac{dv}{dx} = v \left[\frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 + 1)(x^2 - 1)} \right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{x^2 + 1}{x^2 - 1} \times \left[\frac{-4x}{(x^2 + 1)(x^2 - 1)} \right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2 - 1)^2} \qquad \ldots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = x^{x \cos x} \left[\cos x (1 + \log x) - x \cdot \sin x \log x \right] - \frac{4x}{(x^2 - 1)^2}$$

Question 11:

Differentiate the function with respect to x: $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Solution:

Let $y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Also, let $u = (x \cos x)^x$ and $v = (x \sin x)^{\frac{1}{x}}$

$\therefore y = u + v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(1)

Then, $u = (x \cos x)^x$

Taking logarithm on both the sides, we obtain

$\Rightarrow \log u = (x \cos x)^x$

$\Rightarrow \log u = x \log (x \cos x)$

$\Rightarrow \log u = x [\log x + \log \cos x]$

$\Rightarrow \log u = x \log x + x \log \cos x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{u} \cdot \dfrac{du}{dx} = \dfrac{d}{dx}(x \log x) + \dfrac{d}{dx}(x \log \cos x)$

$\Rightarrow \dfrac{du}{dx} = u \left[\left\{ \log x \cdot \dfrac{d}{dx}(x) + x \cdot \dfrac{d}{dx}(\log x) \right\} + \left\{ \log \cos x \cdot \dfrac{d}{dx}(x) + x \cdot \dfrac{d}{dx}(\log \cos x) \right\} \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[\left(\log x \cdot 1 + x \cdot \dfrac{1}{x} \right) + \left\{ \log \cos x \cdot 1 + x \cdot \dfrac{1}{\cos x} \cdot \dfrac{d}{dx}(\cos x) \right\} \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[(\log x + 1) + \left\{ \log \cos x + \dfrac{x}{\cos x} \cdot (-\sin x) \right\} \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[(1 + \log x) + (\log \cos x - x \tan x) \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[(1 - x \tan x) + (\log x + \log \cos x) \right]$

$\Rightarrow \dfrac{du}{dx} = (x \cos x)^x \left[1 - x \tan x + \log (x \cos x) \right]$ ...(2)

$v = (x \sin x)^{\frac{1}{x}}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log \left(x \sin x \right)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x} \log \left(x \sin x \right)$$

$$\Rightarrow \log v = \frac{1}{x} \left(\log x + \log \sin x \right)$$

$$\Rightarrow \log v = \frac{1}{x} \log x + \frac{1}{x} \log \sin x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{x} \log x \right) + \frac{d}{dx} \left[\frac{1}{x} \log \left(\sin x \right) \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left[\log x . \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} . \frac{d}{dx} \left(\log x \right) \right] + \left[\log \left(\sin x \right) . \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} . \frac{d}{dx} \left\{ \log \left(\sin x \right) \right\} \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \left[\log x . \left(-\frac{1}{x^2} \right) + \frac{1}{x} . \frac{1}{x} \right] + \left[\log \left(\sin x \right) . \left(-\frac{1}{x^2} \right) + \frac{1}{x} . \frac{1}{\sin x} . \frac{d}{dx} \left(\sin x \right) \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \frac{1}{x^2} \left(1 - \log x \right) + \left[-\frac{\log \left(\sin x \right)}{x^2} + \frac{1}{x \sin x} . \cos x \right]$$

$$\Rightarrow \frac{dv}{dx} = \left(x \sin x \right)^{\frac{1}{x}} \left[\frac{1 - \log x}{x^2} + \frac{-\log \left(\sin x \right) + x \cot x}{x^2} \right]$$

$$\Rightarrow \frac{dv}{dx} = \left(x \sin x \right)^{\frac{1}{x}} \left[\frac{1 - \log x - \log \left(\sin x \right) + x \cot x}{x^2} \right]$$

$$\Rightarrow \frac{dv}{dx} = \left(x \sin x \right)^{\frac{1}{x}} \left[\frac{1 - \log \left(x \sin x \right) + x \cot x}{x^2} \right] \qquad \dots (3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = \left(x \cos x \right)^x \left[1 - x \tan x + \log \left(x \cos x \right) \right] + \left(x \sin x \right)^{\frac{1}{x}} \left[\frac{x \cot x + 1 - \log \left(x \sin x \right)}{x^2} \right]$$

Question 12:

Find $\dfrac{dy}{dx}$ of the function $x^y + y^x = 1$

Solution:

The given function is $x^y + y^x = 1$

Let, $x^y = u$ and $y^x = v$

$\therefore u + v = 1$

$\Rightarrow \dfrac{du}{dx} + \dfrac{dv}{dx} = 0$ \qquad ...(1)

Then, $u = x^y$

Taking logarithm on both the sides, we obtain

$\Rightarrow \log u = \log\left(x^y\right)$

$\Rightarrow \log u = y \log x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{u}\dfrac{du}{dx} = \log x \dfrac{dy}{dx} + y.\dfrac{d}{dx}\left(\log x\right)$

$\Rightarrow \dfrac{du}{dx} = u\left[\log x \dfrac{dy}{dx} + y.\dfrac{1}{x}\right]$

$\Rightarrow \dfrac{du}{dx} = x^y\left[\log x \dfrac{dy}{dx} + \dfrac{y}{x}\right]$ \qquad ...(2)

Now, $v = y^x$

Taking logarithm on both the sides, we obtain

$\Rightarrow \log v = \log\left(y^x\right)$

$\Rightarrow \log v = x \log y$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{v}\dfrac{dv}{dx} = \log y.\dfrac{d}{dx}\left(x\right) + x.\dfrac{d}{dx}\left(\log y\right)$

$\Rightarrow \dfrac{dv}{dx} = v\left[\log y.1 + x.\dfrac{1}{y}.\dfrac{dy}{dx}\right]$

$\Rightarrow \dfrac{dv}{dx} = y^x\left[\log y + \dfrac{x}{y}.\dfrac{dy}{dx}\right]$ \qquad ...(3)

Therefore, from (1), (2) and (3);

$$x^y \left[\log x \frac{dy}{dx} + \frac{y}{x} \right] + y^x \left[\log y + \frac{x}{y} \cdot \frac{dy}{dx} \right] = 0$$

$$\Rightarrow \left(x^y \log x + xy^{x-1} \right) \frac{dy}{dx} = -\left(yx^{y-1} + y^x \log y \right)$$

$$\therefore \frac{dy}{dx} = \frac{-\left(yx^{y-1} + y^x \log y \right)}{\left(x^y \log x + xy^{x-1} \right)}$$

Question 13:

Find $\dfrac{dy}{dx}$ of the function $y^x = x^y$

Solution:

The given function is $y^x = x^y$

Taking logarithm on both the sides, we obtain

$$x \log y = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y) = \log x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x}$$

$$\Rightarrow \log y + \frac{x}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + \frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y} - \log x \right) \frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \left(\frac{x - y \log x}{y} \right) \frac{dy}{dx} = \frac{y - x \log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \left(\frac{y - x \log y}{x - y \log x} \right)$$

Question 14:

Find $\dfrac{dy}{dx}$ of the function $\left(\cos x \right)^y = \left(\cos y \right)^x$

Solution:

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$y \log \cos x = x \log \cos y$

Differentiating both sides with respect to x, we obtain

$\log \cos x . \dfrac{dy}{dx} + y . \dfrac{d}{dx}(\log \cos x) = \log \cos y . \dfrac{d}{dx}(x) + x . \dfrac{d}{dx}(\log \cos y)$

$\Rightarrow \log \cos x . \dfrac{dy}{dx} + y . \dfrac{1}{\cos x} . \dfrac{d}{dx}(\cos x) = \log \cos y . 1 + x . \dfrac{1}{\cos y} \dfrac{d}{dx}(\cos y)$

$\Rightarrow \log \cos x . \dfrac{dy}{dx} + \dfrac{y}{\cos x} . (-\sin x) = \log \cos y + \dfrac{x}{\cos y} . (-\sin y) . \dfrac{dy}{dx}$

$\Rightarrow \log \cos x . \dfrac{dy}{dx} - y \tan x = \log \cos y - x \tan y \dfrac{dy}{dx}$

$\Rightarrow (\log \cos x + x \tan y)\dfrac{dy}{dx} = y \tan x + \log \cos y$

$\therefore \dfrac{dy}{dx} = \dfrac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$

Question 15:

Find $\dfrac{dy}{dx}$ of the function $xy = e^{(x-y)}$

Solution:

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$\log(xy) = \log\left(e^{x-y}\right)$

$\Rightarrow \log x + \log y = (x - y)\log e$

$\Rightarrow \log x + \log y = (x - y) \times 1$

$\Rightarrow \log x + \log y = (x - y)$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right)\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

Solution:

The given function is $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)} \cdot \frac{d}{dx}\left[f(x)\right] = \frac{d}{dx} \log(1+x) + \frac{d}{dx} \log(1+x^2)$$

$$+ \frac{d}{dx} \log(1+x^4) + \frac{d}{dx} \log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)} \cdot f'(x) = \frac{1}{1+x} \cdot \frac{d}{dx}(1+x) + \frac{1}{1+x^2} \cdot \frac{d}{dx}(1+x^2)$$

$$+ \frac{1}{1+x^4} \cdot \frac{d}{dx}(1+x^4) + \frac{1}{1+x^8} \cdot \frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{1+x} + \frac{1}{1+x^2} \cdot 2x + \frac{1}{1+x^4} \cdot 4x^3 + \frac{1}{1+x^8} \cdot 8x^7\right]$$

$$\therefore f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Hence,

$$f'(1) = (1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1} + \frac{2(1)}{1+1^2} + \frac{4(1)^3}{1+1^4} + \frac{8(1)^7}{1+1^8}\right]$$

$$= 2 \times 2 \times 2 \times 2 \left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$$

$$= 16\left(\frac{15}{2}\right)$$

$$= 120$$

Question 17:

Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below.

   (i)   By using product rule
   (ii) By expanding the product to obtain a single polynomial.
   (iii) By logarithmic differentiation.

Do they all give the same answer?

Solution:

Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

   (i)   By using product rule

Let $u = \left(x^2 - 5x + 8\right)$ and $v = x^3 + 7x + 9$

$\therefore y = uv$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx}.v + u.\dfrac{dv}{dx}$ (product rule)

$\Rightarrow \dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^2 - 5x + 8\right).\left(x^3 + 7x + 9\right) + \left(x^2 - 5x + 8\right).\dfrac{d}{dx}\left(x^3 + 7x + 9\right)$

$\Rightarrow \dfrac{dy}{dx} = \left(2x - 5\right)\left(x^3 + 7x + 9\right) + \left(x^2 - 5x + 8\right)\left(3x^2 + 7\right)$

$\Rightarrow \dfrac{dy}{dx} = 2x\left(x^3 + 7x + 9\right) - 5\left(x^3 + 7x + 9\right) + x^2\left(3x^2 + 7\right)$

$-5x\left(3x^2 + 7\right) + 8\left(3x^2 + 7\right)$

$\Rightarrow \dfrac{dy}{dx} = \left(2x^4 + 14x^2 + 18x\right) - 5x^3 - 35x - 45 + \left(3x^4 + 7x^2\right)$

$\Rightarrow \dfrac{dy}{dx} = -15x^3 - 35x + 24x^2 + 56$

$\therefore \dfrac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$

(ii)   By expanding the product to obtain a single polynomial.

$y = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)$

$= x^2\left(x^3 + 7x + 9\right) - 5x\left(x^3 + 7x + 9\right) + 8\left(x^3 + 7x + 9\right)$

$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$

$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

Therefore,

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72\right)$

$= \dfrac{d}{dx}\left(x^5\right) - 5\dfrac{d}{dx}\left(x^4\right) + 15\dfrac{d}{dx}\left(x^3\right) - 26\dfrac{d}{dx}\left(x^2\right) + 11\dfrac{d}{dx}\left(x\right) + \dfrac{d}{dx}\left(72\right)$

$= 5x^4 - 5\left(4x^3\right) + 15\left(3x^2\right) - 26\left(2x\right) + 11\left(1\right) + 0$

$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$

(iii)   By logarithmic differentiation.

$y = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)$

Taking logarithm on both the sides, we obtain

$\log y = \log\left(x^2 - 5x + 8\right) + \log\left(x^3 + 7x + 9\right)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}\log\left(x^2 - 5x + 8\right) + \frac{d}{dx}\log\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}.\frac{d}{dx}\left(x^2 - 5x + 8\right) + \frac{1}{x^3 + 7x + 9}.\frac{d}{dx}\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x^2 - 5x + 8}.\left(2x - 5\right) + \frac{1}{x^3 + 7x + 9}.\left(3x^2 + 7\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{\left(2x - 5\right)\left(x^3 + 7x + 9\right) + \left(3x^2 + 7\right)\left(x^2 - 5x + 8\right)}{\left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2x\left(x^3 + 7x + 9\right) - 5\left(x^3 + 7x + 9\right) + 3x^2\left(x^2 - 5x + 8\right) + 7\left(x^2 - 5x + 8\right)$$

$$\Rightarrow \frac{dy}{dx} = 2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^5 - 15x^3 + 24x^2 + 7x^2 - 35x + 56$$

$$\Rightarrow \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:
If u, v and w are functions of x, then show that

$$\frac{d}{dx}\left(u.v.w\right) = \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

Solution:

Let $y = u.v.w = u.\left(v.w\right)$

By applying product rule, we get

$$\frac{dy}{dx} = \frac{du}{dx}.\left(v.w\right) + u.\frac{d}{dx}\left(v.w\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}.\left(v.w\right) + u\left[\frac{dv}{dx}.w + v.\frac{dw}{dx}\right] \qquad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$$

Taking logarithm on both the sides of the equation $y = u.v.w$, we obtain

$\log y = \log u + \log v + \log w$

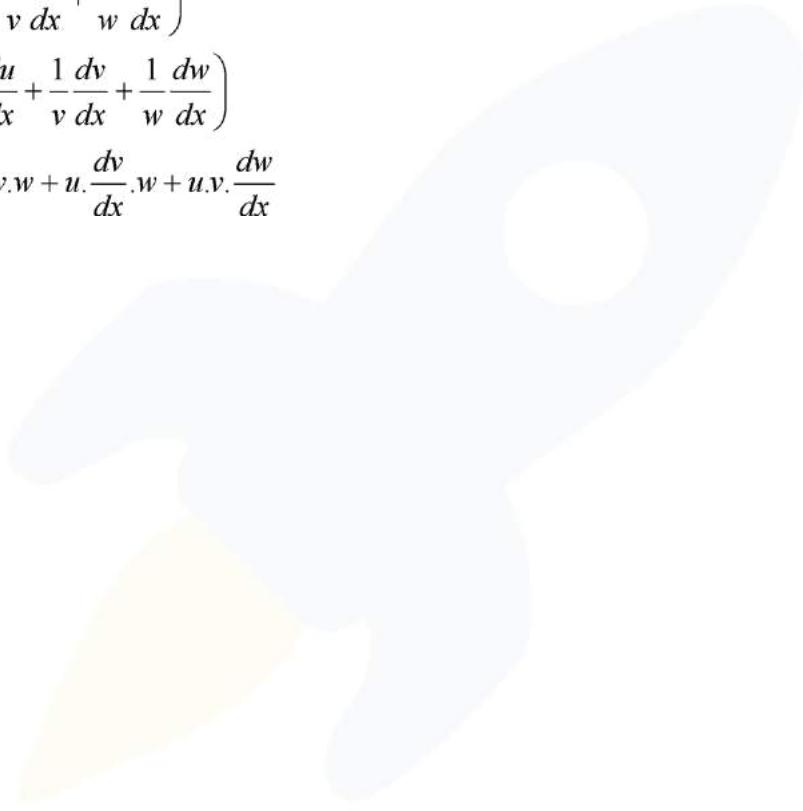Differentiating both sides with respect to x , we obtain

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$$

EXERCISE 5.6

Question 1:

If x and y are connected parametrically by the equations $x = 2at^2, y = at^4$, without eliminating the parameter, find $\dfrac{dy}{dx}$

Solution:

Given, $x = 2at^2, y = at^4$
Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a.\frac{d}{dt}\left(t^2\right) = 2a.2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a.\frac{d}{dt}\left(t^4\right) = a.4t^3 = 4at^3$$

$$\therefore \frac{dy}{dt} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If x and y are connected parametrically by the equations $x = a\cos\theta, y = b\cos\theta$, without eliminating the parameter, find $\dfrac{dy}{dx}$

Solution:

Given, $x = a\cos\theta, y = b\cos\theta$
Then,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) = a(-\sin\theta) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\cos\theta) = b(-\sin\theta) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

If x and y are connected parametrically by the equations $x = \sin t, y = \cos 2t$, without eliminating the parameter, find $\dfrac{dy}{dx}$

Solution:

Given, $x = \sin t, y = \cos 2t$

Then, $\dfrac{dx}{dt} = \dfrac{d}{dt}(\sin t) = \cos t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(\cos 2t) = -\sin 2t \cdot \dfrac{d}{dt}(2t) = -2\sin 2t$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{-2\sin 2t}{\cos t} = \dfrac{-2 \cdot 2\sin t \cos t}{\cos t} = -4\sin t$

Question 4:

If x and y are connected parametrically by the equations $x = 4t, y = \dfrac{4}{t}$, without eliminating the parameter, find $\dfrac{dy}{dx}$

Solution:

Given, $x = 4t, y = \dfrac{4}{t}$

$\dfrac{dx}{dt} = \dfrac{d}{dt}(4t) = 4$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\left(\dfrac{4}{t}\right) = 4 \cdot \dfrac{d}{dt}\left(\dfrac{1}{t}\right) = 4 \cdot \left(\dfrac{-1}{t^2}\right) = \dfrac{-4}{t^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \dfrac{\left(\dfrac{-4}{t^2}\right)}{4} = \dfrac{-1}{t^2}$

Question 5:

If x and y are connected parametrically by the equations $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$, without eliminating the parameter, find $\dfrac{dy}{dx}$

Solution:

Given, $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$

Then,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta - \cos 2\theta) = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}(\cos 2\theta)$$

$$= -\sin\theta - (-2\sin 2\theta) = 2\sin 2\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta) = \frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\sin 2\theta)$$

$$= \cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

Question 6:

If x and y are connected parametrically by the equations $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, without eliminating the parameter, find $\dfrac{dy}{dx}$

Solution:

Given, $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$

Then, $\dfrac{dx}{d\theta} = a\left[\dfrac{d}{d\theta}(\theta) - \dfrac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a[0 + (-\sin\theta)] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2\sin^2\dfrac{\theta}{2}} = \frac{-\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

If x and y are connected parametrically by the equations $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$, without eliminating the parameter, find $\dfrac{dy}{dx}$

Solution:

Given, $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}, y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t}.\dfrac{d}{dt}\left(\sin^3 t\right) - \sin^3 t.\dfrac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.3\sin^2 t.\dfrac{d}{dt}\left(\sin t\right) - \sin^3 t \times \dfrac{1}{2\sqrt{\cos 2t}}.\dfrac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}.\sin^2 t.\cos t - \dfrac{\sin^3 t}{2\sqrt{\cos 2t}}.\left(-2\sin 2t\right)}{\cos 2t}$$

$$= \frac{3\cos 2t.\sin^2 t \cos t + \sin^3 t.\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t}.\dfrac{d}{dt}\left(\cos^3 t\right) - \cos^3 t.\dfrac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.3\cos^2 t.\dfrac{d}{dt}\left(\cos t\right) - \cos^3 t.\dfrac{1}{2\sqrt{\cos 2t}}.\dfrac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}.\cos^2 t\left(-\sin t\right) - \cos^3 t.\dfrac{1}{\sqrt{\cos 2t}}.\left(-2\sin 2t\right)}{\cos 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t.\sin 2t}{\cos 2t.\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{\dfrac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{\cos 2t.\sqrt{\cos 2t}}}{\dfrac{3\cos 2t.\sin^2 t.\cos t + \sin^3 t \sin 2t}{\cos 2t.\sqrt{\cos 2t}}}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{3\cos 2t.\sin^2 t.\cos t + \sin^3 t \sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \left(2\sin t \cos t\right)}{3\cos 2t.\sin^2 t.\cos t + \sin^3 t \left(2\sin t \cos t\right)}$$

$$= \frac{\sin t \cos t \left[-3\cos 2t.\cos t + 2\cos^3 t\right]}{\sin t \cos t \left[3\cos 2t \sin t + 2\sin^3 t\right]}$$

$$= \frac{\left[-3\left(2\cos^2 t - 1\right)\cos t + 2\cos^3 t\right]}{\left[3\left(1 - 2\sin^2 t\right)\sin t + 2\sin^3 t\right]} \qquad \begin{bmatrix} \cos 2t = \left(2\cos^2 t - 1\right) \\ \cos 2t = \left(1 - 2\sin^2 t\right) \end{bmatrix}$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t} \qquad \begin{bmatrix} \cos 3t = 4\cos^3 t - 3\cos t \\ \sin 3t = 3\sin t - 4\sin^2 t \end{bmatrix}$$

$$= \frac{-\cos 3t}{\sin 3t} = -\cot 3t$$

Question 8:

If x and y are connected parametrically by the equations $x = a\left(\cos t + \log \tan \dfrac{t}{2}\right), y = a\sin t$, without eliminating the parameter, find $\dfrac{dy}{dx}$

Solution:

Given, $x = a\left(\cos t + \log \tan \dfrac{t}{2}\right), y = a\sin t$

Then,

$$\frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2}\right) \right]$$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2}\right) \right]$$

$$= a\left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2}\right) \right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2} \right]$$

$$= a\left[-\sin t + \frac{1}{2\sin \frac{t}{2}\cos \frac{t}{2}} \right]$$

$$= a\left(-\sin t + \frac{1}{\sin t} \right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t} \right)$$

$$= a\left(\frac{\cos^2 t}{\sin t} \right)$$

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a\cos t$$

Therefore,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t} \right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

If x and y are connected parametrically by the equations $x = a\sec\theta, y = b\tan\theta$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a\sec\theta, y = b\tan\theta$
Then,

$$\frac{dx}{d\theta} = a.\frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b.\frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$$

Therefore,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b}{a}\sec\theta\cot\theta$$

$$= \frac{b\cos\theta}{a\cos\theta\sin\theta}$$

$$= \frac{b}{a}\times\frac{1}{\sin\theta}$$

$$= \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

If x and y are connected parametrically by the equations $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$
Then,

$$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right]$$

$$= a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta]$$

$$= a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta.\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$$= a\theta\sin\theta$$

Therefore,

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)}$$

$$= \frac{a\theta \sin \theta}{a\theta \cos \theta}$$

$$= \tan \theta$$

Question 11:

If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$, show that $\dfrac{dy}{dx} = -\dfrac{y}{x}$

Solution:

Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

Hence,

$$x = \sqrt{a^{\sin^{-1} t}} = \left(a^{\sin^{-1} t}\right)^{\frac{1}{2}} = a^{\frac{1}{2}\sin^{-1} t} \quad \text{and} \quad y = \sqrt{a^{\cos^{-1} t}} = \left(a^{\cos^{-1} t}\right)^{\frac{1}{2}} = a^{\frac{1}{2}\cos^{-1} t}$$

Consider $x = a^{\frac{1}{2}\sin^{-1} t}$

Taking log on both sides, we get

$$\log x = \frac{1}{2}\sin^{-1} t \log a$$

Therefore,

$$\Rightarrow \frac{1}{x}.\frac{dx}{dt} = \frac{1}{2}\log a.\frac{d}{dt}\left(\sin^{-1} t\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x}{2}\log a.\frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{x \log a}{2\sqrt{1-t^2}}$$

Now, $y = a^{\frac{1}{2}\cos^{-1} t}$

Taking log on both sides, we get

$$\log x = \frac{1}{2}\cos^{-1} t \log a$$

Therefore,

$$\Rightarrow \frac{1}{y}.\frac{dy}{dt} = \frac{1}{2}\log a.\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dy}{dt} = \frac{y}{2}\log a.\frac{-1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-y\log a}{2\sqrt{1-t^2}}$$

Hence,

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{\left(\dfrac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\dfrac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}$$

EXERCISE 5.7

Question 1:

Find the second order derivative of the function $x^2 + 3x + 2$

Solution:

Consider, $y = x^2 + 3x + 2$
Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2)$$
$$= 2x + 3 + 0$$
$$= 2x + 3$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3)$$
$$= \frac{d}{dx}(2x) + \frac{d}{dx}(3)$$
$$= 2 + 0$$
$$= 2$$

Question 2:

Find the second order derivative of the function x^{20}

Solution:

Consider, $y = x^{20}$
Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^{20})$$
$$= 20x^{19}$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19})$$
$$= 20\frac{d}{dx}(x^{19})$$
$$= 20.19.x^{18}$$
$$= 380x^{18}$$

Question 3:

Find the second order derivative of the function $x \cos x$

Solution:

Consider, $y = x \cos x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x.\cos x)$$

$$= \cos x.\frac{d}{dx}(x) + x\frac{d}{dx}(\cos x)$$

$$= \cos x.1 + x(-\sin x)$$

$$= \cos x - x \sin x$$

Therefore,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}[\cos x - x \sin x]$$

$$= \frac{d}{dx}(\cos x) - \frac{d}{dx}(x \sin x)$$

$$= -\sin x - \left[\sin x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\sin x)\right]$$

$$= -\sin x - (\sin x + x \cos x)$$

$$= -(x \cos x + 2 \sin x)$$

Question 4:

Find the second order derivative of the function $\log x$

Solution:

Let $y = \log x$
Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x)$$

$$= \frac{1}{x}$$

Therefore,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{-1}{x^2}$$

Question 5:

Find the second order derivative of the function $x^3 \log x$

Solution:

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right]$$

$$= \log x . \frac{d}{dx}(x^3) + x^3 . \frac{d}{dx}(\log x)$$

$$= \log x . 3x^2 + x^3 . \frac{1}{x} = \log x . 3x^2 + x^2$$

$$= x^2(1 + 3\log x)$$

Therefore,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left[x^2(1 + 3\log x)\right]$$

$$= (1 + 3\log x) . \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(1 + 3\log x)$$

$$= (1 + 3\log x) . 2x + x^2 . \frac{3}{x}$$

$$= 2x + 6\log x + 3x$$

$$= 5x + 6x \log x$$

$$= x(5 + 6\log x)$$

Question 6:

Find the second order derivative of the function $e^x \sin 5x$

Solution:

Let $y = e^x \sin 5x$
Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x \sin 5x\right)$$

$$= \sin 5x \times \frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}\left(\sin 5x\right)$$

$$= \sin 5x.e^x + e^x.\cos 5x.\frac{d}{dx}\left(5x\right)$$

$$= e^x \sin 5x + e^x \cos 5x.5$$

$$= e^x \left(\sin 5x + 5\cos 5x\right)$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^x \left(\sin 5x + 5\cos 5x\right)\right]$$

$$= \left(\sin 5x + 5\cos 5x\right).\frac{d}{dx}\left(e^x\right) + e^x.\frac{d}{dx}\left(\sin 5x + 5\cos 5x\right)$$

$$= \left(\sin 5x + 5\cos 5x\right)e^x + e^x\left[\cos 5x.\frac{d}{dx}\left(5x\right) + 5\left(-\sin 5x\right).\frac{d}{dx}\left(5x\right)\right]$$

$$= e^x \left(\sin 5x + 5\cos 5x\right) + e^x \left(5\cos 5x - 25\sin 5x\right)$$

$$= e^x \left(10\cos 5x - 24\sin 5x\right)$$

$$= 2e^x \left(5\cos 5x - 12\sin 5x\right)$$

Question 7:

Find the second order derivative of the function $e^{6x}\cos 3x$

Solution:

Let $y = e^{6x}\cos 3x$
Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cos 3x\right) = \cos 3x.\frac{d}{dx}\left(e^{6x}\right) + e^{6x}.\frac{d}{dx}\left(\cos 3x\right)$$

$$= \cos 3x.e^{6x}.\frac{d}{dx}\left(6x\right) + e^{6x}.\left(-\sin 3x\right).\frac{d}{dx}\left(3x\right)$$

$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \qquad\qquad ...(1)$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6.\frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3.\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$= 6.\left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3.\left[\sin 3x.\frac{d}{dx}\left(e^{6x}\right) + e^{6x}.\frac{d}{dx}\left(\sin 3x\right)\right] \qquad \left[\text{using } (1)\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x.e^{6x}.6 + e^{6x}.\cos 3x.3\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Question 8:

Find the second order derivative of the function $\tan^{-1} x$

Solution:

Let $y = \tan^{-1} x$
Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right)$$

$$= \frac{1}{1+x^2}$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}\left(1+x^2\right)^{-1}$$

$$= (-1).\left(1+x^2\right)^{-2}.\frac{d}{dx}\left(1+x^2\right) = \frac{-1}{\left(1+x^2\right)^2} \times 2x$$

$$= \frac{-2x}{\left(1+x^2\right)^2}$$

Question 9:

Find the second order derivative of the function $\log\left(\log x\right)$

Solution:

Consider, $y = \log\left(\log x\right)$
Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$

$$= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \cdot \frac{1}{x} = (x \log x)^{-1}$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[(x \log x)^{-1}\right]$$

$$= (-1) \cdot (x \log x)^{-2} \frac{d}{dx}(x \log x)$$

$$= \frac{-1}{(x \log x)^2} \cdot \left[\log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)\right]$$

$$= \frac{-1}{(x \log x)^2} \cdot \left[\log x \cdot 1 + x \cdot \frac{1}{x}\right]$$

$$= \frac{-(1 + \log x)}{(x \log x)^2}$$

Question 10:

Find the second order derivative of the function $\sin(\log x)$

Solution:

Let $y = \sin(\log x)$
Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin x(\log x)\right]$$

$$= \cos(\log x) \cdot \frac{d}{dx}(\log x)$$

$$= \frac{\cos(\log x)}{x}$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$= \frac{x \cdot \frac{d}{dx}\left[\cos(\log x)\right] - \cos(\log x) \cdot \frac{d}{dx}(x)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x\left[-\sin(\log x) \cdot \frac{d}{dx}(\log x)\right] - \cos(\log x) \cdot 1}{x^2}$$

$$= \frac{-x\sin(\log x) \cdot \frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-\left[\sin(\log x) + \cos(\log x)\right]}{x^2}$$

Question 11:

If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Solution:

Given, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x)$$

$$= 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$

$$= 5(-\sin x) - 3\cos x$$

$$= -(5\sin x + 3\cos x)$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(5\sin x + 3\cos x)\right]$$

$$= -\left[5 \cdot \frac{d}{dx}(\sin x) + 3 \cdot \frac{d}{dx}(\cos x)\right]$$

$$= -\left[5\cos x + 3(-\sin x)\right]$$

$$= -\left[5\cos x - 3\sin x\right]$$

$$= -y$$

Thus, $\frac{d^2y}{dx^2} + y = 0$

Hence proved.

Question 12:

If $y = \cos^{-1} x$, find $\dfrac{d^2 y}{dx^2}$ in terms of y alone.

Solution:

Given, $y = \cos^{-1} x$

Then,

$$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\cos^{-1} x\right)$$

$$= \dfrac{-1}{\sqrt{1-x^2}}$$

$$= -\left(1-x^2\right)^{\frac{-1}{2}}$$

Therefore,

$$\dfrac{d^2 y}{dx^2} = \dfrac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$= -\left(-\dfrac{1}{2}\right).\left(1-x^2\right)^{\frac{-3}{2}}.\dfrac{d}{dx}\left(1-x^2\right)$$

$$= \dfrac{1}{2\sqrt{\left(1-x^2\right)^3}} \times (-2x)$$

$$\dfrac{d^2 y}{dx^2} = \dfrac{-x}{\sqrt{\left(1-x^2\right)^3}} \qquad \ldots(1)$$

But we need to calculate $\dfrac{d^2 y}{dx^2}$ in terms of y

$\Rightarrow y = \cos^{-1} x$

$\Rightarrow x = \cos y$

Putting $x = \cos y$ in equation (1), we get

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{(1-\cos^2 y)^3}}$$

$$= \frac{-\cos y}{\sqrt{(\sin^2 y)^3}}$$

$$= \frac{-\cos y}{\sin^3 y}$$

$$= \frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y}$$

$$= -\cot y.\operatorname{cosec}^2 y$$

Question 13:

If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2 y_2 + x y_1 + y = 0$

Solution:

Given, $y = 3\cos(\log x) + 4\sin(\log x)$

Then,

$$y_1 = 3.\frac{d}{dx}\big[\cos(\log x)\big] + 4.\frac{d}{dx}\big[\sin(\log x)\big]$$

$$= 3.\left[-\sin(\log x).\frac{d}{dx}(\log x)\right] + 4.\left[\cos(\log x).\frac{d}{dx}(\log x)\right]$$

$$= \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$= \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

Therefore,

$$y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x.\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}\{x\}'}{x^2}$$

$$= \frac{x.\left[4\{\cos(\log x)\}' - \{3\sin(\log x)\}'\right] - \{4\cos(\log x) - 3\sin(\log x)\}.1}{x^2}$$

$$= \frac{x.\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x.\left[-4\sin(\log x)\frac{1}{x} - 3\cos(\log x)\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

Thus,

$$x^2 y_2 + x y_1 + y = \left[x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) \atop + 3\cos(\log x) + 4\sin(\log x)\right]$$

$$= \left[-\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) \atop + 3\cos(\log x) + 4\sin(\log x)\right]$$

$$= 0$$

Hence proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2 y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.

Solution:

Given, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A.\frac{d}{dx}\left(e^{mx}\right) + B.\frac{d}{dx}\left(e^{nx}\right)$$

$$= A.e^{mx}.\frac{d}{dx}(mx) + B.e^{nx}.\frac{d}{dx}(nx)$$

$$= Ame^{mx} + Bne^{nx}$$

Therefore,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right)$$

$$= Am.\frac{d}{dx}\left(e^{mx}\right) + Bn.\frac{d}{dx}\left(e^{nx}\right)$$

$$= Am.e^{mx}.\frac{d}{dx}\left(mx\right) + Bn.e^{nx}.\frac{d}{dx}\left(nx\right)$$

$$= Am^2 e^{mx} + Bn^2 e^{nx}$$

Thus,

$$\frac{d^2 y}{dx^2} - \left(m+n\right)\frac{dy}{dx} + mny = Am^2 e^{mx} + Bn^2 e^{nx} - \left(m+n\right).\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2 e^{mx} + Bn^2 e^{nx} - Am^2 e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2 e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\dfrac{d^2 y}{dx^2} = 49y$

Solution:

Given, $y = 500e^{7x} + 600e^{-7x}$
Then,

$$\frac{dy}{dx} = 500.\frac{d}{dx}\left(e^{7x}\right) + 600.\frac{d}{dx}\left(e^{-7x}\right)$$

$$= 500.e^{7x}.\frac{d}{dx}\left(7x\right) + 600.e^{-7x}.\frac{d}{dx}\left(-7x\right)$$

$$= 3500e^{7x} - 4200e^{-7x}$$

Therefore,

$$\frac{d^2 y}{dx^2} = 3500e^{7x}.\frac{d}{dx}\left(e^{7x}\right) - 4200.\frac{d}{dx}\left(e^{-7x}\right)$$

$$= 3500.e^{7x}.\frac{d}{dx}\left(7x\right) - 4200.e^{-7x}.\frac{d}{dx}\left(-7x\right)$$

$$= 7 \times 3500.e^{7x} + 7 \times 4200.e^{-7x}$$

$$= 49 \times 500.e^{7x} + 49 \times 600e^{-7x}$$

$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$

$$= 49y$$

Hence proved.

Question 16:

If $e^y (x+1) = 1$, show that $\dfrac{d^2 y}{dx^2} = \left(\dfrac{dy}{dx} \right)^2$

Solution:

Given, $e^y (x+1) = 1$

$\Rightarrow e^y (x+1) = 1$

$\Rightarrow e^y = \dfrac{1}{x+1}$

Taking log on both sides, we get

$y = \log \dfrac{1}{(x+1)}$

Differentiating with respect to x, we get

$\dfrac{dy}{dx} = (x+1) \dfrac{d}{dx} \left(\dfrac{1}{x+1} \right)$

$\qquad = (x+1). \dfrac{-1}{(x+1)^2}$

$\qquad = \dfrac{-1}{x+1}$

Therefore,

$\dfrac{d^2 y}{dx^2} = \dfrac{d}{dx} \left(\dfrac{-1}{x+1} \right) = -\left(\dfrac{-1}{(x+1)^2} \right)$

$\qquad = \dfrac{1}{(x+1)^2} = \left(\dfrac{-1}{x+1} \right)^2$

$\qquad = \left(\dfrac{dy}{dx} \right)^2$

Hence proved.

Question 17:

If $y = \left(\tan^{-1} x \right)^2$, show that $\left(x^2 + 1 \right)^2 y_2 + 2x \left(x^2 + 1 \right) y_1 = 2$

Solution:

Given, $y = \left(\tan^{-1} x \right)^2$

Then,

$$\Rightarrow y_1 = 2\tan^{-1} x \frac{d}{dx}\left(\tan^{-1} x\right)$$

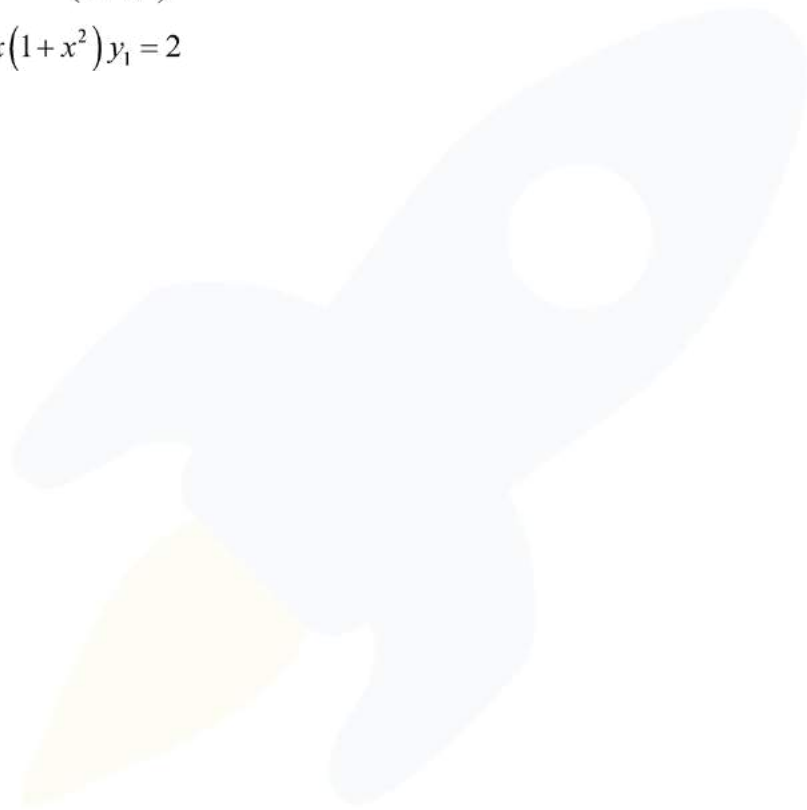$$\Rightarrow y_1 = 2\tan^{-1} x.\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right) y_1 = 2\tan^{-1} x$$

Again, differentiating with respect to x, we get

$$\Rightarrow \left(1+x^2\right) y_2 + 2xy_1 = 2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2 y_2 + 2x\left(1+x^2\right)y_1 = 2$$

Hence proved.

EXERCISE 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$

Solution:

Given, $f(x) = x^2 + 2x - 8$, being polynomial function is continuous in $[-4, 2]$ and also differentiable in $(-4, 2)$.

$$f(-4) = (-4)^2 + 2.(-4) - 8$$
$$= 16 - 8 - 8$$
$$= 0$$

$$f(2) = (2)^2 + 2 \times 2 - 8$$
$$= 4 + 4 - 8$$
$$= 0$$

Therefore, $f(-4) = f(2) = 0$

The value of $f(x)$ at -4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

$$f(x) = x^2 + 2x - 8$$

Therefore, $f'(x) = 2x + 2$

Hence,

$$f'(c) = 0$$
$$2c + 2 = 0$$
$$c = -1$$

Thus, $c = -1 \in (-4, 2)$

Hence, Rolle's Theorem is verified.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these examples?

(i) $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x)=[x]$ for $x\in[-2,2]$

(iii) $f(x)=x^2-1$ for $x\in[1,2]$

Solution:

By Rolle's Theorem, $f:[a,b]\to \mathbf{R}$,

If

(a) f is continuous on $[a,b]$

(b) f is continuous on (a,b)

(c) $f(a)=f(b)$

Then, there exists some $c\in(a,b)$ such that $f'(c)=0$

Thus, Rolle's Theorem is not applicable to those functions that do not satisfy any of three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,9]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x=5$ and $x=9$

Therefore, $f(x)$ is not continuous in $[5,9]$

Also, $f(5)=[5]=5$ and $f(9)=[9]=9$

Thus, $f(5)\neq f(9)$

The differentiability of f in $(5,9)$ is checked as follows.

Let n be an integer such that $n\in(5,9)$

The LHD of f at $x=n$ is

$$\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

The RHD of f at $x=n$ is

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since LHD and RHD of f at $x=n$ are not equal, f is not differentiable at $x=n$

Therefore, f is not differentiable in $(5,9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Thus, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x \in [5,9]$.

(ii) $f(x)=[x]$ for $x \in [-2,2]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x = -2$ and $x = 2$

Therefore, $f(x)$ is not continuous in $[-2,2]$

Also, $f(-2)=[-2]=-2$ and $f(2)=[2]=2$

Thus, $f(-2) \neq f(2)$

The differentiability of f in $(-2,2)$ is checked as follows.

Let n be an integer such that $n \in (-2,2)$

The LHD of f at $x = n$ is

$$\lim_{h \to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x = n$ is

$$\lim_{h \to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h \to 0^+} \frac{n-1-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since LHD and RHD of f at $x = n$ are not equal, f is not differentiable at $x = n$

Therefore, f is not differentiable in $(-2,2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Thus, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x \in [-2,2]$

(iii) $f(x)=x^2 -1$ for $x \in [1,2]$

Since, f being a polynomial function is continuous in $[1,2]$ and is differentiable in $(1,2)$.

Thus,

$f(1)=(1)^2 -1=0$

$f(2)=(2)^2 -1=3$

Therefore, $f(1) \neq f(2)$

Since, f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Question 3:

If $f : [-5, 5] \rightarrow \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Solution:

Given, $f : [-5, 5] \rightarrow \mathbf{R}$ is a differentiable function.
Since every differentiable function is a continuous function, we obtain

(i) f is continuous on $[-5, 5]$

(ii) f is continuous on $(-5, 5)$

Thus, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$\Rightarrow f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10 f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.
Therefore, $f'(c) \neq 0$
Thus,

$$\Rightarrow 10 f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Hence proved.

Question 4:

Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the integral $[a, b]$, where $a = 1$ and $b = 4$.

Solution:

Given, $f(x) = x^2 - 4x - 3$

f, being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$, whose derivative is $2x - 4$.

Thus,

$f(1) = 1^2 - 4 \times 1 - 3 = -6$

$f(4) = 4^2 - 4 \times 4 - 3 = -3$

Therefore,

$$\frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1}$$

$$= \frac{-3 - (-6)}{3}$$

$$= \frac{3}{3}$$

$$= 1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$
Hence,

$\Rightarrow f'(c) = 1$

$\Rightarrow 2c - 4 = 1$

$\Rightarrow c = \dfrac{5}{2}$ $\left[\text{where } c = \dfrac{5}{2} \in (1, 4) \right]$

Thus, mean value theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$ where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Solution:

Given, f is $f(x) = x^3 - 5x^2 - 3x$

f, being a polynomial function, is continuous in $[1, 3]$ and is differentiable in $(1, 3)$, whose derivative is $3x^2 - 10x - 3$

Thus,

$f(1) = 1^3 - 5 \times 1^2 - 3 \times 1 = -7$

$f(3) = 3^3 - 5 \times 3^2 - 3 \times 3 = -27$

Therefore,

$$\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1}$$
$$= \frac{-27-(-7)}{3-1}$$
$$= -10$$

Mean Value Theorem states that there exists a point $c \in (1,3)$ such that $f'(c) = -10$

Hence,

$$\Rightarrow f'(c) = -10$$
$$\Rightarrow 3c^2 - 10c - 3 = -10$$
$$\Rightarrow 3c^2 - 10c + 7 = 0$$
$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$
$$\Rightarrow 3c(c-1) - 7(c-1) = 0$$
$$\Rightarrow (c-1)(3c-7) = 0$$
$$\Rightarrow c = 1, \frac{7}{3} \qquad \left[\text{where } c = \frac{7}{3} \in (1,3)\right]$$

Thus, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1,3)$ is the only point for which $f'(c) = 0$.

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given

(i) $f(x) = [x]$ for $x \in [5,9]$

(ii) $f(x) = [x]$ for $x \in [-2,2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1,2]$

Solution:

Mean Value Theorem states that for a function $f:[a,b] \to \mathbf{R}$, if

(a) f is continuous on $[a,b]$

(b) f is continuous on (a,b)

Then there exists some $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Thus, Mean Value Theorem is not applicable to those functions that do not satisfy any of three conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5,9]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x = 5$ and $x = 9$

Therefore, $f(x)$ is not continuous in $[5,9]$

The differentiability of f in $(5,9)$ is checked as follows.

Let n be an integer such that $n \in (5,9)$

The LHD of f at $x = n$ is

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x = n$ is

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since LHD and RHD of f at $x = n$ are not equal, f is not differentiable at $x = n$

Therefore, f is not differentiable in $(5,9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Thus, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [5,9]$

(ii) $f(x) = [x]$ for $x \in [-2,2]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x = -2$ and $x = 2$

Therefore, $f(x)$ is not continuous in $[-2,2]$

The differentiability of f in $(-2,2)$ is checked as follows.

Let n be an integer such that $n \in (-2,2)$

The LHD of f at $x = n$ is

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x = n$ is

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{x \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since LHD and RHD of f at $x = n$ are not equal, f is not differentiable at $x = n$

Therefore, f is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Thus, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Since, f being a polynomial function is continuous in $[1, 2]$ and is differentiable in $(1, 2)$
It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

It can be proved as follows.
We have, $f(x) = x^2 - 1$
Then,

$$f(1) = (1)^2 - 1 = 0,$$
$$f(2) = (2)^2 - 1 = 3$$

Therefore,

$$\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1}$$
$$= 3$$

Hence, $f'(x) = 2x$

Thus,

$$\Rightarrow f'(c) = 3$$
$$\Rightarrow 2c = 3$$
$$\Rightarrow c = \frac{3}{2}$$
$$\Rightarrow c = 1.5 \qquad \left[\text{where } 1.5 \in [1, 2] \right]$$

MISCELLANEOUS EXERCISE

Question 1:

Differentiate with respect to x the function $\left(3x^2 - 9x + 5\right)^9$.

Solution:

Let $y = \left(3x^2 - 9x + 5\right)^9$

Using chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^2 - 9x + 5\right)^9$$

$$= 9\left(3x^2 - 9x + 5\right)^8 \cdot \frac{d}{dx}\left(3x^2 - 9x + 5\right)$$

$$= 9\left(3x^2 - 9x + 5\right)^8 \cdot (6x - 9)$$

$$= 9\left(3x^2 - 9x + 5\right)^8 \cdot 3(2x - 3)$$

$$= 27\left(3x^2 - 9x + 5\right)^8 (2x - 3)$$

Question 2:

Differentiate with respect to x the function $\sin^3 x + \cos^6 x$.

Solution:

Let $y = \sin^3 x + \cos^6 x$

Using chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^3 x\right) + \frac{d}{dx}\left(\cos^6 x\right)$$

$$= 3\sin^2 x \cdot \frac{d}{dx}\left(\sin x\right) + 6\cos^5 x \cdot \frac{d}{dx}\left(\cos x\right)$$

$$= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x)$$

$$= 3\sin x \cos x \left(\sin x - 2\cos^4 x\right)$$

Question 3:

Differentiate with respect to x the function $\left(5x\right)^{3\cos 2x}$.

Solution:

Let $y = (5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$$\log y = 3\cos 2x \log 5x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x . \frac{d}{dx}(\cos 2x) + \cos 2x . \frac{d}{dx}(\log 5x)\right]$$

$$\frac{dy}{dx} = 3y\left[\log 5x .(-\sin 2x) . \frac{d}{dx}(2x) + \cos 2x . \frac{1}{5x} . \frac{d}{dx}(5x)\right]$$

$$= 3y\left[-2\sin 2x . \log 5x + \frac{\cos 2x}{x}\right]$$

$$= y\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

$$= (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

Question 4:

Differentiate with respect to x the function $\sin^{-1}\left(x\sqrt{x}\right), \ 0 \le x \le 1$.

Solution:

Let $y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx} \sin^{-1}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{3}{2} \cdot x^{\frac{1}{2}}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5:

Differentiate with respect to x the function $\dfrac{\cos^{-1}\dfrac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$.

Solution:

Let $y = \dfrac{\cos^{-1}\dfrac{x}{2}}{\sqrt{2x+7}}$

Using quotient rule, we get

$$\frac{dy}{dx} = \frac{\sqrt{2x+7} \cdot \frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right) \cdot \frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)\right] - \left(\cos^{-1}\frac{x}{2}\right) \cdot \frac{1}{2\sqrt{2x+7}} \cdot \frac{d}{dx}(2x+7)}{2x+7}$$

$$= \frac{\sqrt{2x+7} \cdot \frac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1}\frac{x}{2}\right) \cdot \frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\left(\sqrt{4-x^2}\right) \cdot (2x+7)} - \frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

Differentiate with respect to x the function $\cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$.

Solution:

Let $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$ ...(1)

Then,

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}=\frac{1+\cos x}{\sin x}$$

$$=\frac{1+2\cos^2\frac{x}{2}-1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes,

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

Thus,

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$=\frac{1}{2}$$

Question 7:

Differentiate with respect to x the function $(\log x)^{\log x},\ x>1$.

Solution:

Let $y=(\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y=\log x.\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\Big[\log x.\log(\log x)\Big]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}\Big[\log(\log x)\Big]$$

$$\Rightarrow \frac{dy}{dx} = y\Bigg[\log(\log x).\frac{1}{x} + \log x.\frac{1}{\log x}.\frac{d}{dx}(\log x)\Bigg]$$

$$\Rightarrow \frac{dy}{dx} = y\Bigg[\frac{1}{x}.\log(\log x) + \frac{1}{x}\Bigg]$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\log x}\Bigg[\frac{1}{x} + \frac{\log(\log x)}{x}\Bigg]$$

Question 8:

Differentiate with respect to x the function $\cos(a\cos x + b\sin x)$, for some constant a and b.

Solution:

Let $y = \cos(a\cos x + b\sin x)$

Using chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x).\frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x).\big[a(-\sin x) + b\cos x\big]$$

$$= (a\sin x - b\cos x).\sin(a\cos x + b\sin x)$$

Question 9:

Differentiate with respect to x the function $(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking log on both the sides, we obtain

$$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$$
$$= (\sin x - \cos x)\log(\sin x - \cos x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x)\log(\sin x - \cos x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x).\frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x).\frac{d}{dx}\log(\sin x - \cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x).(\cos x + \sin x) + (\sin x - \cos x).\frac{1}{(\sin x - \cos x)}.\frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\left[(\cos x + \sin x).\log(\sin x - \cos x) + (\cos x + \sin x)\right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right]$$

Question 10:

Differentiate with respect to x the function $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

Solution:

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u$, $x^a = v$, $a^x = w$ and $a^a = s$

Therefore,

$$\Rightarrow y = u + v + w + s$$
$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \qquad ...(1)$$

Now, $u = x^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log x^x$$
$$\Rightarrow \log u = x \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x . \frac{d}{dx}(x) + x . \frac{d}{dx}(\log x)$$

$$\frac{du}{dx} = u\left[\log x . 1 + x . \frac{1}{x}\right]$$

$$= x^x\left[\log x + 1\right] = x^x\left(1 + \log x\right) \qquad \dots(2)$$

Now, $v = x^a$
Hence,

$$\frac{dv}{dx} = \frac{d}{dx}(x^a)$$

$$= ax^{a-1} \qquad \dots(3)$$

Now, $w = a^x$
Taking logarithm on both the sides, we obtain

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w}\frac{dw}{dx} = \log a . \frac{d}{dx}(x)$$

$$\frac{dw}{dx} = w \log a$$

$$= a^x \log a \qquad \dots(4)$$

Now, $s = a^a$
Since a is constant, a^a is also a constant.
Hence,

$$\frac{ds}{dx} = 0 \qquad \dots(5)$$

From (1), (2), (3), (4) and (5), we obtain

$$\frac{dy}{dx} = x^x\left(1 + \log x\right) + ax^{a-1} + a^x \log a + 0$$

$$= x^x\left(1 + \log x\right) + ax^{a-1} + a^x \log a$$

Question 11:

Differentiate with respect to x the function $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$.

Solution:

Let $y = x^{x^2-3} + (x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$
Therefore,

$y = u + v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ $\qquad \ldots(1)$

Now, $u = x^{x^2-3}$
Taking logarithm on both the sides, we obtain

$\log u = \log\left(x^{x^2-3}\right)$

$\qquad = \left(x^2 - 3\right)\log x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{u}\dfrac{du}{dx} = \log x \cdot \dfrac{d}{dx}\left(x^2 - 3\right) + \left(x^2 - 3\right) \cdot \dfrac{d}{dx}(\log x)$

$\Rightarrow \dfrac{1}{u}\dfrac{du}{dx} = \log x \cdot 2x + \left(x^2 - 3\right) \cdot \dfrac{1}{x}$

$\Rightarrow \dfrac{du}{dx} = x^{x^2-3}\left[\dfrac{x^2 - 3}{x} + 2x\log x\right]$ $\qquad \ldots(2)$

Now, $v = (x-3)^{x^2}$
Taking logarithm on both the sides, we obtain

$\log v = \log(x-3)^{x^2}$

$\qquad = x^2\log(x-3)$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{v}\dfrac{dv}{dx} = \log(x-3) \cdot \dfrac{d}{dx}\left(x^2\right) + \left(x^2\right) \cdot \dfrac{d}{dx}\left[\log(x-3)\right]$

$\Rightarrow \dfrac{1}{v}\dfrac{dv}{dx} = \log(x-3) \cdot 2x + x^2 \cdot \dfrac{1}{x-3} \cdot \dfrac{d}{dx}(x-3)$

$\Rightarrow \dfrac{dv}{dx} = v\left[2x\log(x-3) + \dfrac{x^2}{x-3} \cdot 1\right]$

$\Rightarrow \dfrac{dv}{dx} = (x-3)^{x^2}\left[\dfrac{x^2}{x-3} + 2x\log(x-3)\right]$ $\qquad \ldots(3)$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y = 12(1-\cos t), x = 10(t-\sin t), \frac{-\pi}{2} < t < \frac{\pi}{2}$

Solution:

The given function is $y = 12(1-\cos t), x = 10(t-\sin t)$

Hence,

$$\frac{dx}{dt} = \frac{d}{dt}[10(t-\sin t)]$$

$$= 10.\frac{d}{dt}(t-\sin t)$$

$$= 10(1-\cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt}[12(1-\cos t)]$$

$$= 12.\frac{d}{dt}(1-\cos t)$$

$$= 12.[0-(-\sin t)]$$

$$= 12\sin t$$

Therefore,

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{12\sin t}{10(1-\cos t)}$$

$$= \frac{12.2\sin\frac{t}{2}.\cos\frac{t}{2}}{10.2\sin^2\frac{t}{2}}$$

$$= \frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}, -1 \le x \le 1$.

Solution:

The given function is $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$

Hence,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1} x + \sin^{-1}\sqrt{1-x^2}\right]$$

$$= \frac{d}{dx}\left(\sin^{-1} x\right) + \frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}\left(1-x^2\right)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}}\left(-2x\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$= 0$$

Question 14:

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for $-1 < x < 1$, prove that $\dfrac{dy}{dx} = -\dfrac{1}{\left(1+x\right)^2}$.

Solution:

The given function is $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we obtain

$$x^2\left(1+y\right) = y^2\left(1+x\right)$$

$$\Rightarrow x^2 + x^2 y = y^2 + xy^2$$

$$\Rightarrow x^2 - y^2 = xy^2 - x^2 y$$

$$\Rightarrow x^2 - y^2 = xy\left(y - x\right)$$

$$\Rightarrow \left(x+y\right)\left(x-y\right) = xy\left(y-x\right)$$

$$\Rightarrow x + y = -xy$$

$$\Rightarrow \left(1+x\right)y = -x$$

$$\Rightarrow y = \frac{-x}{\left(1+x\right)}$$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx} = -\left[\frac{(1+x)\frac{d}{dx}(x) - (x).\frac{d}{dx}(1+x)}{(1+x)^2}\right]$$

$$= -\frac{(1+x) - x}{(1+x)^2}$$

$$= -\frac{1}{(1+x)^2}$$

Hence proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$ for $c > 0$, prove that $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ is a constant independent of a and b.

Solution:

The given function is $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right] + \frac{d}{dx}\left[(y-b)^2\right] = \frac{d}{dx}(c^2)$$

$$\Rightarrow 2(x-a).\frac{d}{dx}(x-a) + 2(y-b).\frac{d}{dx}(y-b) = 0$$

$$\Rightarrow 2(x-a).1 + 2(y-b).\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x-a)}{y-b} \qquad ...(1)$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= -\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a) - (x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) - (x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) - (x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$= -\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3}\right]$$

Hence,

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$$= \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$$= -c$$

$-c$ is a constant and is independent of a and b.

Hence proved.

Question 16:

If $\cos y = x\cos(a+y)$ with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$.

Solution:

The given function is $\cos y = x\cos(a+y)$
Therefore,

$$\Rightarrow \frac{d}{dx}[\cos y] = \frac{d}{dx}\big[x\cos(a+y)\big]$$

$$\Rightarrow -\sin y \frac{dy}{dx} = \cos(a+y).\frac{d}{dx}(x) + x.\frac{d}{dx}\big[\cos(a+y)\big]$$

$$\Rightarrow -\sin y \frac{dy}{dx} = \cos(a+y) + x.\Big[-\sin(a+y)\Big]\frac{dy}{dx}$$

$$\Rightarrow \big[x\sin(a+y) - \sin y\big]\frac{dy}{dx} = \cos(a+y) \qquad \ldots(1)$$

Since, $\cos y = x\cos(a+y) \Rightarrow x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) becomes,

$$\left[\frac{\cos y}{\cos(a+y)}.\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow \big[\cos y.\sin(a+y) - \sin y.\cos(a+y)\big].\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Hence proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$.

Solution:

The given function is $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$
Therefore,

$$\frac{dx}{dt} = a.\frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t.\frac{d}{dx}(t) + t.\frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t]$$

$$= at\cos t$$

$$\frac{dy}{dt} = a.\frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t.\frac{d}{dt}(t) + t.\frac{d}{dt}(\cos t)\right\}\right]$$

$$= a[\cos t - \{\cos t - t\sin t\}]$$

$$= at\sin t$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t.\frac{dt}{dx}$$

$$= \sec^2 t.\frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x, and find it.

Solution:

It is known that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when $x \geq 0, f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0, f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Question 19:

Using mathematical induction prove that $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Solution:

To prove: $P(n): \dfrac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.
For $n = 1$,

$$P(1): \dfrac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

Therefore, $P(n)$ is true for $n = 1$.

Let $P(k)$ is true for some positive integer k.

That is, $P(k): \dfrac{d}{dx}(x^k) = kx^{k-1}$

It has to be proved that $P(k+1)$ is also true.
Consider

$$\dfrac{d}{dx}(x^{k+1}) = \dfrac{d}{dx}(x \cdot x^k)$$

$$= x^k \cdot \dfrac{d}{dx}(x) + x \cdot \dfrac{d}{dx}(x^k) \qquad [\text{By applying product rule}]$$

$$= x^k \cdot 1 + x \cdot k \cdot x^{k-1}$$

$$\dfrac{d}{dx}(x^{k+1}) = x^k + kx^k$$

$$= (k+1) \cdot x^k$$

$$= (k+1) \cdot x^{(k+1)-1}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

Solution:

Given, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

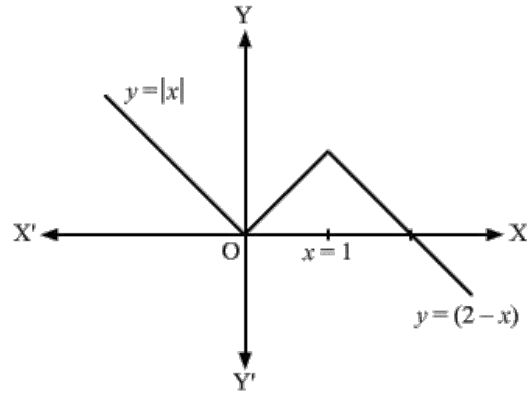Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right] = \frac{d}{dx}(\sin A \cos B) + \frac{d}{dx}(\cos A \sin B)$$

$$\Rightarrow \cos(A+B).\frac{d}{dx}(A+B) = \cos B.\frac{d}{dx}(\sin A) + \sin A.\frac{d}{dx}(\cos B) + \sin B.\frac{d}{dx}(\cos A) + \cos A.\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B).\frac{d}{dx}(A+B) = \cos B.\cos A\frac{dA}{dx} + \sin A(-\sin B)\frac{dB}{dx} + \sin B(-\sin A).\frac{dA}{dx} + \cos A \cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B).\left[\frac{dA}{dx} + \frac{dB}{dx}\right] = (\cos A \cos B - \sin A \sin B).\left[\frac{dA}{dx} + \frac{dB}{dx}\right]$$

$$\Rightarrow \cos(A+B) = \cos A \cos B - \sin A \sin B$$

Question 21:
Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer?

Solution:

Consider, $y = \begin{cases} |x| & -\infty < x \le 1 \\ 2-x & 1 \le x \le \infty \end{cases}$

It can be seen from the above graph that the given function is continuous everywhere but not differentiable at exactly two points which are 0 and 1.

Question 22:

If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Solution:

Given, $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$\Rightarrow y = (mc - nb)f(x) - (lc - na)g(x) + (lb - ma)h(x)$

Then,

$\dfrac{dy}{dx} = \dfrac{d}{dx}\big[(mc - nb)f(x)\big] - \dfrac{d}{dx}\big[(lc - na)g(x)\big] + \dfrac{d}{dx}\big[(lb - ma)h(x)\big]$

$= (mc - nb)f'(x) - (lc - na)g'(x) + (lb - ma)h'(x)$

$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Thus, $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ proved.

Question 23:

If $y = e^{a \cos^{-1} x}, -1 \le x \le 1$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2 y = 0$

Solution:

The given function is $y = e^{a \cos^{-1} x}$
Taking logarithm on both the sides, we obtain

$\Rightarrow \log y = a \cos^{-1} x \log e$

$\Rightarrow \log y = a \cos^{-1} x$

Differentiating both sides with respect to x, we obtain

$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = a \cdot \dfrac{-1}{\sqrt{1-x^2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-ay}{\sqrt{1-x^2}}$

By squaring both the sides, we obtain

$\Rightarrow \left(\dfrac{dy}{dx}\right)^2 = \dfrac{a^2 y^2}{1-x^2}$

$\Rightarrow (1-x^2)\left(\dfrac{dy}{dx}\right)^2 = a^2 y^2$

Again, differentiating both sides with respect to x, we obtain

$\Rightarrow \left(\dfrac{dy}{dx}\right)^2 \dfrac{d}{dx}(1-x^2) + (1-x^2) \times \dfrac{d}{dx}\left[\left(\dfrac{dy}{dx}\right)^2\right] = a^2 \dfrac{d}{dx}(y^2)$

$\Rightarrow \left(\dfrac{dy}{dx}\right)^2 (-2x) + (1-x^2) \times 2\dfrac{dy}{dx}\cdot\dfrac{d^2y}{dx^2} = a^2 . 2y. \dfrac{dy}{dx}$

$\Rightarrow -x\dfrac{dy}{dx} + (1-x^2)\dfrac{d^2y}{dx^2} = a^2 . y \qquad \left[\dfrac{dy}{dx} \ne 0\right]$

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2 y = 0$

Hence proved.

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